Tim Espin, Numerical Analysis Talk: Approximation by Polynomials

Introduction

- One recurring character in this course has been Céa's lemma:

(0.1) $\|u - u_h\|_{H^1} \leq C \|u - v_h\|_{H^1} \quad \forall v_h \in H_h$, where

u is the actual solution, u_h is the FEM approximation, and H_h is our approximation space.

(Proved by Céa in 1964 as part of his PhD thesis.)

- We have spent a lot of time on the ~~right~~ left-hand-side of (0.1) in this course. In this talk I would like to focus on the ~~LHS~~ RHS: how well can we approximate u in H_h.

In particular, can we achieve ~~LHS~~ HS $\to 0$ as $h \to 0$ in some sense? We will work mainly in 2D.

Contents

→ Short exposition of polynomial approximation methods, e.g. Lagrange interpolation.

→ THE MAIN RESULT & ~~show~~ a corollary.

→ Rough Proof of the MAIN RESULT

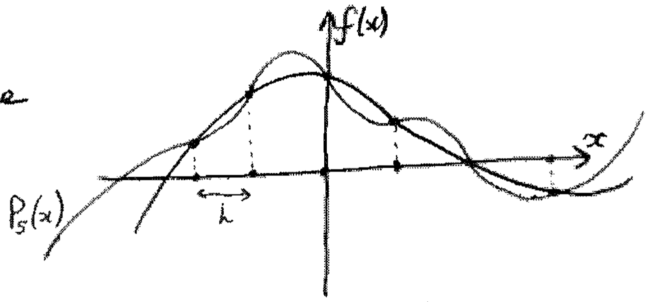
1. Lemma
2. Regular meshes
3. Shape regular meshes.

Polynomial Interpolation - Methods

- Well known method: Lagrange interpolation.

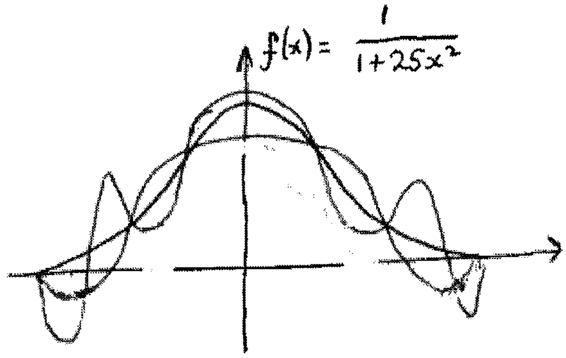
  In 1D you get an unique k^{th}-degree polynomial by specifying $k+1$ points (k roots plus one "amplitude").

  There is a formula, and the method generalises to higher dimensions.



Sketch of 5^{th} degree intepolating Polynomial

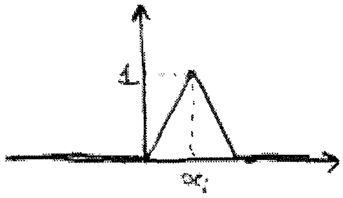
- BUT the formula is cumbersome, and you get problems such as RUNGE'S PHENOMENON (similar to Gibbs Phenomenon for Fourier Series)



- So instead, make use of splines (piecewise polynomials).

  Sacrifice "classical" differentiability in order to avoid Runge's phenomenon, and make implimentation on the computer easier.

- Already seen the 1D basis splines.



THE MAIN RESULT

(Based on Braess, Chapter I·6)

• <u>Theorem 1:</u> Let $t \geq 2$, and suppose $\mathcal{T}_h$ is a shape regular triangulation of Ω. Then there exists a constant $c = c(\Omega, \kappa, t)$ such that

$$(*) \quad \| u - I_h u \|_{m,h} \leq c h^{t-m} \| u \|_{\dot{H}^t(\Omega)}$$

for all $m = 0, \dots, t$, and $u \in H^t(\Omega)$.

Here $I_h u$ is the piecewise polynomial interpolant of u of degree $t-1$.

