

Magnitude 2019, Edinburgh 4–5. July 2019

A threshold of length in magnitude homology

Masahiko Yoshinaga (Hokkaido)

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What I want to discuss:

1. For a metric space X , there is a constant $m_x \geq 0$ s.t. magnitude homology MH^l is different in nature according to $l < m_x$ or $l \geq m_x$.
2. What is the "categorification of magnitude weighting" ??

Plan

1. Order complex of a poset.
2. Magnitude homology with fixed endpoints.
3. A threshold m_x .
4. A categorification of weighting.

1. Order complex of a poset

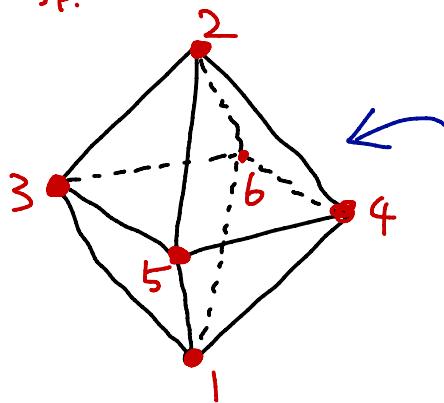
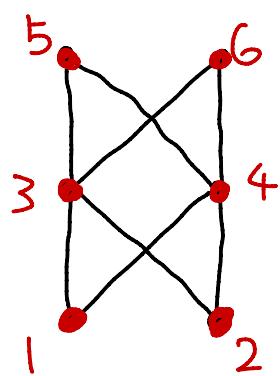
1. Order complex of a poset

P : a poset $\xrightarrow{\text{order complex}}$ $\Delta(P)$: $\xrightarrow{\sim}$ $C_*(\Delta(P))$:

(a.k.a. classifying sp.)

$\Delta(P)$: a Simplicial complex

the reduced chain complex ($C_{-1} = \mathbb{Z}$).



A triangulation of S^2 (octahedron)

0-simplices: $\{1\}, \{2\}, \dots, \{6\}$

1-simplices: $\{1 < 4\}, \{1 < 3\}, \{1 < 6\}, \{1 < 5\}, \dots$

2-simplices: $\{1 < 3 < 5\}, \{1 < 3 < 6\}, \{1 < 4 < 6\}, \dots$

2. Magnitude homology with fixed endpoints

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Let (X, d) be a metric space.

Def. $MC_n^l(X) := \bigoplus_{\alpha} \mathbb{Z} \cdot \langle \alpha \rangle,$

where $\alpha = (x_0, x_1, \dots, x_n)$ runs $x_i \in X$,
 $x_{i-1} \neq x_i$, $\sum_{i=1}^n d(x_{i-1}, x_i) = l$.

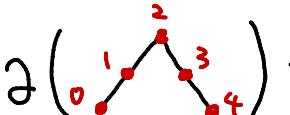
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Recall: $\alpha(0, 1, 2, 3, 4) =$



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Recall: $\alpha \left(\begin{array}{c} 2 \\ 1 \\ 0 \\ 3 \\ 4 \end{array} \right) = \left(\begin{array}{c} 2 \\ 1 \\ 0 \\ 3 \\ 4 \end{array} \right) - \left(\begin{array}{c} 2 \\ 0 \\ 3 \\ 4 \end{array} \right) + \left(\begin{array}{c} 2 \\ 1 \\ 0 \\ 3 \\ 4 \end{array} \right)$

$$- \left(\begin{array}{c} 2 \\ 1 \\ 0 \\ 4 \end{array} \right) + \left(\begin{array}{c} 2 \\ 1 \\ 0 \\ 3 \\ \vdots \end{array} \right)$$

2. Magnitude homology with fixed endpoints

Let (X, d) be a metric space.

