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A threshold of length in magnitude homology

Masahiko Yoshinaga (Hokkaido)

English

Japanese

English	Japanese
- th -	s , z , t

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- th -	s , z , t
king	王 (ou , oh , o)

English	Japanese
-th-	s, z, t
king	王 (ou, oh, o)
King Arthur	A - SA - O アーサー王

What I want to discuss:

1. For a metric space X, there is a constant $m_X \geq 0$ s.t. magnitude homology $MH^\ell_\bullet$ is different in nature according to $\ell < m_X$ or $\ell \geq m_X$.

2. What is the "categorification of magnitude weighting"??

Plan

1. Order complex of a poset.
2. Magnitude homology with fixed endpoints.
3. A threshold m_x.
4. A categorification of weighting.

1. Order complex of a poset

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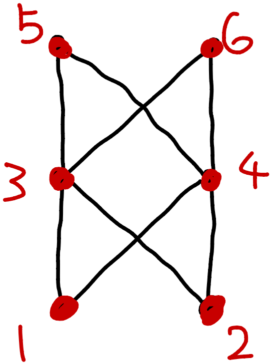
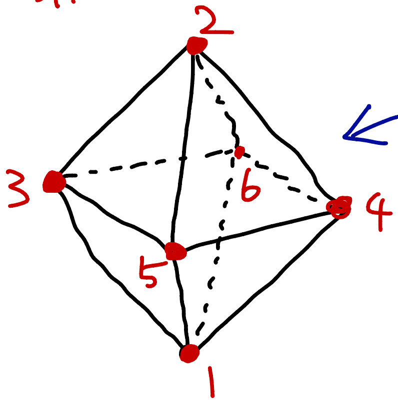
P : a poset $\rightsquigarrow$ $\Delta(P)$: $\rightsquigarrow$ $C_\bullet(\Delta(P))$:

order complex (a.k.a. classifying sp.)

a simplicial complex

the reduced chain complex ($C_{-1} = \mathbb{Z}$).





A triangulation of S^2 (octahedron)

0-simplices : $\{1\}, \{2\}, \cdots, \{6\}$

1-simplices : $\{1<4\}, \{1<3\}, \{1<6\}, \{1<5\}, \cdots$

2-simplices : $\{1<3<5\}, \{1<3<6\}, \{1<4<6\}, \cdots$

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Let (X, d) be a metric space.

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where $\mathbf{x} = (x_0, x_1, \cdots, x_n)$ runs $x_i \in X$,

$x_{i-1} \neq x_i$, $\sum_{i=1}^{n} d(x_{i-1}, x_i) = \ell$.

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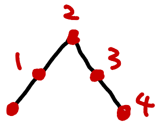
<u>Recall:</u> $\partial\Big($ 

$\Big) =$

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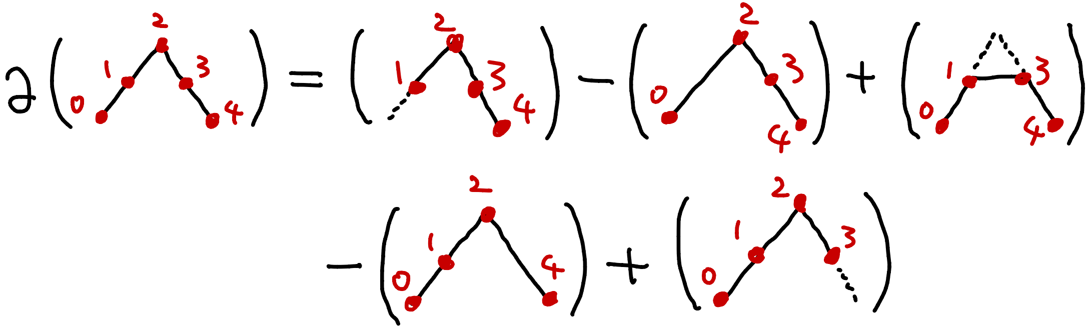

