The magnitude of an infinite metric space

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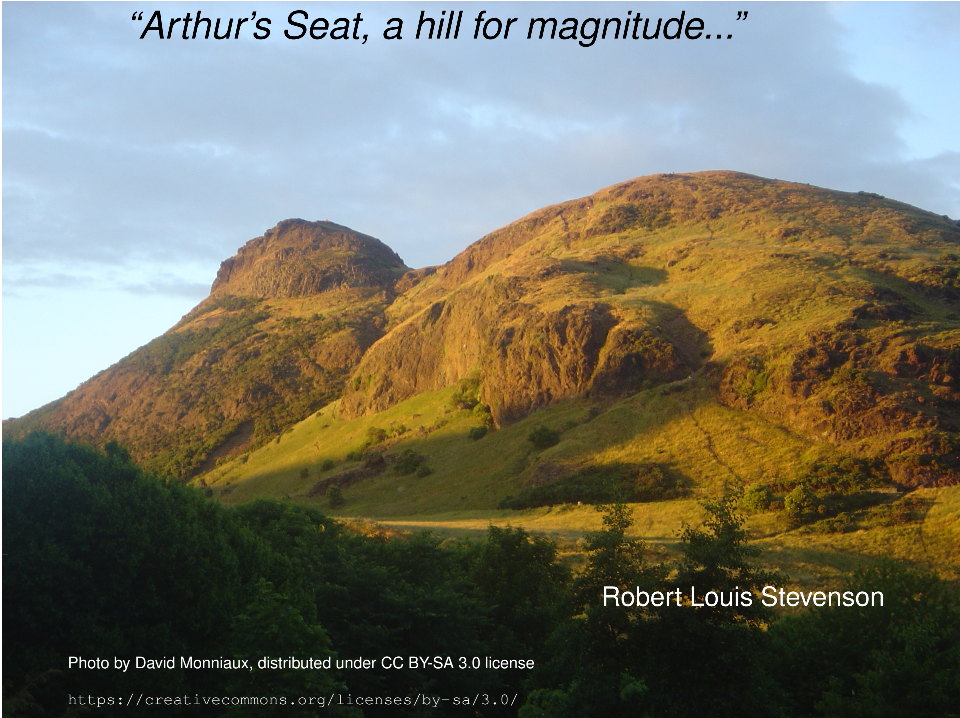
“Arthur’s Seat, a hill for magnitude...”
Robert Louis Stevenson
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The magnitude of a finite metric space

For a finite metric space (A, d), let $Z_A(a, b) = e^{-d(a,b)}$.
A **weighting** for A is a $w \in \mathbb{R}^A$ such that $\forall a \in A$,

$$\sum_{b \in A} Z_A(a, b) w(b) = 1.$$

If A possesses a weighting, then the **magnitude** of A is

$$|A| := \sum_{b \in A} w(b).$$

For $t > 0$, we write tA for the metric space (A, td), and let $0A := \{*\}$. The **magnitude function** of A is the (partially defined) function

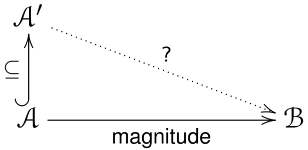
$$t \mapsto |tA|$$

for $t \geq 0$.

How to extend these definitions to infinite spaces?

Finite metric spaces are already a big and important topic!
But most interesting metric spaces aren't finite.

We'd like to complete a commutative triangle



where

- $\mathcal{A} = \{$(nice) finite metric spaces$\}$,
- $\mathcal{A}'$ is some larger class of metric spaces, and
- $\mathcal{B} = \mathbb{R}$, or
  $\mathcal{B} = \{$(nice, partially defined) functions $[0, \infty) \to \mathbb{R}\}$.

General strategies to extend a definition

- **Generalize:** State the definition in such a way that it automatically applies to $A \in \mathcal{A}'$.
  Major challenges: It may be unclear how canonical such a generalization is.
  The generalization itself tends to limit how large $\mathcal{A}'$ is.

- **Approximate** spaces in $\mathcal{A}'$ by spaces in $\mathcal{A}$ and take a limit.
  Major challenges: We need some kind of continuity properties to be sure this gives a well-defined extension.
  This is often ill-suited to explicit computations.

- **Reformulate:** Find an equivalent definition that makes it easier to apply either of the strategies above.
  Major challenge: This also tends to limit $\mathcal{A}'$, then faces the same challenges as the strategies above.