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$$x_{i-1} \neq x_i, \quad \sum_{i=1}^n d(x_{i-1}, x_i) = l.$$

Recall: $\mathfrak{X} = (0, 1, 2, 3, 4)$

$$\begin{aligned} \mathfrak{X} &= \left(\begin{array}{c} 2 \\ | \\ 1 \\ | \\ 0 \end{array} \right) - \left(\begin{array}{c} 2 \\ | \\ 3 \\ | \\ 4 \end{array} \right) + \left(\begin{array}{c} 2 \\ | \\ 3 \\ | \\ 4 \end{array} \right) \\ &\quad - \left(\begin{array}{c} 2 \\ | \\ 4 \end{array} \right) + \left(\begin{array}{c} 2 \\ | \\ 3 \\ | \\ 4 \end{array} \right) \end{aligned}$$

Remove "Shorter than l "

Observation \mathfrak{X} and \mathfrak{X}' have same endpoints.

2. Magnitude homology with fixed endpoints

Let (X, d) be a metric space.

Def. $\text{MC}_n^l(X) := \bigoplus_{\mathfrak{C}} \mathbb{Z} \cdot \langle \mathfrak{C} \rangle,$

where $\mathfrak{C} = (x_0, x_1, \dots, x_n)$ runs $x_i \in X,$

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Remove
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Observation \mathfrak{C} and \mathfrak{C}' have same endpoints.

2. Magnitude homology with fixed endpoints

Recall: $\partial(\text{graph}) = (\text{graph}) - (\text{graph}) + (\text{graph}) - (\text{graph}) + (\text{graph})$

Remove "Shorter than l "

The diagram illustrates the boundary operator ∂ on a graph with 5 vertices labeled 0, 1, 2, 3, 4. The edges are labeled 1, 2, 3, 4. The graph consists of a central triangle with vertices 1, 2, 3 and edges 1, 2, 3, and four additional edges connecting vertex 0 to each of the other vertices (0,1), (0,2), (0,3), and (0,4). The boundary operator ∂ is defined as the sum of the graphs formed by removing edges from the original graph. Specifically, it is shown as the sum of the original graph minus the graph where edge (1,2) is removed, plus the graph where edges (1,2) and (3,4) are removed, minus the graph where edges (1,2), (3,4), and (1,3) are removed, plus the graph where edges (1,2), (3,4), and (1,3) are removed again, and finally plus the graph where edges (1,2), (3,4), and (1,3) are removed again. A pink highlight shows the edges (1,2), (3,4), (1,3), (2,4), and (0,1) being added or removed.

Observation ∞ and $\partial\infty$ have same endpoints.

Def. For $a, b \in X$, chains from a to b

$$MC_n^l(X : a, b) = \bigoplus_{x, x_0=a, x_n=b} \mathbb{Z} \cdot \langle \infty \rangle$$

Already appeared in Hepworth-Willerton

$$MC_n^l(X : a, -) = \bigoplus_{x, x_0=a} \mathbb{Z} \langle \infty \rangle.$$

chains from a .

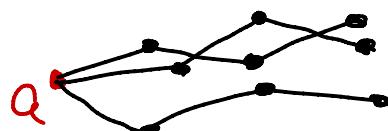
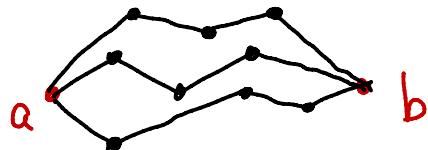
2. Magnitude homology with fixed endpoints

Def. For $a, b \in X$, $MC_n^l(x; a, b) = \bigoplus_{x, x_0=a, x_n=b} \mathbb{Z} \langle x \rangle$

$$MC_n^l(x; a, -) = \bigoplus_{x, x_0=a} \mathbb{Z} \langle x \rangle.$$

Rem If $l < d(a, b)$, then $MC_n^l(x; a, b) = 0$.

