Magnitude of odd balls

Simon Willerton
University of Sheffield

ICMS
4th July 2019

Magnitude of finite metric spaces

Finite metric space (A, d):

Magnitude of finite metric spaces

Finite metric space (A, d): weight $w\colon A \to \mathbb{R}$ such that

$$\sum_a \mathrm{e}^{-\mathrm{d}(a,b)} w(a) = 1 \qquad \text{for every } b \in A.$$

Magnitude of finite metric spaces

Finite metric space (A, d): weight $w\colon A \to \mathbb{R}$ such that

$$\sum_a \mathrm{e}^{-\mathrm{d}(a,b)} w(a) = 1 \qquad \text{for every } b \in A.$$

Define the magnitude by $|A| = \sum_a w(a)$.

Magnitude of finite metric spaces

Finite metric space (A, d): weight $w\colon A \to \mathbb{R}$ such that

$$\sum_a \mathrm{e}^{-\mathrm{d}(a,b)} w(a) = 1 \qquad \text{for every } b \in A.$$

Define the magnitude by $|A| = \sum_a w(a)$.

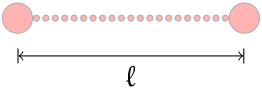
Extending to infinite metric spaces

What happens when try to approximate an infinite subset of $\mathbb{R}^n$?

Extending to infinite metric spaces

What happens when try to approximate an infinite subset of $\mathbb{R}^n$?

$L_m :=$ m pts on the length ℓ interval

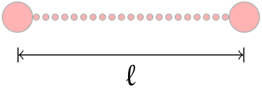


As $m \to \infty$, $|L_m| \to \frac{\ell}{2} + 1$.

Extending to infinite metric spaces

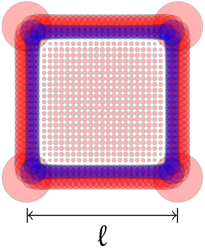
What happens when try to approximate an infinite subset of $\mathbb{R}^n$?

$L_m :=$ m pts on the length ℓ interval



As $m \to \infty$, $|L_m| \to \frac{\ell}{2} + 1$.

$S_m :=$ m^2 pts on the width ℓ square

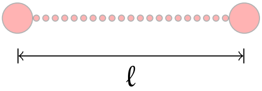


As $m \to \infty$, $|S_m| \simeq \frac{\ell^2}{2\pi} + \ldots$.

Extending to infinite metric spaces

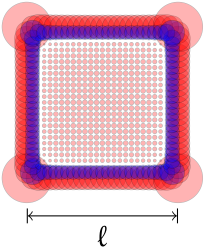
What happens when try to approximate an infinite subset of $\mathbb{R}^n$?

$L_m := m$ pts on the length ℓ interval



As $m \to \infty$, $|L_m| \to \frac{\ell}{2} + 1$.

$S_m := m^2$ pts on the width ℓ square



As $m \to \infty$, $|S_m| \simeq \frac{\ell^2}{2\pi} + \dots$.

Definition/Theorem. (Meckes) If $X \subset \mathbb{R}^n$ is compact and $A_m \to X$ in the Hausdorff topology then we can define $|X| := \lim_m |A_m|$.

Distributions

A distribution on $\mathbb{R}^n$ is a linear functional on some suitable class of functions. Write $\langle w, f \rangle$ for the evaluation of a distribution w on a function f.
E.g.

(i) For each signed measure μ we have an associated distribution with

$$\langle \mu, f \rangle := \int_{\mathbb{R}^n} f \, \mathrm{d}\, \mu.$$

(ii) For a cooriented, smooth, codim 1 submanifold $\Sigma \subset \mathbb{R}^n$, and $i \in \mathbb{N}$

$$\langle w_i, f \rangle := \int_{\Sigma} \frac{\partial^i}{\partial \nu^i} f(\mathbf{x}) \, \mathrm{d}\, \mathbf{x},$$

where $\frac{\partial}{\partial \nu}$ means derivative in the normal direction to the submanifold.

Weight distributions

(Meckes) Let $X \subset \mathbb{R}^n$ be compact, convex with non-empty interior (n odd). A **weight distribution** w for X is a distribution (in $H^{-(n+1)/2}(\mathbb{R}^n)$) supported on X such that

$$\langle w, e^{-\mathrm{d}(\mathbf{s},\cdot)} \rangle = 1 \quad \text{for every } \mathbf{s} \in X.$$

The magnitude of X is given by $|X| = \langle w, \mathbf{1} \rangle$.

Weight distributions

(Meckes) Let $X \subset \mathbb{R}^n$ be compact, convex with non-empty interior (n odd).
A **weight distribution** w for X is a distribution (in $H^{-(n+1)/2}(\mathbb{R}^n)$) supported on X such that

$$\langle w, e^{-\mathrm{d}(\mathbf{s},\cdot)} \rangle = 1 \quad \text{for every } \mathbf{s} \in X.$$

The magnitude of X is given by $|X| = \langle w, \mathbf{1} \rangle$.

