37 a) „$\Rightarrow$" Sei $\omega = df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \Rightarrow \omega_i = \frac{\partial f}{\partial x_i} \overset{f\in C^2}{\Rightarrow} \partial_{x_j}\omega_i = \frac{\partial^2 f}{\partial x_j \partial x_i} \underset{\text{(Satz von Schwarz)}}{=} \frac{\partial^2 f}{\partial x_i \partial x_j} = \partial_{x_i}\omega_j$

„$\Leftarrow$" Sei $f(x) = \sum_{i=1}^n \int_0^1 dt\, (x_i - x_{0i})\, \omega_i(x_0 + t(x-x_0))$

$$\Rightarrow df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \text{ mit}$$

$$\frac{\partial f}{\partial x_j} = \sum_{i=1}^n \partial_{x_j} \int_0^1 dt\, \underbrace{(x_i - x_{0i})\, \omega_i(x_0+t(x-x_0))}_{=:f(x,t)}$$

- $\forall x$: $f(x,\cdot)$ integrierbar auf $[0,1]$, da stetig ($\Rightarrow$ messbar) und beschränkt
- $x \mapsto \partial_{x_j} f(x,t)$ stetig in x
- $|\partial_{x_j} f| \leq \max|\partial_{x_j} f| \in L^1([0,1])$

$\Rightarrow$ (Satz 2.23) $\partial_{x_j}\int = \int \partial_{x_j}$

$$= \sum_{i=1}^n \int_0^1 dt\, \partial_{x_j}\{(x_i - x_{0i})\, \omega_i(x_0+t(x-x_0))\}$$

$$\underset{\text{Kettenregel}}{=} \int_0^1 dt\, \omega_j(x_0+t(x-x_0)) + \sum_{i=1}^n \int_0^1 (x_i - x_{0i}) \overbrace{\frac{\partial \omega_i}{\partial x_j}}^{\overset{\text{Voraussetzung}}{=} \frac{\partial \omega_j}{\partial x_i}}\Bigg|_{x_0+t(x-x_0)} \cdot \underbrace{\partial_{x_j}(x_{0j} + t(x_j - x_{0j}))}_{=t}$$

$$= \int_0^1 dt\, \omega_j(x_0+t(x-x_0)) + \int_0^1 dt\, t \sum_{i=1}^n (x_i - x_{0i}) \frac{\partial \omega_j}{\partial x_i}\Bigg|_{x_0+t(x-x_0)}$$

$$= \int_0^1 dt\, \frac{d}{dt}\left(t\, \omega_j(x_0+t(x-x_0))\right)$$

$$= \omega_j(x)$$

b) $F = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} \overset{!}{=} \nabla V = \begin{pmatrix} \partial_{x_1} V \\ \vdots \\ \partial_{x_n} V \end{pmatrix} \Leftrightarrow \omega = \sum_{i=1}^n F_i\, dx_i = \sum_{i=1}^n \partial_{x_i} V\, dx_i = dV$

c) $df = \sum_{i=1}^n \partial_{x_i} f\, dx_i$, Ist $f(x) = g(r) \Rightarrow \partial_{x_i} f = (\partial_r g)\, \underbrace{\partial_{x_i} r}_{\frac{x_i}{r}} = \frac{x_i}{r}\partial_r g$

$\omega_i \overset{!}{=} \partial_{x_i} f \Rightarrow x_i \phi = \frac{x_i}{r} g' \Rightarrow g' = r\phi \Rightarrow g(r) = \int_{r_0}^r ds\, s\,\phi(s)$

$f(x) = g(\|x\|)$

$$\partial_{x_i} f \underset{\text{Kettenregel}}{=} \partial_s g(s)\big|_{s=\|x\|} \underbrace{\frac{\partial \|x\|}{\partial x_i}}_{= \partial_{x_i}\sqrt{x_1^2+\dots+x_n^2} = \frac{x_i}{\sqrt{x_1^2+\dots+x_n^2}} = \frac{x_i}{\|x\|}} = \|x\|\,\phi(\|x\|)\frac{x_i}{\|x\|} = x_i \phi(\|x\|) = \omega_i$$

$$\partial_s g(s)\big|_{s=\|x\|} = \partial_s \int_{r_0}^s dr\, r\phi(r)\Big|_{s=\|x\|} = s\,\phi(s)\big|_{s=\|x\|} = \|x\|\,\phi(\|x\|)$$