An easy generalization of the original definition

Let (A, d) be any metric space. A **weight measure** for A is a signed measure w on A such that $\forall a \in A$,

$$\int_A e^{-d(a,b)}\, dw(b) = 1.$$

If A possesses a weight measure, then the **magnitude** of A is

$$|A| := w(A).$$

Examples (Willerton)

$[0, t]$ has weight measure $w = \frac{1}{2}(\delta_0 + \lambda_{[0,t]} + \delta_t)$, and so

$$|[0, t]| = 1 + \frac{1}{2}t.$$

If A is a compact, homogeneous n-dimensional Riemannian manifold, then

$$w = \frac{\mathrm{vol}}{\int_A e^{-d(a,b)}\, d\,\mathrm{vol}(a)}$$

is independent of $b \in A$ and is a weight measure, and

$$|tA| = \frac{1}{n!\,\mathrm{vol}(B^n)} \left(\mathrm{vol}(A)t^n + \frac{n+1}{6}\,\mathrm{tsc}(A)t^{n-2} + O\left(t^{n-4}\right) \right)$$

as $t \to \infty$.

How widely does this generalization apply?

The following types of spaces possess weight measures:

- Compact subsets of $\mathbb{R}$.
- Compact ultrametric spaces
  $(d(a,b) \leq \max\{d(a,c), d(c,b)\})$.
- Compact homogeneous spaces.

In fact, all of the above possess positive weight measures.

But: many spaces don't possess weight measures.
Compact convex sets of dimension ≥ 2 probably don't.

Reformulation: positive definiteness and negative type

A finite metric space (A, d) is **positive definite** if the matrix Z_A is positive definite.

A general metric space is **positive definite** if each finite subset is positive definite.

A is of **negative type** if tA is positive definite for each $t > 0$.

Suppose A is a finite metric space.
If A is of negative type then the magnitude function of A is defined everywhere on $[0, \infty)$.
The converse is at least almost true.

Reformulation: positive definiteness and negative type

The following all have negative type (and their subspaces, too):

- $\mathbb{R}^n$ with the Euclidean metric.
- L_p for $1 \leq p \leq 2$.
- S^{n-1}.
- Hyperbolic space.

The following don't:

- $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$ if $n, p > 2$.
- Non-simply connected compact Riemannian manifolds of dimension ≥ 2.

What's that good for?

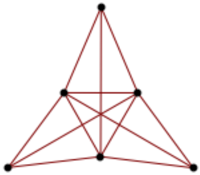
Theorem

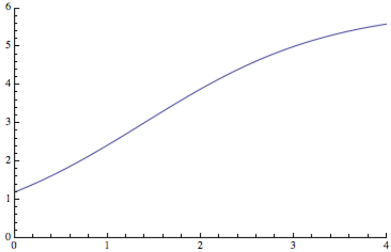
Magnitude is lower semicontinuous on

$$\mathcal{A} = \{\textit{finite positive definite metric spaces}\}$$

equipped with the Gromov–Hausdorff topology.

It's not continuous:





Approximation

Corollary

There is a canonical (maximal) lower semicontinuous extension of magnitude from

$$\mathcal{A} = \{\textit{finite positive definite metric spaces}\}$$

to

$$\mathcal{A}' = \{\textit{compact positive definite metric spaces}\}.$$

For $A \in \mathcal{A}'$, this extension is given by

$$|A| = \sup\{|B| \mid B \subseteq A \textit{ finite}\} \in [1, \infty].$$

Moreover, if $\{A_n\}$ is any sequence of compact subsets of $A \in \mathcal{A}'$ with $A_n \to A$, then

$$|A| = \lim_{n\to\infty} |A_n| .$$

Agreement

Proposition

If A is compact, positive definite, and possesses a weight measure, then the two proposed definitions of $|A|$ coincide.

Encouraged by this (and the lack of competing suggestions) we have adopted both of the above definitions of magnitude on their respective domains.

For metric spaces which are not positive definite and don't possess a weight measure, the situation is less clear...

The most basic open question

If A is compact and positive definite, then is $|A| < \infty$?

We know various additional sufficient conditions, but no counterexamples to the general question.

Aside: How necessary is compactness?

Compactness shows up naturally here:

- Compact spaces are approximated by finite spaces in the Gromov–Hausdorff topology.
- If A is positive definite and totally bounded with completion $\overline{A}$, then

$$\sup\{|B| \mid B \subseteq A \text{ finite}\} = |\overline{A}|.$$

- If A is positive definite and unbounded, then

$$\sup\{|B| \mid B \subseteq A \text{ finite}\} = \infty.$$

On the other hand:
Say $A = \{a_n \mid n \in \mathbb{N}\}$ with $d(a_i, a_j) = 1$ for $i \neq j$.
A is not compact, but

$$\sup\{|B| \mid B \subseteq tA \text{ finite}\} = e^t < \infty.$$

Magnitude functions

So far I've talked only about extending the definition of magnitude as opposed to magnitude functions.

