Introduction to Magnitude Homology

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Outline

Euler characteristic for cell complexes

A cell structure on a space is a decomposition into cells (vertices, edges, faces, etc.) such that each n-cell is topologically an n-ball.

Theorem

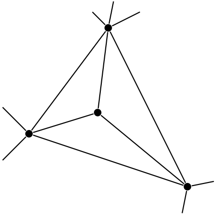
For any space X admitting at least one finite cell structure, there is a number $\chi(X)$, called its Euler characteristic, such that for any finite cell structure on X we have

$$c_0 - c_1 + c_2 - c_3 + \cdots = \chi(X)$$

where c_n = the number of n-cells.

The number $\chi(X)$ is a fairly coarse invariant, while the c_n's are not invariants at all (they depend on the cell structure).
Is there something in between?

Modifying a cell structure



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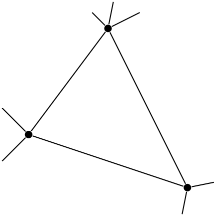


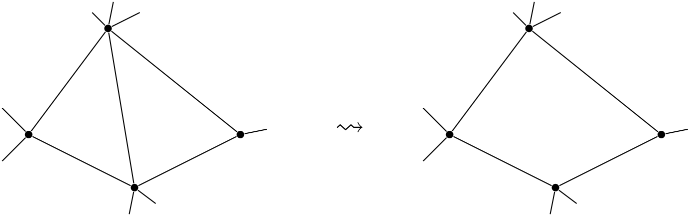
c_0 $\quad$ $c_0' = c_0 - 1$

c_1 $\quad$ $c_1' = c_1 - 3$

c_2 $\quad$ $c_2' = c_2 - 2$

$$c_0 - c_1 + c_2 \ = \ c_0' - c_1' + c_2'$$

Modifying a cell structure, bis



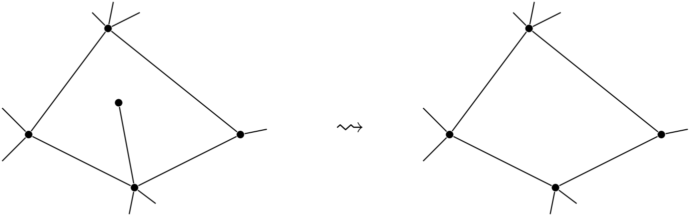
c_0 | $c_0' = c_0$

c_1 | $c_1' = c_1 - 1$

c_2 | $c_2' = c_2 - 1$

$$c_0 - c_1 + c_2 \; = \; c_0' - c_1' + c_2'$$

Modifying a cell structure, ter



$$c_0 \qquad\qquad c_0' = c_0 - 1$$

$$c_1 \qquad\qquad c_1' = c_1 - 1$$

$$c_2 \qquad\qquad c_2' = c_2$$

$$c_0 - c_1 + c_2 \;=\; c_0' - c_1' + c_2'$$

Chains

Idea

We can "cancel" an $(n+1)$-cell with an n-cell on its boundary.

To avoid arbitrary choices, let's try to instead cancel it with "its whole boundary at once". What is that?

- A finite collection of n-cells.
- Not ordered — a finite set?
- One n-cell can appear more than once — a finite multiset?
- Need to retain orientations — a finite signed multiset.

Definition

The n^{th} chain group $C_n(X)$ of a cell structure is the free abelian group generated by the set of n-cells.

Note that c_n is the rank of $C_n(x)$.

Homology

Definition

The boundary map $d_{n+1} : C_{n+1}(X) \to C_n(X)$ sends each $(n+1)$-cell generator to the sum of the n-cells on its boundary (with multiplicities and signs).

The signs work out so that the composite

$$C_{n+1}(X) \xrightarrow{d_{n+1}} C_n(X) \xrightarrow{d_n} C_{n-1}(X)$$

is zero. (An algebraic version of the geometric fact $\partial\partial M = \emptyset$.) Thus if we define

$$Z_n(X) = \ker(d_n) \qquad B_n(X) = \operatorname{im}(d_{n+1})$$

then $B_n(X) \subseteq Z_n(X)$.

