The Maximum Diversity of a Compact Metric Space

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Objectives

Part I: Measuring diversity

- Introduce a family of distance-sensitive entropies $(D_q)_{q \in [0,\infty]}$.

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- Define a real-valued invariant, closely related to magnitude.

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Part II: Maximum diversity and magnitude

- Define a real-valued invariant, closely related to magnitude.

Part III: Uniform measures

- Propose a uniform measure for a wide class of metric spaces.

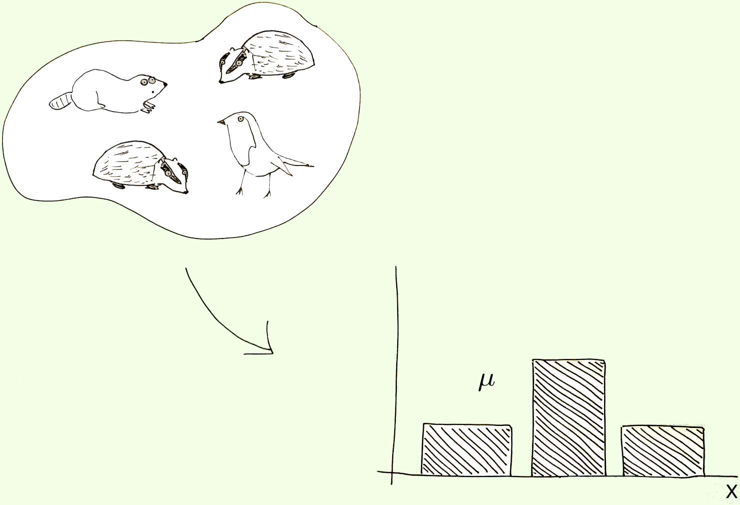
Provenance

- Leinster and Cobbold, *Measuring diversity: the importance of species similarity*. Ecology 93 (2012).
- Leinster and Meckes, *Maximizing diversity in biology and beyond*. Entropy 18(3) (2016).
- Meckes, *Magnitude, diversity, capacities, and dimensions of metric spaces*. Potential Analysis 42(2) (2015).
- Leinster and Roff, *The maximum diversity of a compact space*. (To appear.)

Part I

Measuring Diversity

Modelling a community



Conventionally, an ecological community is modelled by a set of species and a probability distribution recording the relative abundance of each species.

Quantifying diversity

Gini-Simpson Index

Species richness

BIOLOGY

Berger-Parker Dominance

DIVERSITY INDICES: SHANNON'S *H* AND *E*

Given a vector of frequencies (counts), f_i the Shannon diversity index is computed as

$$H = \frac{n \log(n) - \sum_{i=1}^{k} f_i \log(f_i)}{n}$$

Simpson's Diversity Index

A community dominated by one or

A single family of entropies

Definition

Let $X = \{1, \ldots, n\}$, let $\mu = (\mu_1, \ldots, \mu_n)$ be a probability distribution on X, and let $q \in [0, \infty]$. The **Rényi entropy of order** q of μ is

$$H_q(\mu) = \frac{1}{1-q} \log \sum_{i \in \mathrm{supp}(\mu)} \mu_i^q$$

for $q \neq 1, \infty$, and

$$H_q(\mu) = \begin{cases} -\sum_{i \in \mathrm{supp}(\mu)} \mu_i \log \mu_i & \text{when } q = 1, \\ -\log \max_{i \in \mathrm{supp}(\mu)} \mu_i & \text{when } q = \infty. \end{cases}$$

In particular, $H_1(\mu)$ is **Shannon entropy**.

Incorporating similarities between species

Now, let $X = \{1, \ldots n\}$ be a set of species, with pairwise similarities recorded in an $n \times n$ matrix K. Let μ be a probability distribution on X.

Definition

For $q \in [0, \infty)$ not equal to 1, the **order-q diversity** of μ is

$$D_q^K(\mu) = \left(\sum_i (K\mu)_i{}^{q-1} \mu_i \right)^{\frac{1}{1-q}}.$$

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When K is the identity matrix, this reduces to Rényi entropy:

$$D_q^I(\mu) = \exp(H_q(\mu)).$$

Moral:

log(diversity) = generalised entropy.