Maybe extending magnitude functions would actually be easier in some contexts — if there is regularity to be exploited.

But: For a general compact space A of negative type, we know basically nothing: e.g.,

Is the magnitude function of A increasing? continuous on $(0, \infty)$?

Reformulating the reformulation

If A is positive definite, we can define two Hilbert spaces:

- $\mathcal{H}$: a space of functions $A \to \mathbb{C}$, which contains $f_a(b) = e^{-d(a,b)}$ for each $a \in A$.
  $\langle f_a, f_b \rangle_{\mathcal{H}} := e^{-d(a,b)}$.
- $\mathcal{W}$: the dual of $\mathcal{H}$.
  Signed measures on A can be identified with elements of $\mathcal{W}$.

A **weighting** for A is a $w \in \mathcal{W}$ such that $\forall a \in A$,

$$w(f_a) = 1.$$

Reformulating the reformulation

Theorem

Suppose A is compact and positive definite. Then A possesses a weighting w if and only if $|A| < \infty$. In that case $|A| = \|w\|^2_{\mathcal{W}}$.

This approach is well-suited to considering subspaces, and can also be dualized:

Theorem

Suppose that A is positive definite and $B \subseteq A$ is compact. Then $|B| < \infty$ if and only if there exists a function $h \in \mathcal{H}$ such that $h(b) = 1 \ \forall b \in B$. In that case

$$|B| = \inf\{\|h\|^2_{\mathcal{H}} \mid h \in \mathcal{H} \text{ and } h(b) = 1 \ \forall b \in B\}.$$

The unique h that achieves this infimum is called the potential function for B.

Normed spaces

If $\|\cdot\|$ is a norm on $\mathbb{R}^n$, the original weighting condition for finite $A \subseteq \mathbb{R}^n$

$$1 = \sum_{b \in A} e^{-\|a-b\|} w(b)$$

involves a convolution.

We should be using Fourier transforms!

This observation fits best with the last reformulation ($\mathcal{H}$ and $\mathcal{W}$).

Recall that $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$ is positive definite for $1 \leq p \leq 2$.
We will focus on $\mathbb{R}^n = \ell_2^n$ (mostly).

Magnitude in Euclidean space

For $\mathbb{R}^n$, $\mathcal{H}$ and $\mathcal{W}$ are classical Sobolev-type spaces.
$\mathcal{H} = H^{(n+1)/2}$ is a space of weakly smooth functions.
$\mathcal{W} = H^{-(n+1)/2}$ is a space of distributions.

$$\begin{aligned}\|f\|_{\mathcal{H}}^2 &= c_n \int_{\mathbb{R}^n} (1 + c_n' \|x\|^2)^{(n+1)/2} \left|\widehat{f}(x)\right|^2 \, dx \\ &= \int_{\mathbb{R}^n} f(x)(I - \Delta)^{(n+1)/2} f(x) \, dx\end{aligned}$$

if n is odd and f is smooth. So we can use PDEs tools!

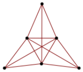
Magnitude in Euclidean space: basic properties (Leinster, Barceló–Carbery, M.)

Suppose $A \subseteq \mathbb{R}^n$ is compact. Then:

- $|A| < \infty$, since there exist smooth compactly supported functions equal to 1 on A.
- $\lim_{t\to\infty} t^{-n} |tA| = c_n \operatorname{vol}_n(A)$.
- Magnitude is continuous on $\{$convex bodies in $\mathbb{R}^n\}$.

The above are all true in any f.d. p.d. normed space.

- The magnitude function of A is continuous on $[0, \infty)$.

This is also true in ℓ_1^n, but false for general spaces of negative type at $t = 0$. 

Magnitude in Euclidean space: computational tools

Classical PDE results lead to the following, if $A \subseteq \mathbb{R}^n$ is sufficiently nice and n is odd.

Proposition (Barceló–Carbery)

The potential function of A is the unique function $h \in H^{(n+1)/2}$ such that $h \equiv 1$ on A and

$$(I - \Delta)^{(n+1)/2} h \equiv 0 \text{ on } \mathbb{R}^n \setminus A.$$

Proposition

The weighting for A is the unique distribution $w \in H^{-(n+1)/2}$ which is supported on A and satisfies

$$w\left(e^{-\|\cdot - a\|}\right) = 1 \ \forall a \in A.$$

Furthermore, $|A| = w(f)$ for any smooth f such that $f \equiv 1$ on A.