$MC_n^l(x; a, b)$ and $MC_n^l(x; a, -)$ become a chain complex. Denote the homology groups by $MH_n^l(x; a, b)$ and $MH_n^l(x; a, -)$.



2. Magnitude homology with fixed endpoints

$\text{MC}_\cdot^l(x; a, b)$ and $\text{MC}_\cdot^l(x; a, -)$ become a chain complex. Denote the homology groups by $\text{MH}_n^l(x; a, b)$ and $\text{MH}_n^l(x; a, -)$.



Prop.

$$\text{MH}_n^l(x) = \bigoplus_{a \in x} \text{MH}_n^l(x; a, -) = \bigoplus_{a, b \in x} \text{MH}_n^l(x; a, b).$$

Prop. If $l < d(a, b)$, then $\text{MH}_n^l(x; a, b) = 0$.

3. A threshold m_x .

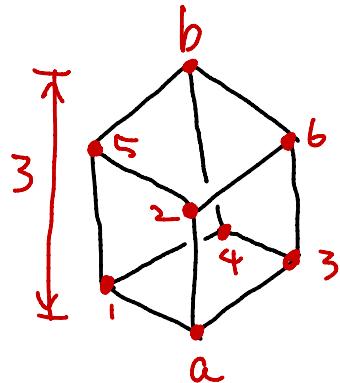
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What happens if $l = d(a, b)$?

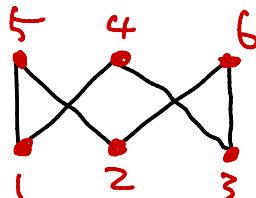
Def. (Interval poset).

$$I(a, b) := \{y \in X \setminus \{a, b\} \mid d(a, b) = d(a, y) + d(y, b)\}$$

$$y_1 \prec y_2 \iff d(a, y_2) = d(a, y_1) + d(y_1, y_2)$$



Interval Poset
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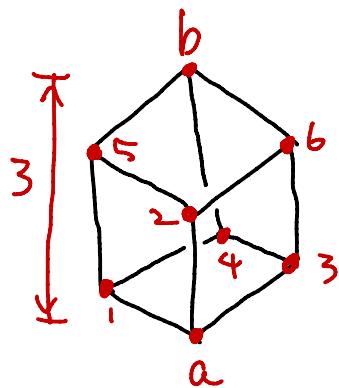


### 3. A threshold $m_x$ .

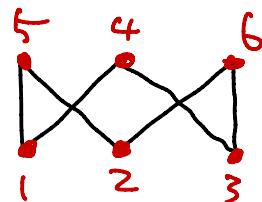
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Interval Poset



$$MC_n^3(x; a, b)$$

$$a = x_0 \prec x_1 \prec x_2 \prec \dots \prec x_n = b$$

$$C_{n-2}(\Delta(I(a,b)))$$

$$x_1 \prec \dots \prec x_{n-1}$$

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$$y_1 \prec y_2 \Leftrightarrow d(a,y_2) = d(a,y_1) + d(y_1,y_2)$$

Prop (Kaneta-Y. arXiv:1803.04247)

Suppose  $l = d(a,b) > 0$ . Then

$$MC_*^l(x:a,b) \xrightarrow{\sim} C_{\bullet-2}(\Delta(I(a,b)))$$

(isomorphic as chain complexes.)

We can generalize this result if "l is short".

