Variational Inequalities

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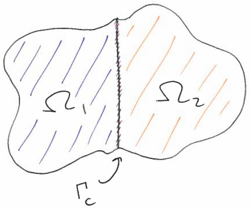
2018 What is. . . Numerical Analysis?

Transmission Problem



Suppose we have a domain $\Omega = \Omega_1 \cup \Omega_2$ and we want to understand the displacements u_1, u_2 materials Ω_1 and Ω_2 which is experiencing force f.

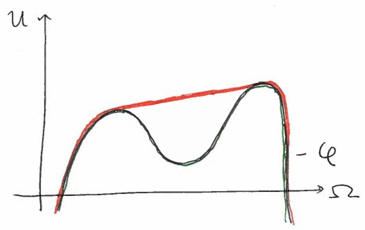
This can be done by minimising the energy

$$E(v_1, v_2) = \sum_{i=1}^{n} \frac{\alpha_i}{2} \int_{\Omega_i} |\nabla v_i|^2 - \int_{\Omega_i} f v_i$$

over $v_i \in H^1(\Omega_i)$ such that $v_1 = v_2$ on Γ_c and $u_i|_{\partial\Omega} = 0$.

Obstacle Problem

Consider a domain/material Ω that we wish to understand under a force f but there is an obstacle, $\varphi \in C^\infty$ that is impassable.



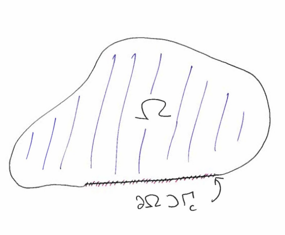
This can be done by minimising the energy

$$E(v) = \int_\Omega |\nabla v|^2 - \int_\Omega fv$$

over the set of $v \in H^1_0(\Omega)$ such that $v \geq \varphi$.

Signorini Problem

Consider a domain/material Ω that we wish to understand as it sits on a surface $\varphi \in C^\infty$ with contact surface $\Gamma_c \subset \partial\Omega$ under a force f.



Similarly, we can do this by minimising the energy

$$E(v) = \int_\Omega |\nabla v|^2 - \int_\Omega fv$$

over the set of $v \in H^1(\Omega)$ such that $v \geq \varphi$ on Γ_c.

Image from http://www.fibre2fashion.com/industry-article/5787/development-of-double-hydrophilic-hydrophobic-surfaces-of-wool-fabric

Differences With Course Material

These examples all have energy described by a symmetric, coercive, continuous bilinear form with continuous functional f. This energy is then minimised over a **convex subset** instead of a vector space.

Lax–Milgram	$\longleftrightarrow$	Lax–Milgram
Weak Formulation	$\longleftrightarrow$	Variational Inequality
Cea's Lemma	$\longleftrightarrow$	Falk's Theorem

Lax–Milgram

Theorem

Suppose $K \subset H$ is closed and convex, $a(\cdot, \cdot)$ is a continuous, coercive symmetric bilinear form and $f \in H^*$. Then

$$E(v) = \frac{1}{2}a(v, v) - \langle f, v \rangle$$

has a unique minimiser.

The proof will be done in 4 steps:

- Show $E(v)$ is bounded below.
- Show an minimising sequence is Cauchy.
- Show the Cauchy limit is in K.
- Uniqueness.

Proof 1/4 — Bounded Below

$$\begin{aligned}
E(v) &= \frac{1}{2}a(v,v) - \langle f, v\rangle \\
&\geq \frac{\alpha}{2}\|v\|^2 - \left|\langle f, v\rangle\right| \\
&\geq \frac{\alpha}{2}\|v\|^2 - \|f\|\|v\| \\
&= \frac{1}{2\alpha}\left(\alpha\|v\| - \|f\|\right)^2 - \frac{1}{2\alpha}\|f\|^2 \\
&\geq -\frac{1}{2\alpha}\|f\|^2
\end{aligned}$$

Let $m := \inf_{v\in K} E(v)$.

Proof 2/4 — Minimising Sequence is Cauchy

Let $(v_n)_{n\in\mathbb{N}}$ be a sequence so that $E(v_n) \to m$. By coercivity,

$$\begin{aligned}
\alpha\|v_n - v_m\|^2 &\leq a(v_n - v_m, v_n - v_m) \\
&= a(v_n, v_n) - a(v_m, v_n) - a(v_n, v_m) + a(v_m, v_m) \\
&= 2a(v_n, v_n) + 2a(v_m, v_m) \\
&\quad - \left(a(v_m, v_n) + a(v_n, v_n) + a(v_n, v_m) + a(v_m, v_m)\right) \\
&= 2a(v_n, v_n) + 2a(v_m, v_m) - a(v_n + v_m, v_n + v_m) \\
&= 4E(v_n) + 4E(v_m) - 8E\left(\frac{1}{2}(v_n + v_m)\right) \\
&\leq 4E(v_n) + 4E(v_m) - 8m.
\end{aligned}$$

Proof 3/4 — Limit Is In K

Since K is closed by assumption, $u := \lim_n v_n \in K$. Moreover, a and f are continuous and so

$$E(v) = \frac{1}{2}a(v, v) - \langle f, v \rangle$$

is continuous. Hence

$$E(u) = E(\lim_n v_n) = \lim_n E(v_n) = m.$$

So u is a minimiser.

