Pseudodifferential equations in polyhedral domains II: hp approximation and graded meshes

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(joint work with H. Gimperlein, E. P. Stephan)

Starting from the regularity results in the talk by H. Gimperlein, this talk discusses the numerical approximation of boundary problems involving nonlocal operators, such as the integral fractional Laplacian. The article [4] surveys recent progress in their numerical analysis.

For elliptic differential boundary value problems in polyhedral domains, explicit singular expansions of their solutions give rise to h and hp discretizations with optimal convergence rates for finite [2] and boundary element methods [11]. In recent work [9, 10], corresponding results have been obtained for the fractional Laplacian and, in [7], for a class of nonlocal transmission problems.

We consider the numerical approximation of the integral fractional Laplacian, defined for a Schwartz function u on $\mathbb{R}^n$ by

$$(-\Delta)^s u(x) = c_{n,s}\, P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy, \qquad (s \in (0,1)).$$

Here $P.V.$ denotes the Cauchy principal value and $c_{n,s} = \frac{2^{2s} s \Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n}{2}}}$. The operator $(-\Delta)^s : \widetilde{H}^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$ is a pseudodifferential operator of order $2s$.

The corresponding fractional Dirichlet problem in a domain $\Omega \subset \mathbb{R}^n$ is given by

$$\begin{aligned} (-\Delta)^s u &= f \text{ in } \Omega, \\ u &= 0 \text{ in } \Omega^C = \mathbb{R}^n \setminus \overline{\Omega}\ , \end{aligned} \tag{1}$$

and its weak formulation is given by: Find $u \in \widetilde{H}^s(\Omega)$ such that for all $v \in \widetilde{H}^s(\Omega)$

$$\frac{c_{n,s}}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dx\, dy = \int_{\mathbb{R}^n} f(x) v(x)\, dx\ . \tag{2}$$

We discuss the Galerkin discretization of (2) by piecewise polynomials $V_h^p \subset \widetilde{H}^s(\Omega)$ of degree p: Find $u_h \in V_h^p$ such that for all $v \in V_h^p$

$$\frac{c_{n,s}}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u_h(x)-u_h(y))(v_h(x)-v_h(y))}{|x-y|^{n+2s}} dx\, dy = \int_{\mathbb{R}^n} f(x) v_h(x)\, dx\ . \tag{3}$$

For $f \in H^{-s}(\Omega)$ there exists a unique solution $u \in \widetilde{H}^s(\Omega)$ to (2) and a unique solution $u_h \in V_h^p$ to (3).

For a polygon $\Omega \subset \mathbb{R}^2$ and sufficiently smooth right hand side f, we obtain a precise decomposition of the solution u near the boundary into edge and corner singular functions and a smooth remainder:

Theorem 1. *The solution of* (2) *has the form:*

$$u = u_{\text{reg}} + \sum_{e\in E} u^e + \sum_{v\in V} u^v + \sum_{v\in V}\sum_{e\in E(v)} u^{ev}. \tag{4}$$

Here $u^e(x)$ admits an asymptotic expansion near the edge e with leading part $u^e(x) \sim \mathrm{dist}(x,e)^s$, $u^v(x)$ admits an asymptotic expansion near the vertex v with leading part $u^v(x) \sim \mathrm{dist}(x,v)^\lambda$, and u^{ev} admits a related asymptotic expansion for the edge-vertex singularity. The remainder term u_{reg} is of higher regularity.

The decomposition follows from an extension of the boundary problem to $\mathbb{R}^3_+$, as popularized by Caffarelli and Silvestre [5]. The resulting local, degenerate mixed boundary value problem in the upper half space $\mathbb{R}^3_+$ allows to study the solution u using classical techniques for mixed boundary problems [6].

We present a careful study of the corner exponent λ generalizing classical results for screen problems for the ordinary Laplacian $-\Delta$ which correspond to $s = \frac{1}{2}$, e.g. in [12]. λ relates to the lowest eigenvalue for a perturbed Laplace-Beltrami operator on S^2_+ with mixed Dirichlet and Neumann boundary conditions. We discuss finite element approximations of λ, as well as highly-accurate semi–analytical approximations, as well as qualitative analytical bounds. In particular, we prove that $\max\{0, s-\frac{1}{2}\} < \lambda < 2s$. The upper and lower bounds are sharp: $\lambda \to \max\{0, s-\frac{1}{2}\}$ as the opening angle at the vertex tends to 2π, while $\lambda \to 2s$ as the opening angle tends to 0.

Theorem 1 allows to derive quasi-optimal convergence rates for Galerkin approximations of the fractional Laplacian, for the h-version on β-graded meshes and hp-versions. In 2d it turns out that the approximation error is dominated by the edge, not corner singularities, a consequence of the above-mentioned bound $\lambda > \max\{0, s - \frac{1}{2}\}$.

Our main approximation result for graded meshes generalises work for convex polyhedral domains by Acosta and Borthagaray [1]:

Theorem 2 ([9])**.** *Let $u \in \widetilde{H}^s(\Omega)$ be a solution to* (1) *and $\Pi_h u$ be the best approximation by p.w. linear functions on a β-graded mesh in the $\widetilde{H}^s(\Omega)$ norm.*
Then for $\beta > 2(2-s)$

$$\|u - \Pi_h u\|_{\widetilde{H}^s(\Omega)} \leq C_{\beta,\varepsilon} h^{\min\{\beta/2, 2-s\}-\varepsilon}.$$

For the hp-method our main theorem generalises results by Bespalov and Heuer for boundary integral equations [3]:

Theorem 3 ([10])**.** *Let $u \in \widetilde{H}^s(\Omega)$ be a solution to* (1) *and $\Pi_{h,p}u$ be the best approximation by piecewise polynomial functions on a quasi-uniform mesh $\mathcal{T}_h$. Then,*

$$\|u - \Pi_{h,p} u\|_{\widetilde{H}^s(\Omega)} \leq C h^{\frac{1}{2}} p^{-1} (1 + \log(ph^{-1}))^\beta,$$

where $\beta \in \mathbb{N}$ relates to the edge and corner behaviour.

The exponential convergence of the *hp*-method on geometrically graded meshes is the content of current work.

Beyond the approximation results, the talk discussed the details of the implementation of *hp*-version finite elements for the integral fractional Laplacian. It illustrated the theoretical results by numerical experiments, to be published in [9, 10].

References

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