Interpreting diversity

For $q \in [0, \infty)$ not equal to 1, the **order-q diversity** of μ is

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So $D_q^K(\mu)$ is the average atypicality of species in X with respect to μ, where 'average' refers to the power mean of order $1 - q$.

Part II

Maximising Diversity on Compact Spaces

Compact spaces with similarities

Definition

Let X be a compact Hausdorff topological space.

A **similarity kernel** on X is a continuous function $K : X \times X \to [0, \infty)$ satisfying $K(x, x) > 0$ for all $x \in X$.

The pair (X, K) is a **space with similarities**.

We say (X, K) is **symmetric** if $K(x, y) = K(y, x)$ for all $x, y \in X$.

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Examples

- A set of species with a similarity matrix.
- A compact metric space X with kernel $K(x, y) = e^{-d(x,y)}$.

Typicality for compact spaces

Definition

Let (X, K) be a space with similarities, and let $P(X)$ be the space of Radon probability measures on X, with the weak* topology.

For each $\mu \in P(X)$ and $x \in X$, define

$$(K\mu)(x) = \int K(x, -)\,\mathrm{d}\mu \in [0, \infty).$$

The function $K\mu : X \to [0, \infty)$ is the **typicality function** of (X, K, μ).

The **atypicality function** of (X, K, μ) is $1/K\mu$.

Diversity for compact spaces

Definition

Let (X, K) be a space with similarities, $\mu \in P(X)$, and $q \in [0, \infty)$ not equal to 1. The **diversity of order** q of μ is

$$D_q^K(\mu) = \left(\int (K\mu)^{q-1} \, d\mu \right)^{1/(1-q)}.$$

The **diversity function of order** q of (X, K) is $D_q^K : P(X) \to (0, \infty)$.

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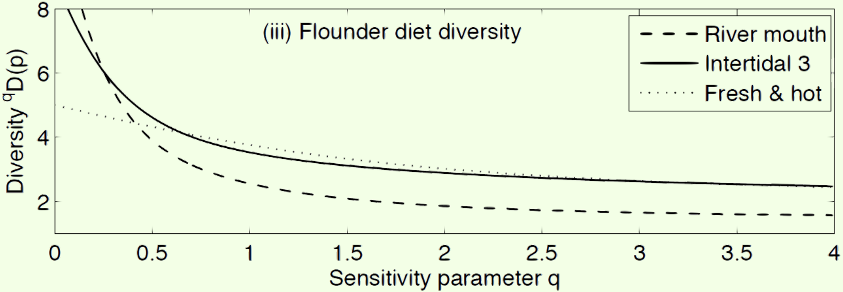
Example

Let X be a compact metric space, and $\mu \in P(X)$. Then

$$D_q(\mu) = \left(\int \left(\int e^{-d(x,y)} \, \mathrm{d}\mu(x) \right)^{q-1} \mathrm{d}\mu(y) \right)^{1/(1-q)}.$$

The viewpoint parameter matters!



Leinster and Cobbold, *Measuring Diversity. . .*, Ecology 93 (2012)

A maximisation theorem

Theorem (Leinster and Roff, 2019)

Let (X, K) be a nonempty symmetric space with similarities.

1. *There exists a probability measure μ on X that maximises $D_q^K(\mu)$ for all $q \in [0, \infty]$ at once.*

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2. *The maximum diversity $\sup_\mu D_q^K(\mu)$ is independent of $q \in [0, \infty]$.*

Definition

The **maximum diversity** of (X, K) is

$$D_{\max}(X) = \sup_{\mu} D_q^K(\mu)$$

for any $q \in [0, \infty]$.

Maximum diversity of metric spaces

Example

The maximum diversity of a compact metric space X is

$$D_{\max}(X) = \sup_{\mu \in P(X)} D_2(\mu) = \sup_{\mu \in P(X)} \frac{1}{\int \int e^{-d(x,y)} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y)}.$$

This quantity is investigated in Meckes (2015).

Part III

Maximum Diversity and Magnitude

Relating diversity to magnitude

Let (X, K) be a space with similarities.

Definition

A measure $\mu \in P(X)$ is **balanced** if $K\mu$ is constant on supp μ.

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A **weight measure** on (X, K) is a signed measure μ such that $K\mu \equiv 1$.