$\Rightarrow df = \omega.$

$\boxed{38}$ a) Übung: $\quad \mathrm{vol}\, G = \int_{\partial G} p\, dV = -\int_{\partial G} V\, dp$

$$\Rightarrow \quad W = \int_{\partial G} \frac{c_V}{Nk_B} V\, dp + \frac{c_p}{Nk_B} p\, dV = \left(-\frac{c_V}{Nk_B} + \frac{c_p}{Nk_B} \right) \mathrm{vol}\, G$$

Ist $\omega = \frac{c_V}{Nk_B} V\, dp + \frac{c_p}{Nk_B} p\, dV$ exakt $\overset{\omega = d\alpha}{\Rightarrow} \int_\gamma \omega = \alpha(\text{Endpunkt}) - \alpha(\text{Anfangspkt})$

$\Rightarrow \int_{\partial G} \omega = 0$, da Endpkt = Anfangspkt.

Aber für $c_V < c_p$ ist $W \neq 0$! $\Rightarrow$ ω nicht exakt.

$$d(g\omega) = \frac{1}{Nk_B} d\left(g\, c_V V\, dp + g\, c_p p\, dV \right)$$

(Definition) $$= \frac{1}{Nk_B} \left(\partial_p (g c_V V)\, dp + \partial_V (g c_V V)\, dV \right) \wedge dp + \frac{1}{Nk_B} \left(\partial_p (g c_p p)\, dp + \partial_V (g c_p p)\, dV \right) \wedge dV$$

(Ausmultiplizieren) $$= \frac{1}{Nk_B} \left(\partial_p (g c_V V) \underbrace{\cancel{dp \wedge dp}}_{=0} + \partial_V (g c_V V)\, dV \wedge dp \right) + \frac{1}{Nk_B} \left(\partial_p (g c_p p) \underbrace{dp \wedge dV}_{= -dV \wedge dp} + \partial_V (g c_p p)\, \underbrace{\cancel{dV \wedge dV}}_{=0} \right)$$

$$= \frac{1}{Nk_B} \left(\partial_V (g c_V V) - \partial_p (g c_p p) \right) dV \wedge dp \overset{!}{=} 0$$

$$\Rightarrow \quad \partial_V (g c_V V) = \partial_p (g c_p p)$$

$$c_V \left(V(\partial_V g) + g \right) = c_p \left(p (\partial_p g) + g \right)$$

$g = g(T) = g\left(\frac{pV}{Nk_B} \right) \Rightarrow \partial_V g(T) = (\partial_T g(T))\, \partial_V T = (\partial_T g(T)) \frac{p}{Nk_B}$, $\quad T = \frac{pV}{Nk_B}$

$$\partial_p g(T) = (\partial_T g(T))\, \partial_p T = (\partial_T g(T)) \frac{V}{Nk_B}$$

$$\Rightarrow c_V \left(\frac{Vp}{Nk_B} \partial_T g + g \right) = c_p \left(\frac{Vp}{Nk_B} \partial_T g + g \right)$$

$$\Leftrightarrow (c_V - c_p)\underbrace{\left(\frac{Vp}{Nk_B}\right)}_{T} \partial_T g + (c_V - c_p)\, g = 0$$

$$\Leftrightarrow \quad g = -\partial_T g \quad \Longrightarrow \quad g(T) = \frac{\text{Konstante}}{T} = Nk_B \frac{\text{Konstante}}{pV}$$

$$g\omega = \text{Konstante}\cdot\left(\frac{c_V}{p}\,dp + \frac{c_p}{V}\,dV\right)$$

physikalisch sinnvoll

Ist exakt! 1. Begründung: $\underbrace{\partial_p (g\omega)_V}_{= \partial_p \frac{c_p}{V} = 0} = \underbrace{\partial_V (g\omega)_p}_{= \partial_V \frac{c_V}{p} = 0} \overset{37a)}{\Longrightarrow} g\omega$ exakt, mit $S = \{(p,V), p>0, V>0\}$