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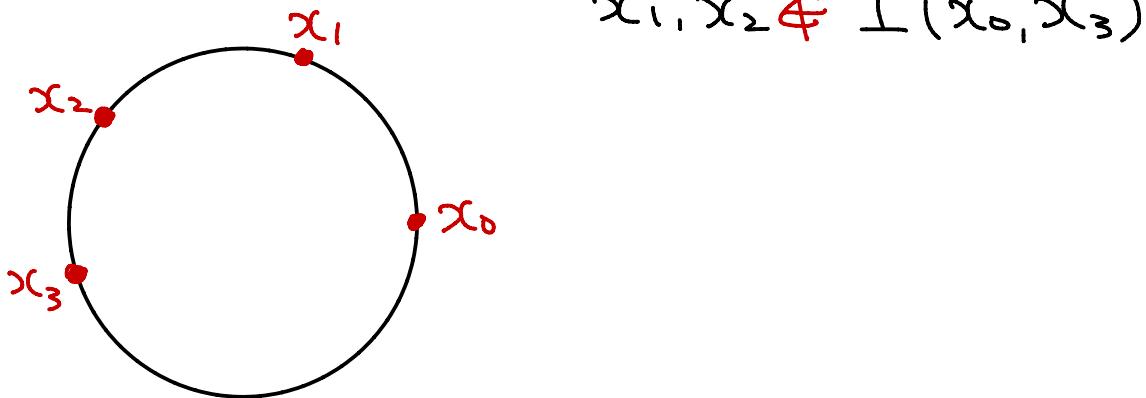
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Def.  $(x_0, x_1, x_2, x_3)$  is a 4-cut

$\overset{\text{def}}{\iff} x_1 \in I(x_0, x_2), x_2 \in I(x_1, x_3)$  and  
 $x_1, x_2 \notin I(x_0, x_3)$