Definition

The n^{th} homology group is the quotient $H_n(X) = Z_n(X)/B_n(X)$.

Abstract homology

Definition

A chain complex is a sequence of abelian groups C_n, with maps $d_{n+1} : C_{n+1} \to C_n$ called differentials, such that each composite $C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$ is zero. We then define

$$Z_n = \ker(d_n) \qquad B_n = \operatorname{im}(d_{n+1}) \quad H_n = Z_n / B_n.$$

We call Z_n the cycles, B_n the boundaries, and H_n the homology, and write* z_n, b_n, and h_n for their ranks.

(Instead of abelian groups we can use vector spaces, modules, graded groups, etc.)

* Confusingly, our h_n's are traditionally denoted b_n (the "Betti numbers").

Homology and Euler characteristic

Theorem

For any finite-rank chain complex we have

$$\sum_n (-1)^n \operatorname{Rank}(C_n) = \sum_n (-1)^n \operatorname{Rank}(H_n).$$

Proof.

The short exact sequence $0 \to Z_n \to C_n \to B_{n-1} \to 0$ implies $c_n = b_{n-1} + z_n$, and hence

$$\begin{aligned} &c_0 - c_1 + c_2 - c_3 + \cdots \\ &\qquad = z_0 - (b_0 + z_1) + (b_1 + z_2) - (b_2 + z_3) + \cdots \\ &\qquad = (z_0 - b_0) - (z_1 - b_1) + (z_2 - b_2) - (z_3 - b_3) + \cdots \\ &\qquad = h_0 - h_1 + h_2 - h_3 + \cdots . \end{aligned}$$

b_{n-1} counts the "pairs of canceled cells" between c_{n-1} and c_n. □

Homology of spaces

Corollary

If X has a finite cell structure, then

$$\chi(X) = \sum_n (-1)^n \operatorname{Rank}(H_n(X)).$$

The homology groups (as opposed to their ranks) have many other variants and useful properties:

- Also make sense for infinite spaces
- Homology with more general coefficients
- Cohomology, with a ring structure
- Functoriality, Kunneth and Mayer-Vietoris theorems
- Spectral sequences
- ...

Outline

Where does magnitude come from?

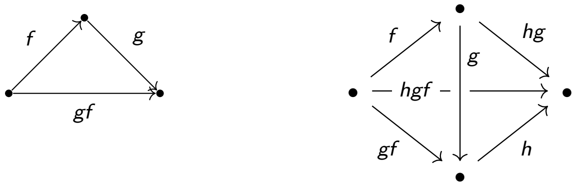
Slogan

Magnitude $=$ homology $+$ summing a geometric series

The nerve of a category

Definition

The nerve of a small category X is a cell complex NX with one n-cell for every composable string of n nonidentity* morphisms in X, with a boundary determined by all partial composites.



* In traditional simplicial-set language, there are also simplices containing identities; but they are degenerate and hence disappear under geometric realization.

Chains in the nerve

Let $\hom'_X(x, y)$ denote the set of nonidentity morphisms $x \to y$.

$$c_n = \Big|\big\{\text{composable strings of } n \text{ nonidentity morphisms}\big\}\Big|$$
$$= \sum_{x_0,\dots,x_n} \Big|\big\{\text{strings of nonidentity morphisms } x_0 \to x_1 \to \cdots \to x_n\big\}\Big|$$
$$= \sum_{x_0,\dots,x_n} \big|\hom'_X(x_0, x_1)\big| \cdot \big|\hom'_X(x_1, x_2)\big| \cdots \big|\hom'_X(x_{n-1}, x_n)\big|$$

Let W denote the square matrix with rows and columns labeled by the objects of X, whose x-y entry is $W_{xy} = \big|\hom'_X(x, y)\big|$. Then

$$(W^n)_{x_0,x_n} = \sum_{x_1,\dots,x_{n-1}} \big|\hom'_X(x_0, x_1)\big| \cdots \big|\hom'_X(x_{n-1}, x_n)\big|$$
$$c_n = \sum_{x_0,x_n} (W^n)_{x_0,x_n}.$$

