**Pseudodifferential equations in polyhedral domains I: Regularity and singular expansions**

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(joint work with R. Mazzeo, N. Louca, E. P. Stephan, J. Stocek, C. Urzua-Torres)

Solutions to elliptic boundary value problems in polyhedral domains exhibit singularities at edges and corners. Their asymptotic behavior has been studied for several decades, with seminal contributions, e.g. by Kondratiev, Dauge or Mazya. Explicit singular expansions give rise to $h$ and $hp$ discretizations with optimal convergence rates for finite [2] and boundary element methods [12].

This talk addressed corresponding questions for boundary problems involving nonlocal operators, such as the integral fractional Laplacian. We refer to the recent survey article [3] for motivation and recent advances.

Sharp results for the regularity of the solutions to such problems in smooth domains have been obtained by Grubb [11]. We here address a recent alternative approach given in [6], which we generalize to domains with corners. We obtain detailed asymptotic expansions for the solution at edges and vertices, which generalize classical results for mixed boundary problems for the standard Laplace equation. Applications to $h$ and $hp$ discretizations with optimal convergence rates are discussed in a second talk at this conference, by J. Stocek. In convex polygons, weighted Sobolev estimates were shown in [1], resulting in optimal convergence rates on graded meshes.

To be specific, we consider the integral fractional Laplacian, defined for a Schwartz function $u$ on $\mathbb{R}^n$ by

$$(-\Delta)^s u(x) = c_{n,s}\, P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy, \qquad (s \in (0,1)).$$

Here $P.V.$ denotes the Cauchy principal value and $c_{n,s} = \frac{2^{2s} s\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n}{2}}}$. The operator $(-\Delta)^s : \widetilde{H}^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$ is a pseudodifferential operator of order $2s$. For $s \to 1^-$ one recovers the Laplacian $-\Delta$.

The corresponding fractional Dirichlet problem in a domain $\Omega \subset \mathbb{R}^n$ is given by

$$\begin{aligned} (-\Delta)^s u &= f \text{ in } \Omega, \\ u &= 0 \text{ in } \Omega^C = \mathbb{R}^n \setminus \overline{\Omega}\ . \end{aligned} \tag{1}$$

For $f \in H^{-s}(\Omega)$ there exists a unique solution $u \in \widetilde{H}^s(\Omega)$, and the purpose of this talk is to discuss higher regularity of $u$ for smoother $f$. Our first observation relates this problem to classical problems for boundary integral equations:

**Theorem 1** ([10])**.** *a) For $s = \frac{1}{2}$, Problem* (1) *is equivalent to the hypersingular integral equation $2Wu = f$ on $\Omega \times \{0\} \subset \mathbb{R}^{n+1}$.*

*b) Problem* (1) *admits an analytic extension to* $s \in \mathbb{C}$. *For* $s = -\frac{1}{2}$, *it is equivalent to the weakly singular integral equation* $2Vu = f$ *on* $\Omega \times \{0\} \subset \mathbb{R}^{n+1}$.

An explicit solution to (1) when $\Omega = \mathcal{B}_1$ is the unit ball goes back to Boggio:

$$(2)\quad u(x) = \int_{\mathcal{B}_1} G_s(x,y) f(y)\, dy\ , G_s(x,y) = k_{n,s}|x-y|^{2s-n} \int_0^{r(x,y)} \frac{t^{s-1}}{(t+1)^{n/2}}\, dt\ ,$$

where $r(x,y) = \dfrac{(1-|x|^2)_+(1-|y|^2)_+}{|x-y|^2}$ and $k_{n,s} = \dfrac{2^{1-2s}}{|\partial\mathcal{B}_1|\Gamma(s)^2}$. Similar exact solution formulas for $V$ and $W$ have been of recent interest [13, 14], with applications to preconditioning. They are recovered by Boggio's formula (2) and Theorem 1. The observation leads to a short proof for this line of results and generalizes them to arbitrary elliptic pseudodifferential boundary problems of order between $-2$ and 2, such as Problem (1).

Specifically, in [10] we consider conforming finite element discretizations of such problems by piecewise constant, resp. linear elements. Boggio's formula (2) defines an operator preconditioner $C$ for the Galerkin matrix $A$ of the boundary problem, which we show to be optimal:

**Theorem 2** ([10])**.** *Under mild assumptions on the triangulation, the condition number of* $CA$ *is independent of the mesh size.*

Boggio's formula also shows that when $\Omega = \mathcal{B}_1$ and $f$ is smooth, the solution $u$ admits an asymptotic expansion near $\partial\mathcal{B}_1$, with leading singular exponent $\nu = s$. This is a special case of the precise boundary behavior obtained in [11] near a smooth boundary $\partial\Omega$. We obtain asymptotic expansions for the solution $u$ of (1) near the boundary and corners of a polygon $\Omega$ for all $s \in (0,1)$:

**Theorem 3** ([6, 7])**.** *Let* $\Omega \subset \mathbb{R}^2$ *be a polygonal domain,* $f \in C^\infty(\overline{\Omega})$. *Then the solution* $u$ *to* (1) *admits an asymptotic expansion near the boundary and the vertices. Near a smooth boundary point the leading singular exponent is given by* $s$: $u(x) \sim \operatorname{dist}(x,\partial\Omega)^s$. *At a vertex* $V$ *the singular exponent* $\lambda$, *with* $u(x) \sim \operatorname{dist}(x,V)^\lambda$, *is the smallest eigenvalue of an elliptic differential operator of second order on* $S^2_+$.

**Remark 1.** This exponent $\lambda$ is an increasing function of $s$ and a decreasing function of the vertex angle. The upper and lower bounds $\lambda < 2s$, $\lambda > \max\{0, s - \frac{1}{2}\}$, are sharp and attained when the vertex angle tends to $2\pi$. Our study of $\lambda$ generalizes classical results for $s = \frac{1}{2}$, e.g. in [9].

The crucial idea in the proof of Theorem 3 is to reformulate (1) as a local, degenerate mixed boundary value problem in the upper half space $\mathbb{R}^3_+$ [6]. The reformulation as a local differential equation allows to study the solution $u$ using pseudodifferential techniques developed for boundary problems on singular spaces [4]. The idea to extend to $\mathbb{R}^3_+$ goes back to Caffarelli and Silvestre (2007), but its use for a precise regularity theory seems to be new.

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