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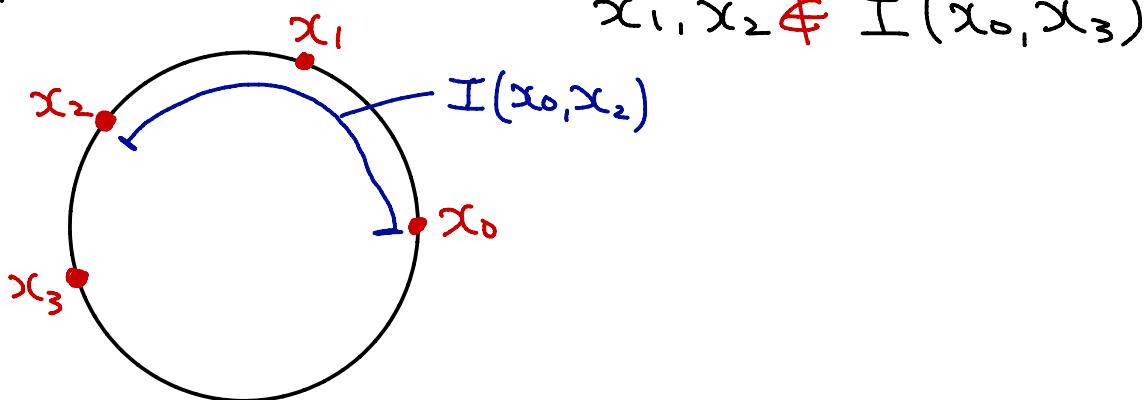
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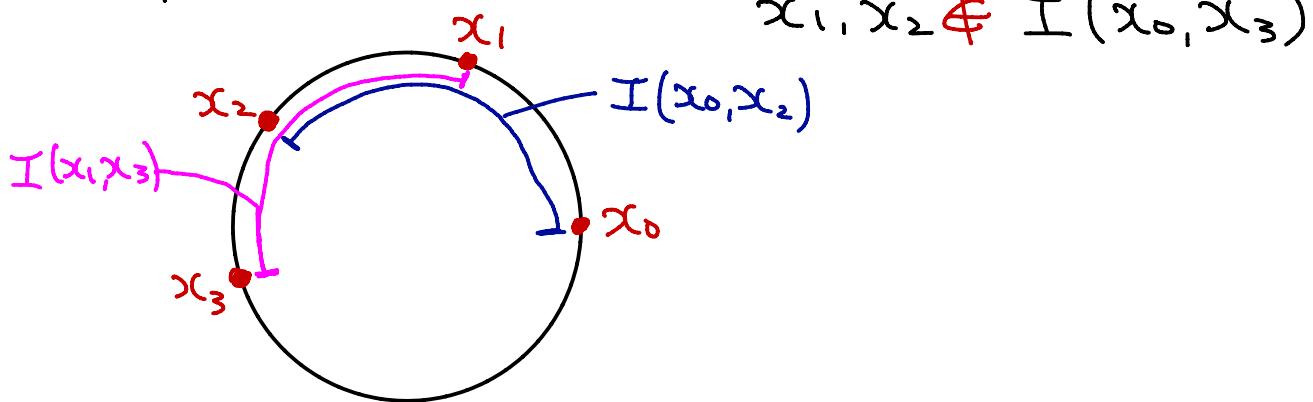
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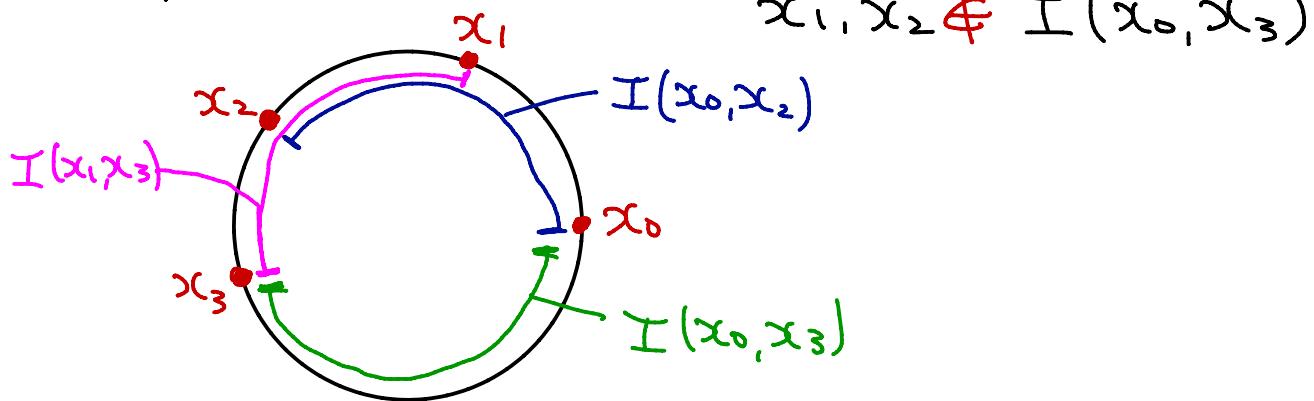
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Def  $m_x := \inf \left\{ \sum_{i=1}^3 d(x_{i-1}, x_i) \mid (x_0, x_1, x_2, x_3) \text{ is a 4-cut} \right\}$

Thm (Kaneta - Y. 2018)

Suppose  $l < m_x$ , Then,

$$MH_n^L(x; a, b) \cong \bigoplus_{k \geq 1} \bigoplus_{\substack{a = a_0, a_1, \dots, a_{n-k} = b \\ \text{s.t. } \sum d(a_{i-1}, a_i) = l, a_i \notin I(a_{i-1}, a_{i+1})}} H_{n-2-k} \left( C_0(a_0, a_1) \otimes \dots \otimes C_0(a_{n-1}, a_n) \right)$$

Roughly, it is homology of order complexes.

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Suppose  $\ell < m_x$ , Then,

$$MH_n^\ell(X; a, b) \cong \bigoplus_{k \geq 1} \bigoplus_{\substack{a = a_0, a_1, \dots, a_k = b \\ \text{s.t. } \sum d(a_{i-1}, a_i) = \ell, a_i \notin I(a_{i-1}, a_{i+1})}} H_{n-2-k} \left( C_0(a_0, a_1) \otimes \dots \otimes C_0(a_{k-1}, a_k) \right)$$

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A conclusion:  $\ell < m_x$ , then  $MH_n^\ell(x)$  is related to poset topology.