Try to calculate the magnitude of an odd ball!
Guess a weight distribution for B_R^n, the radius R ball of dimension $n = 2p+1$.

$$\langle w, f \rangle = \frac{1}{n!\,\omega_n} \left(\int_{\mathbf{x}\in B_R^n} f \,\mathrm{d}\mathbf{x} + \sum_{i=0}^{p} \beta_i(R) \int_{\mathbf{x}\in S_R^{n-1}} \frac{\partial^i}{\partial \nu^i} f \,\mathrm{d}\mathbf{x} \right)$$

Weight distributions

(Meckes) Let $X \subset \mathbb{R}^n$ be compact, convex with non-empty interior (n odd). A **weight distribution** w for X is a distribution (in $H^{-(n+1)/2}(\mathbb{R}^n)$) supported on X such that

$$\langle w, e^{-\mathrm{d}(\mathbf{s},\cdot)} \rangle = 1 \quad \text{for every } \mathbf{s} \in X.$$

The magnitude of X is given by $|X| = \langle w, \mathbf{1} \rangle$.

Try to calculate the magnitude of an odd ball!

Guess a weight distribution for B_R^n, the radius R ball of dimension $n = 2p+1$.

$$\langle w, f \rangle = \frac{1}{n!\,\omega_n} \left(\int_{\mathbf{x} \in B_R^n} f \, \mathrm{d}\mathbf{x} + \sum_{i=0}^{p} \beta_i(R) \int_{\mathbf{x} \in S_R^{n-1}} \frac{\partial^i}{\partial \nu^i} f \, \mathrm{d}\mathbf{x} \right)$$

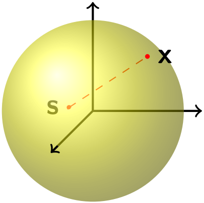
Need to solve the weight equation for every $\mathbf{s} \in B_R^n$ to find $(\beta_i(R))_{i=0}^p$.

Then

$$|B_R^n| = \tfrac{1}{n!} \left(R^n + n\beta_0(R) R^{n-1} \right).$$

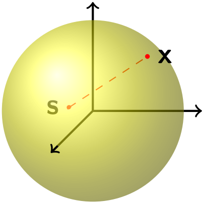
The Key Integral

$$\frac{1}{n!\,\omega_n}\int_{\mathbf{x}\in S_R^{n-1}} e^{-|\mathbf{x}-\mathbf{s}|}\,\mathrm{d}\mathbf{x} \quad \text{for } \mathbf{s}\in B_R^n$$



The Key Integral

$$\frac{1}{n!\,\omega_n}\int_{\mathbf{x}\in S_R^{n-1}} e^{-|\mathbf{x}-\mathbf{s}|}\,\mathrm{d}\mathbf{x} \quad \text{for } \mathbf{s}\in B_R^n$$



Theorem

For $n = 2p+1$, $R > 0$ and $s = |\mathbf{s}| < R$, then

$$\frac{1}{n!\,\omega_n}\int_{\mathbf{x}\in S_R^{n-1}} e^{-|\mathbf{x}-\mathbf{s}|}\,\mathrm{d}\mathbf{x} \;=\; \frac{(-1)^p e^{-R}}{2^p p!}\sum_{i=0}^{p}\binom{p}{i}\chi_{p+i}(R)\tau_i(s).$$

Reverse Bessel polynomials

$$\chi_0(R) = 1;$$
$$\chi_1(R) = R;$$
$$\chi_2(R) = R^2 + R;$$
$$\chi_3(R) = R^3 + 3R^2 + 3R$$
$$\chi_4(R) = R^4 + 6R^3 + 15R^2 + 15R.$$

modified spherical Bessel functions-ish

$$\tau_0(s) = \cosh(s);$$
$$\tau_1(s) = -\frac{\sinh(s)}{s};$$
$$\tau_2(s) = \frac{\cosh(s)}{s^2} - \frac{\sinh(s)}{s^3};$$
$$\tau_3(s) = -\frac{\sinh(s)}{s^3} + \frac{3\cosh(s)}{s^4} - \frac{3\sinh(s)}{s^5}$$

Solving the weight equations

Trying to solve the weight equation for every $s \in S_R^{n-1}$ gives a linear system.