Euler characteristic of the nerve

Assume that the c_n's are eventually zero, so all sums are finite.

$$\begin{aligned}
\chi(NX) &= c_0 - c_1 + c_2 - c_3 + \cdots \\
&= \sum_{x_0} (W^0)_{x_0,x_0} - \sum_{x_0,x_1} (W^1)_{x_0,x_1} + \sum_{x_0,x_2} (W^2)_{x_0,x_2} - \cdots \\
&= \sum_{x,y} \left(W^0 - W^1 + W^2 - W^3 + \cdots \right)_{x,y} \\
&= \sum_{x,y} \left(\frac{1}{1+W} \right)_{x,y}
\end{aligned}$$

- By $\frac{1}{1+W}$ we mean $(\mathbf{1} + W)^{-1}$, where $\mathbf{1}$ is the identity matrix.
- $Z = \mathbf{1} + W$ is the matrix with $Z_{xy} = |\hom_X(x,y)|$, the number of *possibly identity* morphisms $x \to y$.

Euler characteristic and magnitude

Definition

If X is a finite category such that the matrix $Z_{xy} = |\hom_X(x, y)|$ is invertible, then its magnitude is the sum of all the entries of Z^{-1}:

$$|X| = \sum_{xy} (Z^{-1})_{xy}$$

Theorem

If the c_n's are eventually zero, then

$$|X| = \chi(NX) = \sum_n (-1)^n \operatorname{Rank}(H_n(NX))$$

Outline

Metric spaces as enriched categories

For a monoidal category $(\mathcal{V}, \otimes)$, we have $\mathcal{V}$-enriched categories.

Example

For $(\mathcal{V}, \otimes) = (\mathsf{Set}, \times)$, a $\mathcal{V}$-enriched category is an ordinary category.

Example (Lawvere)

For $(\mathcal{V}, \otimes) = ([0, \infty), +)$, $\mathcal{V}$-enriched categories include metric spaces.

Magnitude of enriched categories

Let R be a ring (often a field) and $|\cdot| : \mathcal{V} \to R$ an (iso-invariant) homomorphism of multiplicative monoids, i.e. $|u \otimes v| = |u| \cdot |v|$. For a $\mathcal{V}$-enriched category X with finitely many objects, define a matrix Z by $Z_{xy} = |\hom_X(x, y)|$.

Definition

The magnitude of X (with respect to $|\cdot|$) is the sum of the entries of Z^{-1}, if it exists.

Example

$\mathcal{V} = \text{FinSet}$, $R = \mathbb{Q}$ or $\mathbb{R}$, $|\cdot| =$ cardinality: yields the magnitude ($\approx$ Euler characteristic) of a finite category.

Example

$\mathcal{V} = [0, \infty)$, $R = \mathbb{R}$, $|u| = e^{-u}$: defines the *magnitude of a finite metric space*.

Towards magnitude homology

Question

For a general $\mathcal{V}$ (e.g. $[0,\infty)$), can the magnitude be recovered from a magnitude homology theory?

Calculating purely formally, we can write

$$\sum_{x,y}(Z^{-1})_{xy} = \sum_{x,y}\left(W^0 - W^1 + W^2 - W^3 + \cdots\right)_{xy}$$

$$= \sum_{x_0}(W^0)_{x_0,x_0} - \sum_{x_0,x_1}(W^1)_{x_0,x_1} + \sum_{x_0,x_2}(W^2)_{x_0,x_2} - \cdots$$

$$= \sum_{n}(-1)^n \sum_{x_0,x_1,\ldots,x_n} W_{x_0,x_1} W_{x_1,x_2} \cdots W_{x_{n-1},x_n}.$$

where $W = Z - \mathbf{1}$.