$\ell \geq m_x$ , then new techniques are required (cf. Gomi's talk.)

Cor. Homology of any ranked poset can be embedded into MH of a graph.

## 4. A categorification of weighting.

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Q. What is a categorification of weighting?

??



magnitude homology  $MH^?_t(x)$

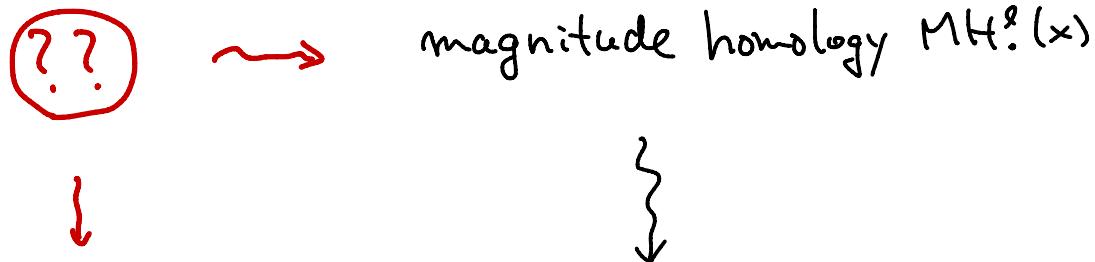


weighting  $w \rightsquigarrow$  magnitude  $\int_x dw$

An Answer :

## 4. A categorification of weighting.

Q. What is a categorification of weighting?



An Answer: It is  $MH_n^l(x; a, -)$ .

(*cf. Shulman's talk*) (may be better  $MC_*^l(x; a, -)$ )

Why?

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Reason 1.  $MH_n^l(x) = \bigoplus_{a \in X} MH_n^l(x; a, -)$

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Reason 1.  $MH_n^l(x) = \bigoplus_{a \in X} MH_n^l(x; a, -)$

Reason 2. (Essentially known)

Cancellation Lemma Let  $a \in X$  and  $l > 0$ .

$$MC_+^l(x; a, -) \cong \bigoplus_{\substack{b \in X \\ \neq}} MC_{-1}^{l - d(a, b)}(x; b, -)$$

Isomorphic as graded Abelian groups.

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## 4. A categorification of weighting.

Cancellation Lemma Let  $a \in X$  and  $\ell > 0$ .

$$MC_{\bullet}^{\ell}(x: a, -) \cong \bigoplus_{b \in X \setminus \{a\}} MC_{\bullet-1}^{\ell - d(a, b)}(x: b, -)$$

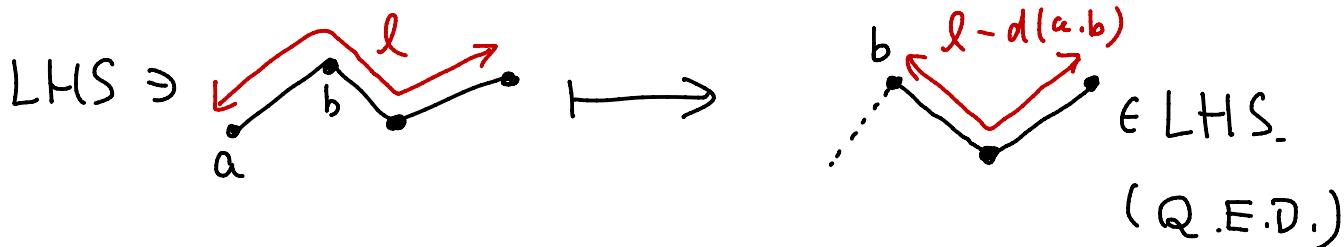
(Proof)

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(Proof) Construct a map as follows :