$$\begin{pmatrix} \chi_p(R) & \delta\chi_p(R) & \dots & \delta^p\chi_p(R) \\ \chi_{p+1}(R) & \delta\chi_{p+1}(R) & \dots & \delta^p\chi_{p+1}(R) \\ \vdots & \vdots & & \vdots \\ \chi_{2p}(R) & \delta\chi_{2p}(R) & \dots & \delta^p\chi_{2p}(R) \end{pmatrix} \begin{pmatrix} \beta_0(R) \\ \beta_1(R) \\ \vdots \\ \beta_p(R) \end{pmatrix} = \begin{pmatrix} \chi_{p+1}(R)/R \\ \chi_{p+2}(R)/R \\ \vdots \\ \chi_{2p+1}(R)/R \end{pmatrix}$$

But remember the magnitude has the following form.

$$|B_R^n| = \frac{1}{n!}\left(R^n + n\beta_0(R)R^{n-1}\right),$$

So we can add this to our linear system.

$$\begin{pmatrix} \chi_p(R) & \delta\chi_p(R) & \dots & \delta^p\chi_p(R) & 0 \\ \chi_{p+1}(R) & \delta\chi_{p+1}(R) & \dots & \delta^p\chi_{p+1}(R) & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \chi_{2p}(R) & \delta\chi_{2p}(R) & \dots & \delta^p\chi_{2p}(R) & 0 \\ -nR^{n-1} & 0 & \dots & 0 & n! \end{pmatrix} \begin{pmatrix} \beta_0(R) \\ \beta_1(R) \\ \vdots \\ \beta_p(R) \\ \left|B_R^n\right| \end{pmatrix} = \begin{pmatrix} \chi_{p+1}(R)/R \\ \chi_{p+2}(R)/R \\ \vdots \\ \chi_{2p+1}(R)/R \\ R^n \end{pmatrix}$$

Now use Cramer's Rule...

The answer

$$|B_R^n| = \frac{\left|\begin{matrix}\text{some matrix of} \\ \text{derivatives of } \chi_i(R)s\end{matrix}\right|}{n!\left|\begin{matrix}\text{some other matrix of} \\ \text{derivatives of } \chi_i(R)s\end{matrix}\right|} = \cdots = \frac{\begin{vmatrix} \chi_2(R) & \chi_3(R) & \dots & \chi_{p+2}(R) \\ \chi_3(R) & \chi_4(R) & \dots & \chi_{p+3}(R) \\ \vdots & & \vdots & \\ \chi_{p+2}(R) & \chi_{p+3}(R) & \dots & \chi_{2p+2}(R) \end{vmatrix}}{n!\,R\begin{vmatrix} \chi_0(R) & \chi_1(R) & \dots & \chi_p(R) \\ \chi_1(R) & \chi_2(R) & \dots & \chi_{p+1}(R) \\ \vdots & & \vdots & \\ \chi_p(R) & \chi_{p+1}(R) & \dots & \chi_{2p}(R) \end{vmatrix}}$$

[Determinants with constant antidiagonals are called Hankel determinants.]

The answer

$$|B_R^n| = \frac{\begin{vmatrix} \text{some matrix of} \\ \text{derivatives of } \chi_i(R)s \end{vmatrix}}{n! \begin{vmatrix} \text{some other matrix of} \\ \text{derivatives of } \chi_i(R)s \end{vmatrix}} = \cdots = \frac{\begin{vmatrix} \chi_2(R) & \chi_3(R) & \dots & \chi_{p+2}(R) \\ \chi_3(R) & \chi_4(R) & \dots & \chi_{p+3}(R) \\ \vdots & & \vdots & \\ \chi_{p+2}(R) & \chi_{p+3}(R) & \dots & \chi_{2p+2}(R) \end{vmatrix}}{n!\, R \begin{vmatrix} \chi_0(R) & \chi_1(R) & \dots & \chi_p(R) \\ \chi_1(R) & \chi_2(R) & \dots & \chi_{p+1}(R) \\ \vdots & & \vdots & \\ \chi_p(R) & \chi_{p+1}(R) & \dots & \chi_{2p}(R) \end{vmatrix}}$$

[Determinants with constant antidiagonals are called Hankel determinants.]

$$\left|B_R^1\right| = R+1$$

$$\left|B_R^3\right| = \frac{R^3+6R^2+12R+6}{3!}$$

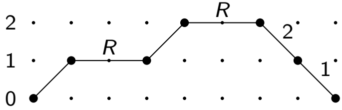
$$\left|B_R^5\right| = \frac{R^6+18\,R^5+135\,R^4+525\,R^3+1080\,R^2+1080\,R+360}{5!\,(R+3)}$$

$$\left|B_R^7\right| = \frac{R^{10}+40R^9+720R^8+\cdots+1814400R^2+1209600R+302400}{7!\,(R^3+12R^2+48R+60)}$$

Lots of things about these not (immediately) explained by the formula...