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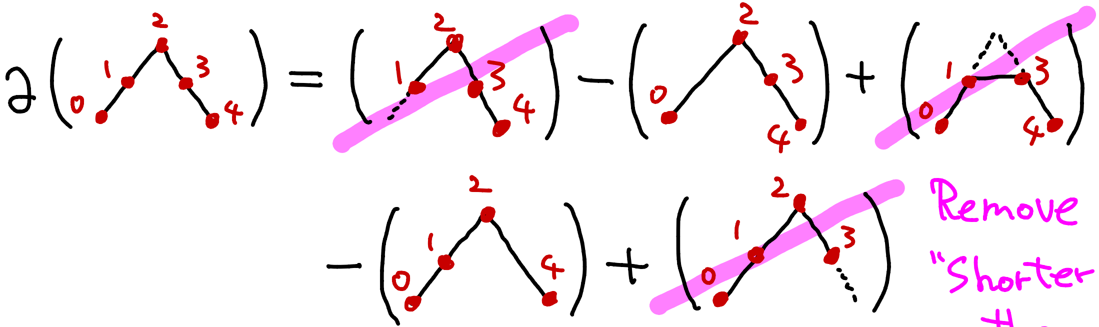

Remove "Shorter than ℓ"

<u>Observation</u> $\boldsymbol{x}$ and $\partial \boldsymbol{x}$ have same endpoints.

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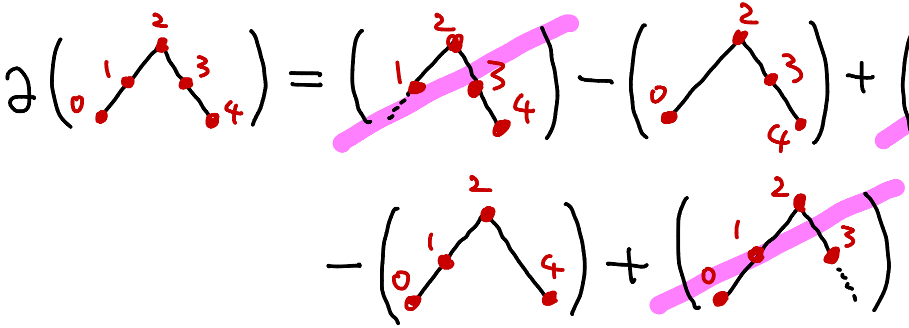

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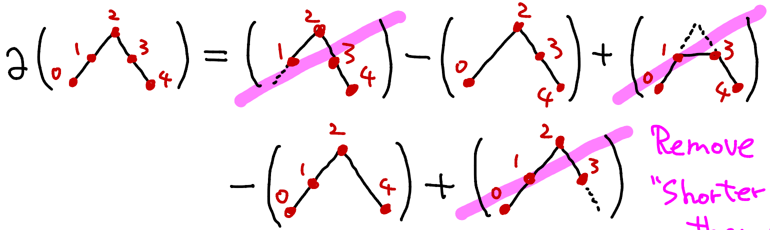


Remove "Shorter than ℓ"

Observation: x and ∂x have same endpoints.

Def. For $a, b \in X$,

chains from a to b

$$MC_n^\ell(X : a, b) = \bigoplus_{x,\ x_0 = a,\ x_n = b} \mathbb{Z} \cdot \langle x \rangle$$

Already appeared in Hepworth-Willerton

$$MC_n^\ell(X : a, -) = \bigoplus_{x,\ x_0 = a} \mathbb{Z} \langle x \rangle.$$

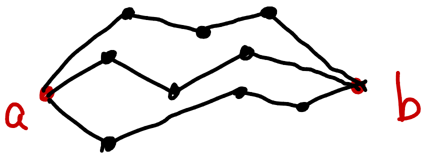
chains from a.

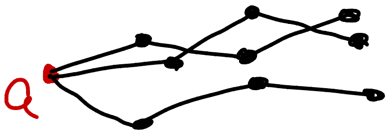
2. Magnitude homology with fixed endpoints

<u>Def.</u> For $a, b \in X$, $$MC_n^\ell(X:a,b) = \bigoplus_{x,\ x_0=a,\ x_n=b} \mathbb{Z}\cdot\langle x\rangle$$
$$MC_n^\ell(X:a,-) = \bigoplus_{x,\ x_0=a} \mathbb{Z}\langle x\rangle .$$

<u>Rem</u> If $\ell < d(a,b)$, then $MC_n^\ell(X:a,b)=0$.

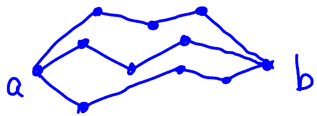
$MC_\bullet^\ell(X:a,b)$ and $MC_\bullet^\ell(X:a,-)$ become a chain complex. Denote the homology groups by $MH_n^\ell(X;a,b)$ and $MH_n^\ell(X:a,-)$.