[• <u>Shape Regularity:</u> A triangulation $\mathcal{T}_h = \{ T_i \}$ is shape regular if $\exists\ \kappa > 0$ such that every $T_i \in \mathcal{T}_h$ contains a circle of radius

$$\rho_{T_i} \geq \frac{\text{"half diameter of } T_i\text{"}}{\kappa} \quad \sim h$$

$\Rightarrow$ The T_i cannot be too thin.]

[• <u>$\dot{H}^t(\Omega)$</u> is the homogeneous Sobolev space of order t.

The (semi) norm is $\| u \|_{\dot{H}^t(\Omega)} = \left(\sum_{|\alpha| = t} \| \partial^\alpha u \|_{L^2}^2 \right)^{1/2}$

• We are also using the mesh-dependen[t]
Which is related to the Sobolev norm:

$$\| u - I_h u \|_{m,h} := \Big(\sum_{T_i \in \tau_h} \| u - I_h u \|_{\ldots}$$

This is because we must be careful
$I_h u$ is in a particular Sobolev s[pace]
triangles T_i, $I_h u$ is C^∞, but
might only be H^m for small eno[ugh]

→ Allows us to get slightly more
In the $t = 2$ case, linear i[...]
only in H^1, but ⊛ holds

[• **Proposition:** If $k \geq 1$, then a piecewise
$v : \bar{\Omega} \to \mathbb{R}$ is $H^k(\Omega)$ iff

- Why is the MAIN RESULT (Theorem 1) important / useful?

- "<u>Corollary 1:</u>" Suppose Ω has sufficiently nice geometry (e.g. convex, polygonal,..). By elliptic regularity, we have

$$\|u\|_{H^2} \lesssim \|f\|_{L^2}, \quad \text{(provided the RHS is finite)}.$$

By Theorem 1, there exists $v_h \in H_h$ (namely $I_h u$), such that

$$\|u - v_h\|_{H^1} \leq ch\|u\|_{\dot{H}^2(\Omega)} \leq ch\|u\|_{H^2(\Omega)} \lesssim Ch\|$$

Combining this with Céa's Lemma gives

$$\|u - u_h\|_{H^1} \leq C \inf_{H_h} \|u - v_h\|_{H^1} \leq Ch\|f\|_{L^2} \longrightarrow 0$$

as $h \to 0$.

Proof of MAIN RESULT

First, we need a lemma.

① Lemma 1: (Special case of the so-called Bramble-Hilbert lemma, '70.)

Let Ω be a domain with Lipschitz boundary. Let $t \geq 2$ (think of this as the regularity of u) and suppose $z_1, \ldots, z_s$ are $s = \frac{t(t+1)}{2}$ (pre-)prescribed points in $\bar{\Omega}$ such that $I_t : H^t \to \mathcal{P}_{t-1}$ is a well-defined interpolation operator. Then $\exists$ a constant $c = c(\Omega, z_1, \ldots, z_s)$ such that

$$\| u - Iu \|_t \leq c \| u \|_{\dot{H}^t(\Omega)} , \quad \forall u \in H^t(\Omega).$$

Sketch Proof: A trick! Define a new norm on $H^t(\Omega)$ by

$$|||v||| := \|v\|_{\dot{H}^t} + \sum_{i=1}^{s} |v(z_i)| .$$

It turns out that this is equivalent to $\|\cdot\|_t$ (see Braess)

It now follows that

$$\|u - Iu\|_t \leq c|||u - Iu|||$$

$$= c\left[\|u - Iu\|_{\dot{H}^t} + \overbrace{\sum_{i=1}^{s} |(u - Iu)(z_i)|}^{=0} \right]$$

$$= c\|u - \underbrace{Iu}_{Iu \text{ is of degree } t-1.}\|_{\dot{H}^t} = c\|u\|_{\dot{H}^t} . \qquad \square$$

② Now suppose our mesh is regular.

$\Rightarrow$ Each triangle T_h is a scaled version of a fixed "reference triangle" T_1 :

$$T_h = hT_1 = \{ x = hy \mid y \in T_1 \}, \quad h \leq 1.$$

For $u \in H^t(T_h)$, define $v \in H^t(T_1)$ by $v(y) = u(hy)$.

