# Recovery of piecewise linear closed curves

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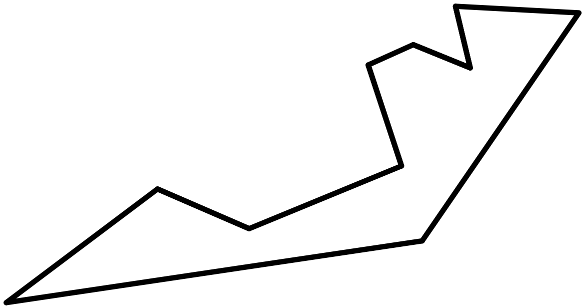
**Theorem (Hepworth)**

*Let $X$ be an extended quasimetric space, for which $\inf\{d(a,b) | a \neq b\}$ is positive. Then $X$ is determined up to isometry by the magnitude cohomology ring $\mathrm{MH}^*_*(X)$.*

Can this class of discrete metric space be enlarged?

# Piecewise linear closed curves

- The topic of this talk will be piecewise linear closed curves
- The metric of $X$ will be the metric inherited from $\mathbb{R}^2$



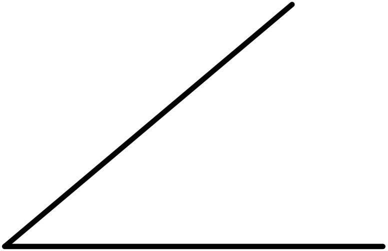
# Piecewise linear closed curves

**Theorem**

*Let $X$ be a piecewise linear closed curve with finitely many self intersections. Then $X$ is determined up to isometry by its magnitude cohomology ring $\mathrm{MH}^*_*(X)$.*

# Piecewise linear closed curves

- The space $X$ consists of finitely many supspaces $X = \bigcup X_i$
- It would be useful to be able to measure angles in $\mathbb{R}^2$.



# The magnitude cohomology ring of curves in $\mathbb{R}^n$

**Definition (Magnitude cohomology)**

The magnitude cohomology groups of a Euclidean space $X$ are given by

$$\mathrm{MH}_l^k(X) = \mathrm{Hom}(\mathrm{MH}_{l,k}(X), \mathbb{Z}).$$

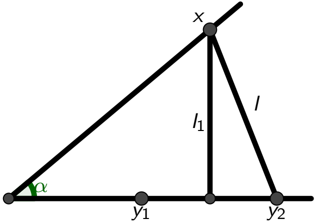
For $\phi \in \mathrm{MH}_{l_1}^{k_1}(X), \psi \in \mathrm{MH}_{l_2}^{k_2}(X)$ let

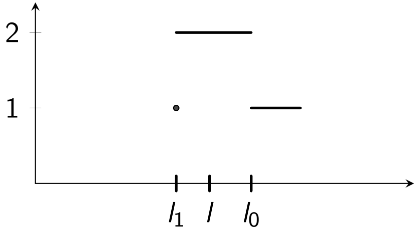
$$\phi\psi(x_0, \dots, x_{k_1+k_2}) := \phi(x_0, \dots, x_{n_1})\psi(x_{n_1}, \dots, x_{n_1+n_2}).$$

# The magnitude cohomology ring of curves in $\mathbb{R}^n$

- The set $X$ can be found in the magnitude cohomology ring:
- $\mathrm{MH}_{0,0}(X) = \langle (x) : x \in X \rangle$.
- For each magnitude homology class corresponding to a point in $X$, there exists a dual magnitude cohomology class.
- Therefore in $\mathrm{MH}^0_0(X)$ there is a set $M$ corresponding to the set $X$.
- Each geometric property on points of $X$ may imply an algebraic result on elements in $M$.

# Example of how to measure angles (The case of acute angles)



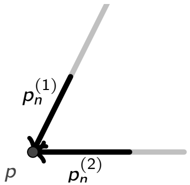




- For $e \in M$, let $N_e(l) = \#\{f \in M : e\,\mathrm{MH}^1_l(X) f \neq 0\}$
- In the above picture, with $e = \delta_x$ : $N_e(l) = 2$.
- For any $e \in M$, the function $N_e : \mathbb{R}_+ \to \mathbb{N}$ has two non-continuities $l_1$ and $l_0$.
- If $x$ is close to the intersection point, then

$$\sin(\alpha) = \frac{l_1}{l_0}$$

# Magnitude distance

- But how can we partition $M$ into $M_i$, corresponding to $X_i$?
- An important step is to identify the corner points.
- A corner point can be identified by considering convergence (There are two types of sequences converging to a corner point, using different directions)
- Therefore we need some kind of "distance measure" on $M$ to detect convergence.