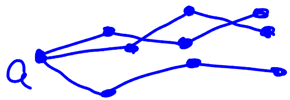




2. Magnitude homology with fixed endpoints

$MC^{\ell}_{\bullet}(X:a,b)$ and $MC^{\ell}_{\bullet}(X:a,-)$ become a chain complex. Denote the homology groups by $MH^{\ell}_{n}(X:a,b)$ and $MH^{\ell}_{n}(X:a,-)$.





Prop.

$$MH^{\ell}_{n}(X) = \bigoplus_{a\in X} MH^{\ell}_{n}(X:a,-) = \bigoplus_{a,b\in X} MH^{\ell}_{n}(X:a,b).$$

Prop. If $\ell < d(a,b)$, then $MH^{\ell}_{n}(X:a,b)=0$.

3. A threshold m_X.

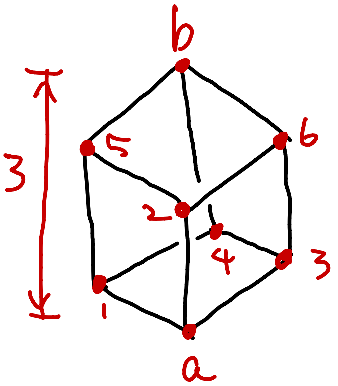
Prop. If $\ell < d(a,b)$, then $MH_n^\ell(X:a,b)=0$.

What happens if $\ell = d(a,b)$?

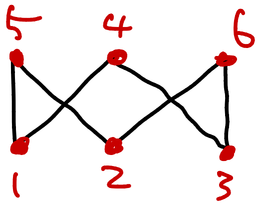
Def. (Interval poset).

$$I(a,b) := \{y \in X\setminus\{a,b\} \mid d(a,b) = d(a,y) + d(y,b)\}$$

$$y_1 < y_2 \iff d(a,y_2) = d(a,y_1) + d(y_1,y_2)$$



Interval poset



3. A threshold m_X.

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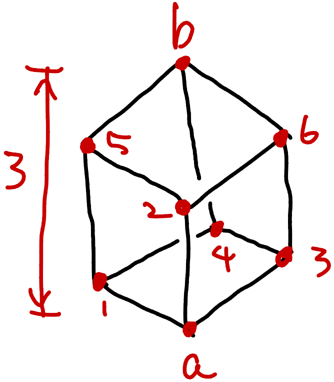

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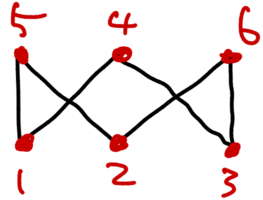

$$MC^3_n(X : a, b) \qquad\qquad C_{n-2}(\Delta(I(a,b)))$$

$$\rotatebox{90}{$\in$}$$

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b \qquad\qquad x_1 < \cdots < x_{n-1}$$

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<u>Prop</u> (Kaneta–Y. arXiv:1803.04247)

Suppose $\ell = d(a,b) > 0$. Then

$$MC_{\bullet}^{\ell}(X : a,b) \xrightarrow{\sim} C_{\bullet-2}(\Delta(I(a,b)))$$

(isomorphic as chain complexes.)

We can generalize this result if "ℓ is short".

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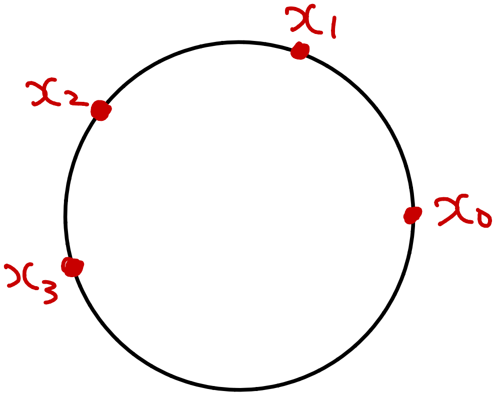
$$y_1 < y_2 \iff d(a,y_2) = d(a,y_1) + d(y_1,y_2)$$

Def. (x_0, x_1, x_2, x_3) is a 4-cut

$\overset{\text{def}}{\iff}$ $x_1 \in I(x_0, x_2)$, $x_2 \in I(x_1, x_3)$ and

$x_1, x_2 \notin I(x_0, x_3)$



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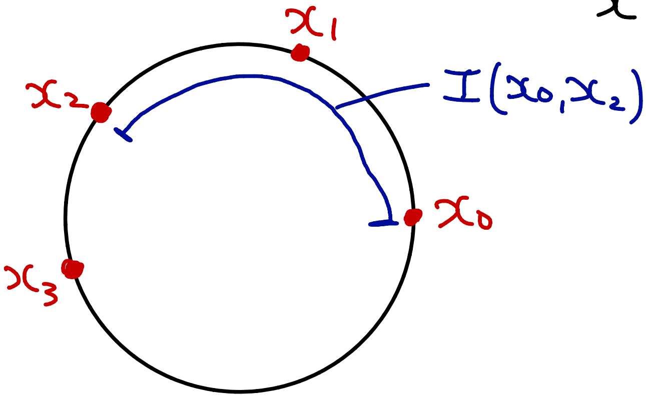