Strategy: work on T_1 and take advantage of scaling properties of Sobolev norms when translating back to T_h

- For multiindices α with $|\alpha| \leq t$, $\partial^\alpha v = h^{|\alpha|} \partial^\alpha u$.

$$\Rightarrow \|v\|^2_{\dot{H}^\ell(T_1)} = \sum_{|\alpha|=\ell} \int_{T_1} (\partial^\alpha v(y))^2 \, dy = \text{change vars}$$

$$= \sum_{|\alpha|=\ell} \int_{T_h} h^{2\ell} (\partial^\alpha u(x))^2 \frac{dx}{h^2}$$

$$= h^{2\ell-2} \|u\|^2_{\dot{H}^\ell(T_h)}.$$

- Now, for $m \leq t$,

$$\|u\|^2_{H^m(T_h)} = \sum_{\ell \leq m} \|u\|^2_{\dot{H}^\ell(T_h)} =$$

$$= \sum_{\ell \leq m} \underbrace{h^{2-2\ell}}_{\leq h^{2-2m}, \text{ as } h \leq 1} \|v\|^2_{\dot{H}^\ell(T_1)} \quad \text{Pull out factor of } h.$$

$$\leq h^{2-2m} \|v\|^2_{H^m(T_1)}.$$

- Since we are working on an individual triangle, and $u - Iu \in H^t(T_h)$, we can substitute:

$$\begin{aligned}
\| u - Iu \|_{H^m(T_h)} &\leq h^{1-m} \| v - Iv \|_{H^m(T_1)} \quad \Big\} \ \| \cdot \|_{H^m} \leq \| \cdot \|_{H^t} \\
&\lesssim h^{1-m} \| v - Iv \|_{H^t(T_1)} \quad \Big\} \text{ lemma} \\
&\leq C h^{1-m} \| v \|_{\dot{H}^t(T_1)} \quad \Big\} \text{ scaling} \\
&= C h^{1-m} h^{t-1} \| u \|_{\dot{H}^t(T_h)} \\
&= C h^{t-m} \| u \|_{\dot{H}^t(T_h)} .
\end{aligned}$$

- We now get the MAIN RESULT by squaring the above, summing over triangles in T_h, and then square-rooting again.

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③ Now, what if our mesh is only shape regular ??
→ Want to do something similar to ②, but we need an improved transformation between triangles....

⤳ "Affine equivalence" does the trick.

(Note: polynomials are invariant under affine transformati

- Consider again a reference triangle T_1, whose precise geometry we will fix later. There exists an invertible matrix B and a "shift" x_0 such that

$$T_h = \{ x_0 + B T_1 \} \quad \text{for each } T_h .$$

• <u>Scaling law:</u> If $u \in H^m(T_h)$, then $v(y) = u(x_0 + By) \in H^m(T_1)$ and $\exists$ a constant $c = c(T_1, m)$ such that

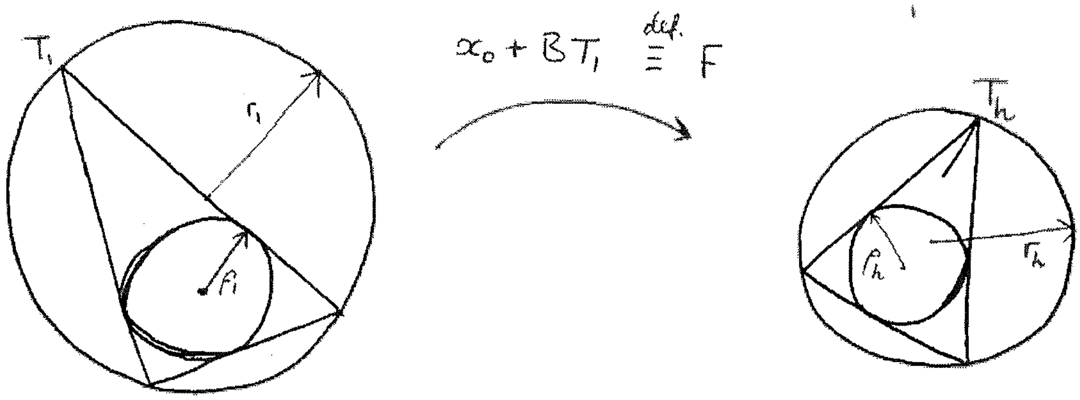
$$\|v\|_{\dot{H}^m(T_1)} \leq c\,\|B\|^m\,|\det B|^{-1/2}\,\|u\|_{\dot{H}^m(T_h)},$$