Towards magnitude homology for metric spaces

For $\mathcal{V} = [0, \infty)$ we have $W_{x,y} = \begin{cases} e^{-d(x,y)} & \text{if } x \neq y \\ e^0 - 1 = 0 & \text{if } x = y \end{cases}$.

Thus, our formal calculation becomes

$$|X| = \sum_n (-1)^n \sum_{x_0 \neq x_1 \neq \cdots \neq x_n} e^{-(d(x_0,x_1)+d(x_1,x_2)+\cdots+d(x_{n-1},x_n))}$$

Problems

1. Does this formal calculation actually make sense?
   (E.g. does the infinite series converge?)
2. Is there a chain complex whose "ranks" are the (non-integers!)

$$\sum_{x_0 \neq x_1 \neq \cdots \neq x_n} e^{-(d(x_0,x_1)+d(x_1,x_2)+\cdots+d(x_{n-1},x_n))} \text{ ?}$$

Yes: Hepworth–Willerton (2015), Leinster–Shulman (2017).

Outline

Universal magnitudes

Any small $\mathcal{V}$ has a universal iso-invariant monoid map to a ring.

$$\begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathbb{Z}[\mathcal{V}] \\ & \searrow & \downarrow{\scriptstyle \phi_R} \\ & & R \end{array}$$

- The elements of $\mathbb{Z}[\mathcal{V}]$ are $\mathbb{Z}$-linear combinations of isomorphism classes $[u]$ with $u \in \mathcal{V}$, where $[u] \cdot [v] := [u \otimes v]$.
- If a $\mathcal{V}$-category X has magnitude over $\mathbb{Z}[\mathcal{V}]$, it is universal:

$$\phi_R(|X|_{\mathbb{Z}[\mathcal{V}]}) = |X|_R.$$

Generalized polynomials and rational functions

When $\mathcal{V} = [0, \infty)$, it is more intuitive to write $[u]$ as q^u, where q is a formal variable, since then instead of $[u][v] = [u + v]$ we have

$$q^u q^v = q^{u+v}.$$

Thus $\mathbb{Z}[[0, \infty)]$ is the ring $\mathbb{Z}[q^{[0,\infty)}]$ of generalized polynomials. This is like the polynomial ring $\mathbb{Z}[q]$ but with exponents in $[0, \infty)$, e.g.

$$1 - 2q^{0.5} + 11q^{\pi}.$$

To invert more matrices, we embed $\mathbb{Z}[q^{[0,\infty)}]$ in its field of fractions, which consists of generalized rational functions, e.g.

$$\frac{1 + 3q - 2q^{0.5} + 11q^{\pi}}{4 + q^{1/3} - 17q^{\sqrt{2}}}.$$

We can also include rational coefficients and negative exponents, so we write it as $\mathbb{Q}(q^{\mathbb{R}})$.

Universal magnitudes of metric spaces

Theorem

Every finite metric space X has magnitude over $\mathbb{Q}(q^{\mathbb{R}})$.

Proof.

$\mathbb{Q}(q^{\mathbb{R}})$ is an ordered field in which positive powers of q are infinitesimal. Since the entries of the matrix Z are $q^{d(x,y)}$, where $d(x,x) = 0$ and $d(x,y) > 0$ for $x \neq y$, they are $q^0 = 1$ along the diagonal and infinitesimal off it. Thus, the determinant of Z is 1 plus a bunch of infinitesimal terms, hence nonzero. □

In fact, Z is even positive definite over $\mathbb{Q}(q^{\mathbb{R}})$, by the Levy-Desplanques theorem.

The magnitude function

A generalized rational function can be evaluated at any nonnegative real number r by plugging it in for the variable q, except for a finite number of singularities where the denominator vanishes.

It thereby defines a partial real-analytic function $[0, \infty) \rightharpoonup \mathbb{R}$.

Evaluating the universal magnitude $|X|_{\mathbb{Q}(q^{\mathbb{R}})}$ at $r = e^{-t}$, for $t \in (0, \infty)$, yields the classical magnitude function $t \mapsto |tX|_{\mathbb{R}}$.