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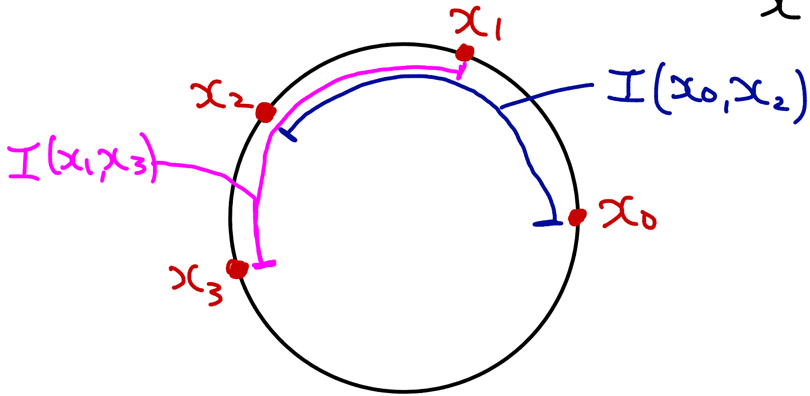

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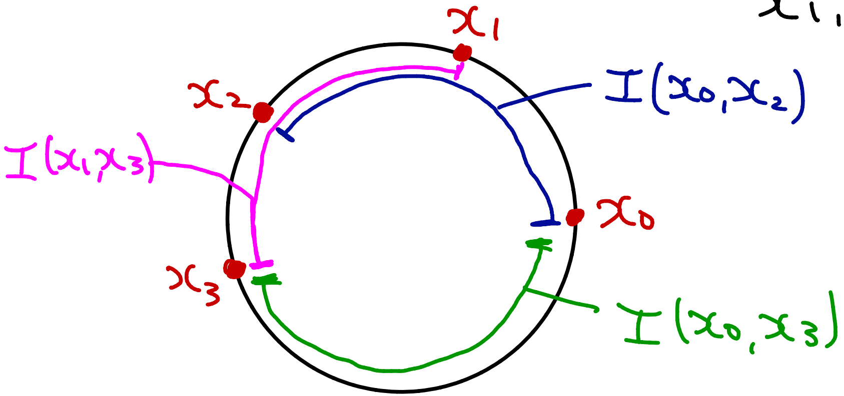

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<u>Def</u> $$m_X := \inf\left\{ \sum_{i=1}^{3} d(x_{i-1}, x_i) \;\middle|\; \begin{array}{l}(x_0, x_1, x_2, x_3) \\ \text{is a 4-cut}\end{array} \right\}$$

<u>Thm</u> (Kaneta - Y. 2018)

Suppose $\ell < m_X$. Then,

$$MH^{\ell}_{n}(X; a, b) \cong \bigoplus_{k \geq 1} \bigoplus_{\substack{a = a_0, a_1, \cdots, a_k = b \\ \text{s.t. } \sum d(a_{i-1}, a_i) = \ell,\ a_i \notin I(a_{i-1}, a_{i+1})}} H_{n-2k}\left(C_\bullet(a_0, a_1) \otimes \cdots \otimes C_\bullet(a_{k-1}, a_k)\right)$$

Roughly, it is homology of order complexes.

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A conclusion: $\ell < m_X$, then $MH^{\ell}_{\bullet}(X)$ is related to poset topology.

$\ell \geq m_X$, then new techniques are required (cf. Gomi's talk.)

<u>Cor.</u> Homology of any ranked poset can be embedded into MH of a graph.

4. A categorification of weighting.

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Q. What is a categorification of weighting?

$$
\begin{tikzcd}
?? \arrow[r, squiggly] \arrow[d, squiggly] & \text{magnitude homology } MH_*^*(X) \arrow[d, squiggly] \\
\text{weighting } w \arrow[r, squiggly] & \text{magnitude } \int_X dw
\end{tikzcd}
$$

An Answer:

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An Answer: It is $MH_n^\ell(X;a,-)$.