For example, a weight measure on a compact metric space satisfies

$$\int e^{-d(x,y)} \, \mathrm{d}\mu(x) = 1 \text{ for all } y \in X.$$

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Corollary

If μ is a maximising measure, its restriction to $\operatorname{supp} \mu$ *is a scalar multiple of a weight measure.*

Relating diversity to magnitude

Definition

Let (X, K) be a symmetric space with similarities. Suppose there exists a weight measure on (X, K). The **magnitude** of (X, K) is

$$|X| = \mu(X)$$

for any weight measure μ on (X, K).

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Lemma

Let μ be a maximising measure on (X, K). Then

$$D_{\max}(X) = |\text{supp } \mu|.$$

Positive definite spaces

Let X be a compact, positive definite metric space. Its magnitude is

$$|X| = \sup\{|Y| \,:\, \text{finite } Y \subseteq X\}.$$

Proposition (Meckes, 2011)

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Consequences

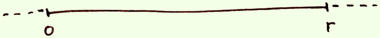
- For all $Y \subseteq X$, we have $|Y| \leq |X|$. So when $X \neq \emptyset$,

$$D_{\max}(X) \leq |X|.$$

- If X admits a nonnegative weight measure μ, its normalisation $\widehat{\mu} \in P(X)$ is the unique maximising measure on X, and

$$D_{\max}(X) = |X|.$$

Example



A line segment $[0, r] \subset \mathbb{R}$ has weight measure $\frac{1}{2}(\delta_0 + \lambda_{[0,r]} + \delta_r)$.

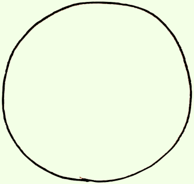
Hence,

$$D_{\max}([0, r]) = |[0, r]| = 1 + \frac{r}{2}$$

and the unique maximising measure on $[0, r]$ is

$$\frac{\delta_0 + \lambda_{[0,r]} + \delta_r}{2 + r}.$$

Example



Suppose X is homogeneous. Then the Haar probability measure μ is the unique maximising measure on X and, for any $y \in X$,

$$D_{\max}(X) = |X| = \frac{1}{\int e^{-d(x,y)}\, d\mu(x)}.$$

Part IV

Uniform Measures

Defining a uniform measure

Let $X = (X, d)$ be a metric space. Write tX for the space (X, td).

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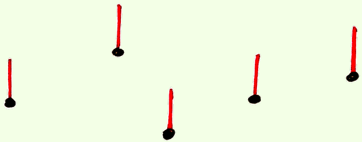
Definition

Let X be a compact metric space. Suppose that tX has a unique maximising measure μ_t for all $t \gg 0$, and that $\lim_{t\to\infty} \mu_t$ exists in $P(X)$. Then the **uniform measure** on X is

$$\mu_X = \lim_{t\to\infty} \mu_t.$$

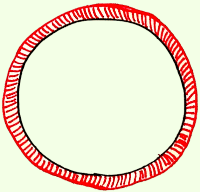
Example



Proposition

On a finite metric space, the uniform measure is the uniform measure.

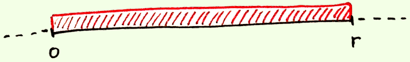
Example



Proposition

On a homogeneous space, the uniform measure is the Haar measure.

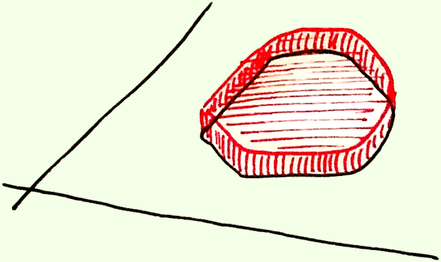
Example



Proposition

On a compact subinterval of $\mathbb{R}$, the uniform measure is Lebesgue measure restricted and normalised.

Example



Proposition

Let $X \subset \mathbb{R}^n$ be a compact subset with nonzero volume. Let $\widehat{\lambda_X}$ be Lebesgue measure restricted to X and normalised. For all $q \in [0, \infty]$,

$$\frac{D_q^{e^{-td}}(\widehat{\lambda_X})}{D_{\max}(tX)} \to 1 \text{ as } t \to \infty.$$

Moral: $\widehat{\lambda_X}$ is approximately maximising at large scales.

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