Generalized power series

Just as $\mathbb{Z}[q]$ and $\mathbb{Q}(q)$ embed respectively in the formal power series ring $\mathbb{Z}[\![q]\!]$ and the formal Laurent series ring $\mathbb{Q}(\!(q)\!)$, the field $\mathbb{Q}(q^{\mathbb{R}})$ embeds in the field of formal Hahn series $\mathbb{Q}(\!(q^{\mathbb{R}})\!)$:

$$\sum_{n<\alpha} a_n q^{b_n} \qquad \alpha \text{ an ordinal}$$

with $a_n \in \mathbb{Q}$ and $b_n \in \mathbb{R}$, such that $b_m < b_n$ whenever $m < n$.

Like $\mathbb{Z}[\![q]\!]$ and $\mathbb{Q}(\!(q)\!)$, the field $\mathbb{Q}(\!(q^{\mathbb{R}})\!)$ has a topology generated by a valuation $\nu(q^b) = b$ (though it differs in that not all Hahn series converge to themselves in this topology).

Finally!

Let $A = (A_r)_{r \in \mathbb{R}}$ be an $\mathbb{R}$-graded abelian group. If each A_r has finite rank and $\{r \mid A_r \neq 0\} \subseteq \mathbb{R}$ is well-ordered, then A has a Hahn rank

$$\mathrm{Rank}_{\mathbb{Q}((q^{\mathbb{R}}))}(A) = \sum_{A_r \neq 0} \mathrm{Rank}(A_r)\, q^r \quad \in \mathbb{Q}((q^{\mathbb{R}})).$$

Thus, we can hope for an $\mathbb{R}$-graded magnitude homology of metric spaces for which the alternating sum of the Hahn ranks converges in $\mathbb{Q}((q^{\mathbb{R}}))$ to the universal magnitude in $\mathbb{Q}(q^{\mathbb{R}}) \subseteq \mathbb{Q}((q^{\mathbb{R}}))$.

Outline

Towards magnitude chains

Recall our formal calculation over $\mathbb{R}$

$$|X|_{\mathbb{R}} = \sum_n (-1)^n \sum_{x_0 \neq x_1 \neq \cdots \neq x_n} e^{-(d(x_0,x_1)+d(x_1,x_2)+\cdots+d(x_{n-1},x_n))}$$

which over $\mathbb{Q}(\!(q^{\mathbb{R}})\!)$ becomes

$$\begin{aligned}|X|_{\mathbb{Q}((q^{\mathbb{R}}))} &= \sum_n (-1)^n \sum_{x_0 \neq x_1 \neq \cdots \neq x_n} q^{d(x_0,x_1)+d(x_1,x_2)+\cdots+d(x_{n-1},x_n)} \\ &= \sum_n (-1)^n \sum_{r \in [0,\infty)} c_{n,r}\, q^r\end{aligned}$$

where

$$c_{n,r} = \left|\left\{x_0 \neq x_1 \neq \cdots \neq x_n \;\middle|\; d(x_0,x_1)+\cdots+d(x_{n-1},x_n) = r\right\}\right|.$$

And the cardinality of a finite set is the rank of the free abelian group on that set!

Magnitude chains

Let X be a metric space, $n \in \mathbb{N}$, and $r \in [0, \infty)$.

Definition

Let $C_{n,r}(X)$ be the free abelian group on the set of $(n+1)$-tuples $x_0 \neq x_1 \neq x_2 \neq \cdots \neq x_n$ such that

$$d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n) = r.$$

Theorem

For a finite metric space X, we have

$$|X|_{\mathbb{Q}((q^{\mathbb{R}}))} = \sum_n (-1)^n \operatorname{Rank}_{\mathbb{Q}((q^{\mathbb{R}}))}(C_{n,*}(X))$$

the sum over n converging in the topology of $\mathbb{Q}((q^{\mathbb{R}}))$.