(cf. Shulman's talk) (may be better $MC_\bullet^\ell(X;a,-)$.)

Why?

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Reason 2.

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Reason 2. (Essentially known)

Cancellation Lemma Let $a \in X$ and $\ell > 0$.

$$MC_\bullet^\ell(X : a, -) \cong \bigoplus_{b \in X} MC_{\bullet - 1}^{\ell - d(a,b)}(X : b, -)$$

Isomorphic as graded Abelian groups.

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(Proof)

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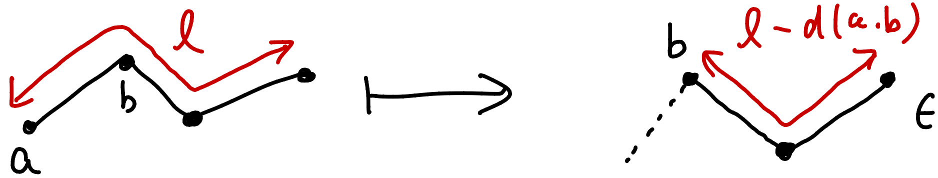
(Proof) Construct a map as follows:

LHS $\ni$



$\in$ LHS.

(Q.E.D.)

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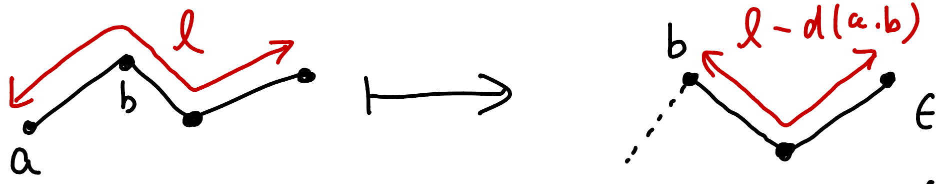
LHS $\ni$  $\in$ LHS.

(Q.E.D.)

Prop. Let X be a finite (enough sparse) metric space.

Then $w(a) := \sum_{\ell \geq 0} \sum_{n \geq 0} (-1)^n e^{-\ell} \cdot \operatorname{rank} MH_n^\ell(X : a, -)$

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$$\sum_{b \in X} e^{-d(a,b)} w(b) = 1.$$

$$LHS = w(a) + \sum_{b \in X \setminus \{a\}} e^{-d(a,b)} w(b)$$

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<u>Prop.</u> Let X be a finite (enough sparse) metric space. Then

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$$\sum_{b \in X} e^{-d(a,b)} w(b) = 1.$$

$$\text{LHS} = w(a) + \sum_{b \in X \setminus \{a\}} e^{-d(a,b)} w(b)$$

$$= 1 + \sum_{\ell > 0} \sum_{n > 0} (-1)^n \cdot e^{-\ell} \cdot \operatorname{rank} MC_n^{\ell}(X ; a, -)$$

$$+ \sum_{b \in X \setminus \{a\}} \sum_{\ell > 0} \sum_{n > 0} (-1)^n \cdot e^{-\ell - d(a,b)} \cdot \operatorname{rank} MC_n^{\ell}(X : b, -)$$

4. A categorification of weighting.

$$\sum_{b\in X} e^{-d(a,b)} w(b) = w(a) + \sum_{b\in X\setminus\{a\}} e^{-d(a,b)} w(b)$$

$$= 1 + \sum_{l>0} \sum_{n>0} (-1)^n \cdot e^{-l} \cdot \operatorname{rank} MC_n^l(X; a, -)$$

$$+ \sum_{b\in X\setminus\{a\}} \sum_{l>0} \sum_{n>0} (-1)^n \cdot e^{-l-d(a,b)} \cdot \operatorname{rank} MC_n^l(X: b, -)$$

$$= 1 + \sum_{l>0} \sum_{n>0} (-1)^n \cdot e^{-l} \cdot \operatorname{rank} MC_n^l(X: a, -)$$

$$+ \sum_{l>0} \sum_{n>0} (-1)^n \cdot e^{-l-d(a,b)} \cdot \sum_{b\in X\setminus\{a\}} \operatorname{rank} MC_n^l(X: b, -)$$

by "Cancellation Lemma"

$$= 1$$

(Q.E.D.)

Summary

1. $m_X := \inf\{4\text{-cut}\}$. If $\ell < m_X$,

   $MH_n^\ell(X)$ is described by order complexes.

2. $MH_n^\ell(X : a, -)$ can be considered as a categorification of the weighting.

   (But, $MC_\bullet^\ell(X : a, -)$ maybe better.)