Differentialformen auf $\mathbb{R}^3$

$U \subseteq \mathbb{R}^3$ offen

$$\begin{array}{lcl}
\Omega^0(U) = \{\omega\} & \longleftrightarrow & \text{Funktionen } \omega \\
d \downarrow & & \downarrow \text{grad} \\
\Omega^1(U) = \{\omega_1 dx_1 + \omega_2 dx_2 + \omega_3 dx_3\} & \longleftrightarrow & \text{Vektorfelder} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\
d \downarrow & & \downarrow \text{rot} \\
\Omega^2(U) = \left\{ \begin{array}{l} \omega_1\, dx_2 \wedge dx_3 \\ + \omega_2\, dx_3 \wedge dx_1 \\ + \omega_3\, dx_1 \wedge dx_2 \end{array} \right\} & \longleftrightarrow & \text{Vektorfelder} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\
d \downarrow & & \downarrow \text{div} \\
\Omega^3(U) = \{\omega\, dx_1 \wedge dx_2 \wedge dx_3\} & \longleftrightarrow & \text{Funktionen } \omega
\end{array}$$

$$d^2 = 0 \quad \longleftrightarrow \quad \begin{array}{l} \text{rot grad} = 0 \\ \text{div rot} = 0 \end{array}$$

Einband vom Jackson (E-Dynamik)

$\nabla \times \nabla \psi = 0 \quad \longleftrightarrow \quad d^2 \psi = 0$

$\nabla \cdot (\nabla \times \mathbf{a}) = 0 \quad \longleftrightarrow \quad d^2 a = 0$

$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a} \quad \longleftrightarrow \quad ?$

$\nabla \cdot (\psi \mathbf{a}) = \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a} \quad \longleftrightarrow \quad d(\psi \wedge a) = a \wedge d\psi + \psi \wedge da$

$\nabla \times (\psi \mathbf{a}) = \nabla \psi \times \mathbf{a} + \psi \nabla \times \mathbf{a} \quad \longleftrightarrow \quad d(\psi \wedge a) = d\psi \wedge a + \psi \wedge da$

$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) \quad \longleftrightarrow \quad ?$

$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \quad \longleftrightarrow \quad d(a \wedge b) = b \wedge da - a \wedge db$

$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} \quad \longleftrightarrow \quad ?$

$\bigcirc \in \Omega^0, \quad \bigcirc \in \Omega^1, \quad \bigcirc \in \Omega^2$

Kovariante E-Dynamik

Hodge $*: \Omega^2(\mathbb{R}^4) \to \Omega^2(\mathbb{R}^4)$

- $\forall f_i \in \Omega^0: * (f_1 \omega_1 + f_2 \omega_2) = f_1 * \omega_1 + f_2 * \omega_2$
- $** \omega = -\omega$
- $*(dt \wedge dx_1) = dx_2 \wedge dx_3$
  $*(dt \wedge dx_2) = dx_3 \wedge dx_1$
  $*(dt \wedge dx_3) = dx_1 \wedge dx_2$

Feldstärketensor + Stromdichte

$$F = -E_1\, dt \wedge dx_1 - E_2\, dt \wedge dx_2 - E_3\, dt \wedge dx_3$$
$$+ B_1\, dx_2 \wedge dx_3 + B_2\, dx_3 \wedge dx_1 + B_3\, dx_1 \wedge dx_2$$
$$\longleftrightarrow \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

$$J = \rho\, dx_1 \wedge dx_2 \wedge dx_3$$
$$- (j_1\, dx_2 \wedge dx_3 + j_2\, dx_3 \wedge dx_1 + j_3\, dx_1 \wedge dx_2) \wedge dt$$

<u>Maxwell:</u>	$\nabla\times E+\partial_t B=0$ $\nabla\cdot B=0$	$\longleftrightarrow$	$\partial^\alpha F^{\beta\gamma}+\partial^\beta F^{\gamma\alpha}+\partial^\gamma F^{\alpha\beta}=0$	$\longleftrightarrow$	$dF=0$
	$\nabla\cdot E=\rho$ $\nabla\times B-\partial_t E=J$	$\longleftrightarrow$	$\partial_\alpha F^{\alpha\beta}=J^\beta$	$\longleftrightarrow$	$d{*}F=J$
<u>Poincaré:</u>	$E=\nabla\phi$ $B=\nabla\times A$	$\longleftrightarrow$	$F^{\alpha\beta}=\partial^\alpha A^\beta-\partial^\beta A^\alpha$	$\longleftrightarrow$	$dF=0\Rightarrow$ $F=dA$
<u>Konti:</u>	$\partial_t\rho+\nabla\cdot J=0$	$\longleftrightarrow$	$\partial_\alpha J^\alpha=0$	$\longleftrightarrow$	$dJ=d^2{*}F=0$
<u>Lorentz:</u>	$\partial_t\phi+\nabla\cdot A=0$	$\longleftrightarrow$	$\partial_\alpha A^\alpha=0$	$\longleftrightarrow$	$d{*}A=0$
<u>Wellen:</u>	$(\partial_t^2-\Delta)\begin{pmatrix}\phi\\A\end{pmatrix}=\begin{pmatrix}\rho\\J\end{pmatrix}$	$\longleftrightarrow$	$\partial_\alpha\partial^\alpha A^\beta=J^\beta$	$\longleftrightarrow$	$d{*}dA=d{*}F=J$
<u>Invarianten:</u>	$E\cdot B$ E^2-B^2	$\longleftrightarrow$	$F^{\alpha\beta}F^{\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta}$ $F_{\alpha\beta}F^{\alpha\beta}$	$\longleftrightarrow$	$F\wedge F$ $F\wedge{*}F$
<u>Lagrange-dichte:</u>	E^2-B^2	$\longleftrightarrow$	$F_{\alpha\beta}F^{\alpha\beta}$	$\longleftrightarrow$	$F\wedge{*}F$