The point is that X has a nonzero minimum distance between distinct points, so the valuations of the ranks go to ∞ as n does.

Magnitude homology determines magnitude

Fact

There are maps $d : C_{n+1,r}(X) \to C_{n,r}(X)$ making $C_{*,*}(X)$ an $\mathbb{R}$-graded chain complex, with magnitude homology groups $H_{n,q}(X)$.

Corollary

For a finite metric space X, we have

$$|X|_{\mathbb{Q}((q^{\mathbb{R}}))} = \sum_n (-1)^n \operatorname{Rank}_{\mathbb{Q}((q^{\mathbb{R}}))}(H_{n,*}(X)).$$

the sum over n converging in the topology of $\mathbb{Q}((q^{\mathbb{R}}))$.

In particular, since the magnitude function is determined by evaluating $|X|_{\mathbb{Q}(q^{\mathbb{R}})}$ at numbers $q = e^{-t}$, the magnitude function is determined by the magnitude homology.

The integral case

Suppose all distances in X are integers (such as if it is a graph with the shortest-path metric). Then:

- We can consider it as enriched over $\mathbb{N}$ instead of $[0, \infty)$.
- Its universal magnitude lies in the field of ordinary rational functions $\mathbb{Q}(q)$.
- Instead of Hahn series we can use Laurent series $\mathbb{Q}(\!(q)\!)$.
- Instead of $\mathbb{R}$-gradings we can use $\mathbb{Z}$-gradings.

Magnitude homology was first worked out in the case of graphs by Hepworth and Willerton, making use of these simplifications.

But what are the differentials?

Everything I've said so far would be true even if the differentials $d : C_{n+1,r}(X) \to C_{n,r}(X)$ were zero! But that would make $H_{n,r}(X) = C_{n,r}(X)$, which is too trivial to be interesting.

Fact which I wish I had time to explain

The action of $d : C_{n+1,r}(X) \to C_{n,r}(X)$ on generators is almost*

$$¿ \quad d([x_0, \ldots, x_{n+1}]) = \sum_{0 \le k \le n+1} (-1)^k \cdot [x_0, \ldots, \widehat{x_k}, \ldots, x_{n+1}] \quad ?$$

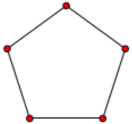
The problem is that $[x_0, \ldots, \widehat{x_k}, \ldots, x_{n+1}]$ may not be a generator of $C_{n,r}(X)$, since its total distance may be $< r$.
In that case we simply omit the k^{th} term from the sum.

(This is not an *ad hoc* definition, but falls naturally out of general homotopical enriched category theory. See arXiv:1711.00802.)

* The notation $\widehat{x_k}$ means to omit that point from the list.

Outline

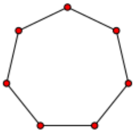
Magnitude homology of some graphs (Hepworth–Willerton)



l \ k	0	1	2	3	4	5	6	7	8	9	10	11
0	5											
1		10										
2			10									
3			10	10								
4				30	10							
5					50	10						
6					20	70	10					
7						80	90	10				
8							180	110	10			
9							40	320	130	10		
10								200	500	150	10	
11									560	720	170	10

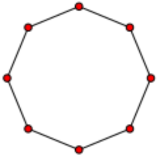
TABLE 1. The ranks of $\mathrm{MH}_{k,l}(C_5)$, the magnitude homology groups of the pictured five-cycle graph, as computed using Sage.

Magnitude homology of some graphs (Hepworth–Willerton)



	0	1	2	3	4	5	6	7	8	9	10	11
0	7											
1		14										
2			14									
3				14								
4			14		14							
5				42		14						
6					70		14					
7						98		14				
8					28		126		14			
9						112		154		14		
10							252		182		14	
11								448		210		14

Magnitude homology of some graphs (Hepworth–Willerton)



	0	1	2	3	4	5	6	7	8	9	10
0	8										
1		16									
2			16								
3				16							
4			8		16						
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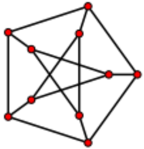
6					16		16				
7						16		16			
8					8		16		16		
9						16		16		16	
10							16		16		16