Proof 4/4 — Minimiser Is Unique

Standard trick — suppose there are two distinct minimisers, u and $\tilde{u}$. Define $(u_n)_{n\in\mathbb{N}}$ by

$$u_n = \begin{cases} u & n \equiv 0 \mod 2 \\ \tilde{u} & n \equiv 1 \mod 2. \end{cases}$$

By the argument in step 2, $(u_n)_{n\in\mathbb{N}}$ is Cauchy. However,

$$\left\|u_n - u_{n+1}\right\| = \left\|u - \tilde{u}\right\| \geq c > 0$$

for every n. So $(u_n)_{n\in\mathbb{N}}$ is not Cauchy. □

Variational Inequality

Theorem

Let V be a vector space with a symmetric, positive definite bilinear form a. In addition, let $f : V \to \mathbb{R}$ be linear and $C \subset V$ be convex. Then u minimises $E(v)$ over C if and only if

$$a(u, h - u) \geq \langle f, h - u \rangle \quad \forall h \in C.$$

Notice that if C is a vector space, then we can let $h = u \pm w$ for any $w \in C$. Hence

$$a(u, w) \geq \langle f, w \rangle \quad \text{and} \quad a(u, -w) \geq \langle u, -w \rangle.$$

So we recover the weak formulation

$$a(u, w) = \langle u, w \rangle \quad \forall w \in \ C.$$

Proof 1/2

Take the usual difference

$$
\begin{aligned}
& E(u+\lambda v) - E(u) \\
= \; & \frac{1}{2} a(u+\lambda v, u+\lambda v) - \langle f, u+\lambda v \rangle - E(u) \\
= \; & \frac{1}{2} a(u,u) - \langle f, u \rangle - E(u) + \lambda \left(a(u,v) - \langle f, v \rangle \right) + \frac{\lambda^2}{2} a(v,v).
\end{aligned}
$$

If u minimise $E(v)$ then this is greater than or equal to 0 for all $u - \lambda v \in C$. Let $v = h - u$, $h \in C$. Then for $\lambda \in [0,1]$,

$$u + \lambda v = (1-\lambda)u + \lambda h \in C.$$

Therefore

$$a(u,v) - \langle f, v \rangle \geq -\frac{\lambda}{2} a(v,v) \quad \forall \lambda \in (0,1].$$

Proof 2/2

Hence

$$a(u, v) - \langle f, v \rangle \geq 0.$$

Since v was chosen to be $h - u$ this is equivalent to

$$a(u, h - u) \geq \langle f, h - u \rangle \quad \forall h \in C.$$

For the reverse implication, we reverse this argument. $\square$

Falk's Theorem

Theorem

- *Let a be a symmetric, coercive, continuous bilinear form on H and $f \in H^*$.*
- *Let $H_h \subset H$ be a finite dimensional subspace. Let $K_h \subset H_h$ be convex and $K \subset H$ be be convex and closed.*
- *Let u_h be the minimiser of $E(v)$ over K_h and let u be its minimiser over K.*

If W is a Hilbert space so that $H \hookrightarrow W = W^* \hookrightarrow H^*$ then

$$\|u - u_h\|_H^2 \leq \frac{C^2}{\alpha^2}\|u - v\|_H^2 + \frac{2}{\alpha}\|f - Au\|_W \left(\|u - v_h\|_{W^*} + \|u_h - v\|_{W^*}\right)$$

for every $v \in K$ and $v_h \in K_h$ when $Au - f \in W$.

Remarks

Where

$$(Au, v) := a(u, v) \quad \forall v \in W$$

An example of such Hilbert spaces, H and W would be

$$H_0^1(\Omega) \subset L^2(\Omega).$$

- If $K = H$ then $f - Au = 0$ because we can find a weak solution and this becomes Cea's Lemma.
- If $K_h \subset K$ then the final term vanishes.

Proof 1/2

By the Variational Inequality,

$$\begin{aligned} a(u, u - v) &\leq \langle f, u - v \rangle \\ a(u_h, u_h - v_h) &\leq \langle f, u_h - v_h \rangle \end{aligned}$$

for all $v \in K, v_h \in K_h$. Summing and subtracting $2a(u, u_h)$ gives

$$a(u, u-v)+a(u_h, u_h-v_h)-2a(u, u_h) \leq \langle f, u-v \rangle+\langle f, u_h-v_h \rangle-2a(u, u_h).$$

Which can be rewritten as

$$\begin{aligned} &a(u - u_h, u - u_h) \\ &\qquad \leq \langle f, u - v_h \rangle + \langle f, u_h - v \rangle - a(u, u_h - v) - a(u, u - v_h) \\ &\qquad\qquad + a(u - u_h, u - v_h). \end{aligned}$$

Proof 2/2

This is equal to

$$
\begin{aligned}
& \langle f - Au, u - v_h \rangle + \langle f - Au, u_h - v \rangle + a(u - u_h, v - v_h) \\
\leq \; & \|f - Au\| \left(\|u - v_h\| + \|u_h - v\|\right) + C\|u - u_h\|\|u - v_h\| \\
\leq \; & \|f - Au\| \left(\|u - v_h\| + \|u_h - v\|\right) \\
& \quad + \frac{1}{2}\left(\sqrt{\alpha}\|u - u_h\|\right)^2 + \frac{1}{2}\left(\frac{C}{\sqrt{\alpha}}\|u - v_h\|\right)^2
\end{aligned}
$$

and bounded below by

$$
\alpha\|u - u_h\|^2 .
$$

□

References

- Heiko's Interface and Contact Problems Lecture Notes — http://www.macs.hw.ac.uk/ hg94/wien16/
- Duvaut & Lions. Inequalities in Mechanics and Physics
- Ciarlet. The Finite Element Method For Elliptic Problems