# Magnitude distance

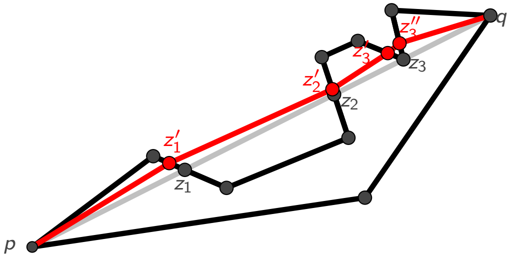
**Definition**

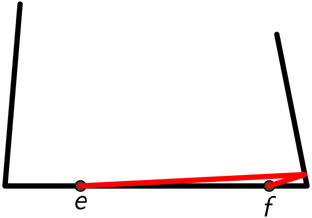
Let $e, f \in M$. Then the magnitude distance between $e$ and $f$ is

$$d_M(e, f) = \inf\{l : \exists n : e\,\mathrm{MH}_l^n(X) f \neq 0\}.$$

# Magnitude distance

- Often $d_M(\delta_p, \delta_q) = d(p, q)$
- But not always ...



# Intervals in $M$

**Definition**

Let $p, q \in X$ and $e = \delta_p, f = \delta_q$. The interval between $p$ and $q$ is given by

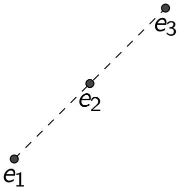
$$I_X(p,q) = \{x \in X : p \prec x \prec q\}$$

and the interval between $e$ and $f$ is given by

$$I_M(e,f) = \{\delta_x : x \in I_X(p,q)\}.$$

Observation: Given $e, f \in M$ one can not immediately determine the set $I_M(e,f)$.

# Intervals in $M$




- $e_1 \Phi_1 e_2 \neq 0$ for $\Phi_1 \in \mathrm{MH}^1_{d(e_1,e_2)}(X)$.
- $e_2 \Phi_2 e_3 \neq 0$ for $\Phi_2 \in \mathrm{MH}^1_{d(e_2,e_3)}(X)$.
- $\Phi_1 \Phi_2 = 0$, iff. $e_1 \prec e_2 \prec e_3$.

## Theorem

*Let $e, f \in M$. Then $I_M(e,f)$ is a finite set if and only if there exists a finite sequence $\{e_n\}_{n=0}^{k}$, such that $e_0 = e$, $e_k = f$ and there exist magnitude cohomology classes $\Phi_i \in \mathrm{MH}^1_{l_i}(X)$, such that for every $i$*

$$e_i \Phi_i e_{i+1} \neq 0$$

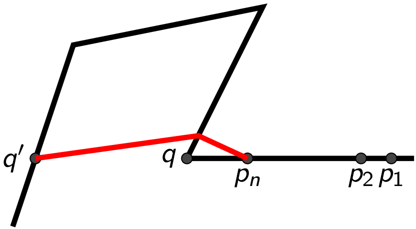
*and $\Phi_i \Phi_{i+1} = 0$.*

# Magnitude convergence for piecewise linear closed curves

**Theorem**

*Let $X$ be a piecewise linear closed curve and let $e_n = \delta_{p_n}$ and $e = \delta_q$. Then $p_n \to q$, if and only if*

$$\forall f \in M : |I_M(e,f)| < \infty : d_M(e_n, f) \to d_M(e,f) = d(e,f).$$

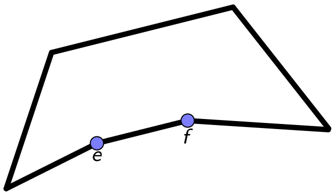



**Definition (Magnitude convergence)**

In those cases, we write $e_n \xrightarrow{M} e$

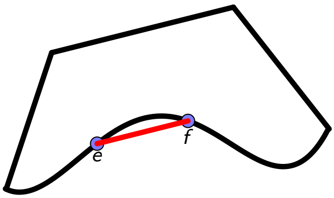
# Where can magnitude distance be used?

- What happens for arbitrary closed curves?



# Where can magnitude distance be used?

- What happens for arbitrary closed curves?
- Magnitude distance gets more precise?
- But also much more unpredictable . . .



# The end

This talk was based on my master thesis.
If you are interested in reading that thesis, feel free to contact me

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