Magnitude homology of some graphs (Hepworth–Willerton)



	0	1	2	3	4	5	6	7	8
0	10								
1		30							
2			30						
3			120	30					
4				480	30				
5					840	30			
6					1440	1200	30		
7						7200	1560	30	
8							17280	1920	30

Magnitude homology of trees

Theorem (Hepworth–Willerton)

If X is a tree, then:

- $H_{0,0}(X)$ *is free on the vertices of* X*.*
- $H_{n,n}(X)$ *is free on the directed edges of* X *for all* $n > 0$*.*
- $H_{n,r}(X) = 0$ *if* $n \neq r$*.*

Magnitude homology distinguishes more graphs than magnitude (Gu)

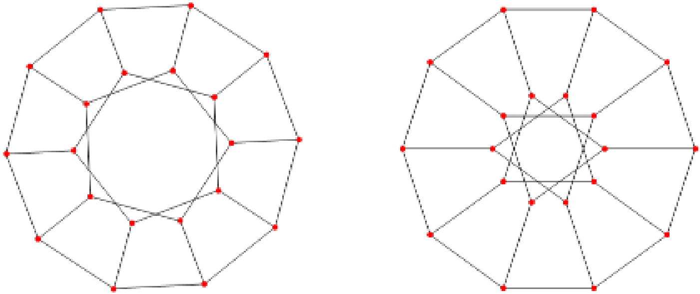


FIGURE 4. Left: dodecahedral graph; Right: Desargues graph

Magnitude homology detects convexity/geodesy

Theorem

$H_{1,*}(X) = 0$ iff X is Menger convex: for any $x \neq z$ there is a y with $x \neq y \neq z$ and $d(x, y) + d(y, z) = d(x, z)$.

If closed and bounded subsets of X are compact, it is Menger convex iff it is geodesic (any two points are connected by a geodesic).

If X is not Menger convex, then $H_{1,r}(X)$ is freely generated by the ordered pairs of points $x \neq z$ with $d(x, z) = r$ that do not have such a y strictly between them.

Magnitude homology detects uniqueness of geodesics

Theorem (Kaneta–Yoshinaga (2018), Jubin (2018), Gomi (2019))

If X is convex in $\mathbb{R}^n$ or uniquely geodesic, $H_{n,*}(X) = 0$ for all $n > 0$.

Theorem (Gomi (2019))

If X is geodesic but not uniquely so, $H_{2,r}(X)$ "counts" the nonuniqueness of length-r geodesics joining pairs of points.

Example

When S^1 has the geodesic metric:

- $H_{2,\pi}(S^1) = \mathbb{Z}[S^1]$ and $H_{2,r}(S^1) = 0$ for all other r.
- $H_{3,r}(S^1) = 0$. (Gomi 2018)

However, $H_{3,*}(X)$ does not always vanish even for geodesic X.

A curious bifurcation

Observation #1

If X has no strictly-collinear triples (such that $x \neq y \neq z$ and $d(x, y) + d(y, z) = d(x, z)$), then all differentials are zero. Thus $H_{*,*}(X) = C_{*,*}(X)$, and magnitude homology is boring.

Observation #2

We only know how to recover magnitude from magnitude homology for finite metric spaces.

- (Finite) graphs have lots of strictly-collinear triples.
- So do geodesic spaces; but they must be infinite.
- A *finite* subset of $\mathbb{R}^n$ almost never has strictly-collinear triples!

Compact metric spaces

The magnitude of finite metric spaces can be generalized to various kinds of compact metric spaces using analysis (e.g. replacing sums by integrals, or taking suprema over finite subsets).

However, the magnitude homology does not determine this analytic magnitude. E.g. $H_{*,*}(B^n)$ knows only the cardinality of B^n, while $|B^n|$ knows its volume, etc.

Question

Is there an "analytic magnitude homology" that does determine the analytic magnitude?