and conversely, $$\|u\|_{\dot{H}^m(T_h)} \leq c\,\|B^{-1}\|^m\,|\det B|^{1/2}\,\|v\|_{\dot{H}^m(T_1)}.$$

• So

$$
\begin{aligned}
\|u - Iu\|_{\dot{H}^\ell(T_h)} &\overset{\text{change var}}{\leq} c\,\|B^{-1}\|^\ell\,|\det B|^{1/2}\,\|v - Iv\|_{\dot{H}^\ell(T_1)} \quad \text{lemma}\\
&\leq C\,\|B^{-1}\|^\ell\,|\det B|^{1/2}\,\|v\|_{\dot{H}^t(T_1)} \quad \text{change var}\\
&\leq C\,\|B^{-1}\|^\ell\,|\cancel{\det B}|^{1/2}\,\|B\|^t\,|\cancel{\det B}|^{-1/2}\,\|u\|_{\dot{H}^t(T_h)}\\
\times \frac{\|B\|^m}{\|B\|^m} \quad &= C\left(\|B\|\cdot\|B^{-1}\|\right)^m\,\|B\|^{t-m}\,\|u\|_{\dot{H}^t(T_h)}.
\end{aligned}
$$

• Now want to use the shape regularity to estimate the factors $\|B\|$ and $\|B\|\cdot\|B^{-1}\|$. $\rightsquigarrow$ Geometry with B!



r_1 = radius of smallest circle containing T_1,

ρ_1 = radius of largest circle inside T_1.

r_h = radius of smallest circle containing T_h,

ρ_h = radius of largest circle inside T_h.

- Now, given $x \in \mathbb{R}^2$ with $\|x\| \leq 2\rho_1$, we find two points $y_1, z_1 \in T_1$ with $y_1 - z_1 = x$.

$\Rightarrow \quad \|B(y_1 - z_1)\| \leq 2r_h$, since $F(y_1), F(z_1) \in T_h$.

$\Rightarrow \quad \|B(x)\| \leq 2r_h \quad \Rightarrow \quad \underline{\|B\| \leq \frac{r_h}{\rho_1}}$.

Similarly, performing the opposite argument, $\underline{\|B^{-1}\| \leq \frac{r_1}{\rho_h}}$.

$$\Rightarrow \boxed{\|B\| \cdot \|B^{-1}\| \leq \frac{r_1 r_h}{\rho_1 \rho_h}}.$$

- We now "fix" T_1 to be a triangle with

$$r_1 = \frac{1}{\sqrt{2}} \quad \text{and} \quad \rho_1 = \frac{1}{2+\sqrt{2}} \geq \frac{2}{7}.$$

- Shape regularity $\Rightarrow \frac{r_h}{\rho_h} \leq \kappa$

$$\Rightarrow \boxed{\|B\| \cdot \|B^{-1}\| \leq (1+\sqrt{2})\kappa}, \quad \text{and} \quad \boxed{\|B\| \leq \frac{7}{2}h \leq 4h}$$

- Putting this all back together gives ....

$$\|u - Iu\|_{\dot{H}^\ell(T_h)} \leq C(\Omega, \kappa)\, h^{t-\ell} \|u\|_{\dot{H}^t(T_h)}.$$

Squaring this, summing over ℓ from 0 to m, and summing over triangles gives exactly the result. $\square$