2. Begründung: $g\omega = \text{Konstante}\cdot d\left(\ln\left(p^{c_V} V^{c_p}\right)\right)$

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b)

$$\underbrace{\nabla\times}_{(=\text{rot})}\left(a\left(\sqrt{x^2+y^2}\right)(-y,x,0)\right) = \begin{pmatrix}\partial_y(0) - \partial_z(a(\dots)x)\\ \partial_z(-a(\dots)y) - \partial_x(0)\\ \partial_x(a(\dots)x) - \partial_y(-a(\dots)y)\end{pmatrix} = \begin{pmatrix}0\\0\\ *\end{pmatrix}$$

$$* = x a'\left(\sqrt{x^2+y^2}\right)\underbrace{\partial_x\sqrt{x^2+y^2}}_{=\frac{x}{\sqrt{x^2+y^2}}} + a\left(\sqrt{x^2+y^2}\right) + y a'\left(\sqrt{x^2+y^2}\right)\underbrace{\partial_y\sqrt{x^2+y^2}}_{=\frac{y}{\sqrt{x^2+y^2}}} + a\left(\sqrt{x^2+y^2}\right)$$

$$= \frac{x^2+y^2}{\sqrt{x^2+y^2}}\, a'\left(\sqrt{x^2+y^2}\right) + 2a\left(\sqrt{x^2+y^2}\right)$$

$$= \sqrt{x^2+y^2}\, a'\left(\sqrt{x^2+y^2}\right) + 2a\left(\sqrt{x^2+y^2}\right), \quad \xi = \sqrt{}$$

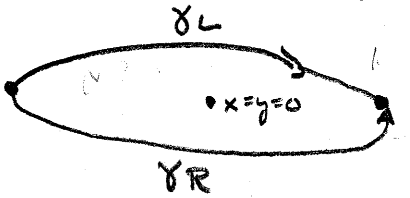
$$\Longrightarrow \text{rot } A = 0 \Leftrightarrow \quad \xi a'(\xi) + 2a(\xi) = 0$$

$$\partial_\xi \ln(a) = \frac{a'}{a} = -\frac{2}{\xi} = \partial_\xi(-2\ln(\xi)) = \partial_\xi\left(\ln\left(\xi^{-2}\right)\right)$$

$$\Longrightarrow \ln(a) = \ln\left(\xi^{-2}\right) + C$$

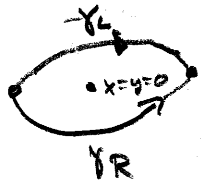
$$\Longrightarrow a = \frac{\tilde{C}}{\xi^2} = \frac{\tilde{C}}{x^2+y^2}, \quad a(1) = \phi \Rightarrow \tilde{C} = \phi$$

$$\Longrightarrow A = \frac{\phi}{x^2+y^2}\begin{pmatrix}-y\\x\\0\end{pmatrix} \quad \Longrightarrow \quad \omega = -\frac{y\phi}{x^2+y^2}\,dx + \frac{x\phi}{x^2+y^2}\,dy$$



$$|\psi_R + \psi_L|^2 = \left|\psi_R^0 e^{i\int_{\gamma_R}\omega} + \psi_L^0 e^{i\int_{\gamma_L}\omega}\right|^2 = \underbrace{\left|e^{i\int_{\gamma_L}\omega}\right|^2}_{=1} \left|\psi_R^0 e^{i\int_{\gamma_R}\omega - i\int_{\gamma_L}\omega} + \psi_L^0\right|^2$$

$$\Rightarrow |\psi_R+\psi_L|^2 = |\psi_R^0 \, e^{i\int_\gamma \omega} + \psi_L^0|$$



$\gamma = \gamma_R - \gamma_L$

Da $\operatorname{rot} A = 0 \overset{\text{Stokes}}{\Longrightarrow} \int_\gamma \omega = \int_{S^1_R} \omega$

gegen die Uhr

$S^1 = \{\gamma(t) : t \in (0, 2\pi)\} \; : \; \gamma(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$

$$\Rightarrow \int_\gamma \omega = \int_0^{2\pi} -\frac{\sin(t)}{\sin^2+\cos^2}\,\phi\, \underbrace{(-\sin(t))\,dt}_{dx} + \frac{\cos(t)}{\sin^2+\cos^2}\,\phi\,(\cos(t))\,dt$$

$$= \phi \int_0^{2\pi} dt = 2\pi\phi$$