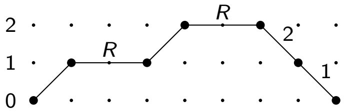
Combinatorial interpretation of reverse Bessel polynomials

A **length ℓ weighted Schröder path** is a lattice path from $(x,0)$ to $(x+\ell,0)$ of a certain form:



Combinatorial interpretation of reverse Bessel polynomials

A **length ℓ weighted Schröder path** is a lattice path from $(x,0)$ to $(x+\ell,0)$ of a certain form:



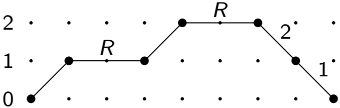
Each Bessel polynomial is a weighted count of such paths.

Theorem (Favreau, Sokal)

$$\chi_{i+1}(R)/R = \sum_{\gamma \text{ length } 2i} w(\gamma)$$

Combinatorial interpretation of reverse Bessel polynomials

A **length ℓ weighted Schröder path** is a lattice path from $(x,0)$ to $(x+\ell,0)$ of a certain form:



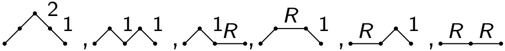
Each Bessel polynomial is a weighted count of such paths.

Theorem (Favreau, Sokal)

$$\chi_{i+1}(R)/R = \sum_{\gamma \text{ length } 2i} w(\gamma)$$

Example

$\chi_3(R)/R = \# \{$



$\} = R^2 + 3R + 3$

Combinatorial interpretation of the determinants

Theorem (Rough version of Lindström-Gessel-Viennot Lemma)

- *Let G be a weighted, directed, acyclic graph.*
- *Suppose $\{K_i\}_{i=0}^k$ and $\{L_j\}_{j=0}^k$ be two sets of vertices in G.*
- *Let $M_{i,j}$ denote the weighted count of paths from K_i to L_j.*
- *Then subject to some condition on the vertices, the determinant*

$$\det[M_{i,j}]_{i,j=0}^k$$

is the weighted count of all disjoint collections of $k+1$ paths joining K_i to L_i for $i=0,\dots,k$.

Each $\chi_i(R)$ is a count of lattice paths.

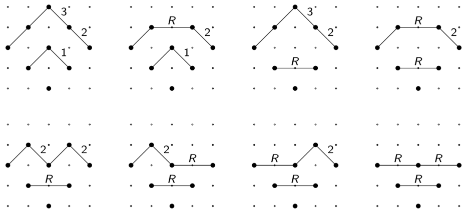
Corollary

- *The Hankel determinants $\det[\chi_{i+j+2}(R)]_{i,j=0}^p$ and $\det[\chi_{i+j}(R)]_{i,j=0}^p$ are counts of disjoint collections of lattice paths.*
- *Thus so are the numerator and denominator of $\left|B_R^n\right|$.*

Combinatorial interpretation of the determinants (ctd)

You end up with very nice expressions for the numerator and denominator. For example,

$$\text{numerator}\left|B_R^3\right| = \#\left\{ \quad \right\}$$



$$= R^3 + 6R^2 + 12R + 6$$

We now have a combinatorial interpretation of each of the coefficients:

$$\left|B_R^1\right| = R + 1$$

$$\left|B_R^3\right| = \frac{R^3+6R^2+12R+6}{3!}$$

$$\left|B_R^5\right| = \frac{R^6+18\,R^5+135\,R^4+525\,R^3+1080\,R^2+1080\,R+360}{5!\,(R+3)}$$

$$\left|B_R^7\right| = \frac{R^{10}+40R^9+720R^8+\cdots+1814400R^2+1209600R+302400}{7!\left(R^3+12R^2+48R+60\right)}$$

The payoff

$$\left|B_R^1\right| = R+1$$
$$\left|B_R^3\right| = \frac{R^3+6R^2+12R+6}{3!}$$
$$\left|B_R^5\right| = \frac{R^6+18\,R^5+135\,R^4+525\,R^3+1080\,R^2+1080\,R+360}{5!\,(R+3)}$$
$$\left|B_R^7\right| = \frac{R^{10}+40R^9+720R^8+\cdots+1814400R^2+1209600R+302400}{7!\left(R^3+12R^2+48R+60\right)}$$

Theorem

- *Both numerator and denominator are monic polynomials of the obvious degrees with positive integer coefficients.*
- $\left|B_R^n\right| \to 1$ *as* $R \to 0$.
- $\left|B_R^n\right| = 1 + \frac{(n-1)!!}{(n-2)!!} R + O(R^2)$ *as* $R \to 0$ *(with Meckes).*
- $\left|B_R^n\right| = \frac{1}{n!}\left(R^n + \frac{n(n+1)}{2}R^{n-1} + \frac{(n-1)n(n+1)^2}{8}R^{n-2} + \dots\right)$ *as* $R \to \infty$.

See Gimperlein-Goffeng later this afternoon. . .