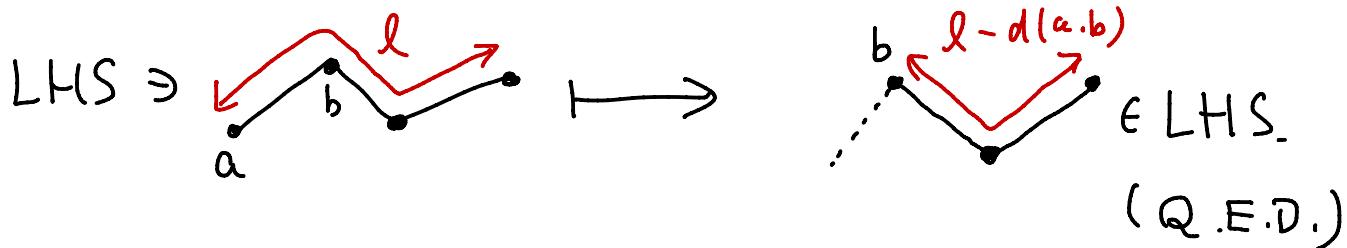


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(Proof) Construct a map as follows :



Prop. Let  $X$  be a finite (enough sparse) metric space.

$$\text{Then } w(a) := \sum_{\ell \geq 0} \sum_{n \geq 0} (-1)^n e^{-\ell} \cdot \text{rank } \text{MH}_n^{\ell}(X: a, -)$$

gives a weighting.

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(Proof) Need to check

$$\sum_{b \in X} e^{-d(a,b)} w(b) = 1.$$

$$\text{LHS} = w(a) + \sum_{b \in X \setminus \{a\}} e^{-d(a,b)} w(b)$$

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$$w(a) := \sum_{\ell \geq 0} \sum_{n \geq 0} (-1)^n e^{-\ell} \cdot \text{rank } \text{MH}_n^\ell(x; a, -) \quad \text{gives a weighting.}$$

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$$\text{LHS} = w(a) + \sum_{b \in X \setminus \{a\}} e^{-d(a, b)} w(b)$$

$$= 1 + \sum_{\ell \geq 0} \sum_{n \geq 0} (-1)^n \cdot e^{-\ell} \cdot \text{rank } \text{MC}_n^\ell(x; a, -)$$

$$+ \sum_{b \in X \setminus \{a\}} \sum_{\ell \geq 0} \sum_{n \geq 0} (-1)^n \cdot e^{-\ell - d(a, b)} \cdot \text{rank } \text{MC}_n^\ell(x; b, -)$$

#### 4. A categorification of weighting.

$$\begin{aligned} \sum_{b \in X} e^{-d(a,b)} w(b) &= w(a) + \sum_{b \in X \setminus \{a\}} e^{-d(a,b)} w(b) \\ &= 1 + \sum_{l>0} \sum_{n>0} (-1)^n \cdot e^{-l} \cdot \text{rank } MC_n^l(x; a, -) \\ &\quad + \sum_{b \in X \setminus \{a\}} \sum_{l>0} \sum_{n>0} (-1)^n \cdot e^{-l-d(a,b)} \cdot \text{rank } MC_n^l(x; b, -) \\ &= 1 + \sum_{l>0} \sum_{n>0} \underline{(-1)^n \cdot e^{-l}} \cdot \underline{\text{rank } MC_n^l(x; a, -)} \\ &\quad + \sum_{l>0} \sum_{n>0} \underline{(-1)^n \cdot e^{-l-d(a,b)}} \cdot \underline{\sum_{b \in X \setminus \{a\}} \text{rank } MC_n^l(x; b, -)} \end{aligned}$$

by "Cancellation Lemma"

$$= 1 \quad (\text{Q.E.D.})$$

## Summary

1.  $m_x := \inf \{4\text{-cut}\}$ . If  $l < m_x$ ,  
 $MH_n^l(x)$  is described by order complexes.
2.  $MH_n^l(x:a,-)$  can be considered as a categorification of the weighting.  
(But,  $MC_n^l(x:a,-)$  maybe better.)