Example: one-dimensional balls

Let $A = [0, t]$. The potential function h satisfies

- $h(0) = h(t) = 1$.
- $\lim_{x \to \pm\infty} h(x) = 0$.
- $h(x) - h''(x) = 0$ for $x \in \mathbb{R} \setminus [0, t]$.

This easily yields

$$h(x) = \begin{cases} e^x & \text{if } x < 0, \\ 1 & \text{if } 0 \leq x \leq t, \\ e^{-(x-t)} & \text{if } x > t, \end{cases}$$

from which we can recover $|[0, t]| = 1 + \frac{t}{2}$.

Examples: balls and shells

Using these tools, the exact magnitude has been computed for odd-dimensional balls and shells

$$A(r, R) := \{x \in \mathbb{R}^n \mid r \leq \|x\| \leq R\}$$

(Barceló–Carbery, Willerton, Gimperlein–Goffeng).

The computations imply that

$$\lim_{r \to R^-} |A(r, R)| > |A(R, R)| = \left| RS^{n-1} \right|.$$

Thus magnitude is not continuous on

$$\{\text{compact subsets of } \mathbb{R}^n\}.$$

Observation: All known cases of discontinuity of magnitude for spaces of negative type involve change of topology.

Digression: Intrinsic volumes

A **valuation** on $\mathbb{R}^n$ is a function

$$v : \{\text{compact convex sets}\} \to \mathbb{R}$$

such that

$$v(K \cup L) = v(K) + v(L) - v(K \cap L)$$

whenever K, L, and $K \cup L$ are all convex.

Theorem (Hadwiger, Klain)

For $0 \leq i \leq n$, there is a unique (up to scalar multiples) continuous, isometry-invariant valuation V_i on $\mathbb{R}^n$ such that $V_i(tK) = t^i K$.

With the normalization $V_i(K) = \mathrm{vol}_i(K)$ for i-dimensional K, V_i is called the i^{th} **intrinsic volume.**

Intrinsic volumes and magnitude

$V_n(K) = \mathrm{vol}_n(K)$
$V_{n-1}(K) \propto$ surface area of K
$V_1(K) \propto$ mean width of K
$V_0(K) = 1 =$ Euler characteristic of K

Theorem

If $A \subseteq \mathbb{R}^n$ is compact and convex, then

$$|A| \leq \sum_{i=0}^{n} 4^{-i}\,\mathrm{vol}_i(B^i)\,V_i(A) \leq \sum_{i=0}^{n} \frac{\mathrm{vol}_i(B_i)}{4^i i!} V_1(A)^i.$$

Corollary

Suppose that $A \subseteq \ell_2$ is compact and convex, and that

$$\sup\{V_1(K) \mid K \subseteq A \textit{ is a finite-dimensional convex body}\} < \infty.$$

Then $|A| < \infty$ and $\lim_{t\to 0^+} |tA| = 1$.

Outline of the proof, I (Leinster)

- There are versions of intrinsic volumes (V_i') adapted to the ℓ_1 metric.
- If $A = \prod_{i=1}^n [a_i, b_i] \subseteq \ell_1^n$, then $|A| = \sum_{i=0}^n 2^{-i} V_i'(A)$.

  Weight measure can be expressed explicitly as a product measure.
- If $A \subseteq \ell_1^n$ is ℓ_1-convex and pixellated, then $|A| = \sum_{i=0}^n 2^{-i} V_i'(A)$.

  Weight measures exist by Groemer's extension theorem from valuation theory.
- If $A \subseteq \ell_1^n$ is compact and convex, then $|A| \leq \sum_{i=0}^n 2^{-i} V_i'(A)$.

  Lower semicontinuity.

Outline of the proof, II (M.)

- The normed space ℓ_2^n can be approximated by explicit subspaces of ℓ_1^N with $N \gg n$.
  Hence a convex set $A \subseteq \ell_2^n$ can be approximated (as a metric space) by a sequence of convex sets $\{A_N \subseteq \ell_1^N\}$.

  Standard facts / techniques in Banach space theory.

- $\lim_{N\to\infty} V_i'(A_N) = \frac{\mathrm{vol}_i(B^i)}{2^i} V_i(A)$

  Explicit computation.

- If $A \subseteq \ell_2^n$ is compact and convex, then
  $|A| \leq \liminf_{N\to\infty} |A_N| \leq \sum_{i=0}^{n} 4^{-i} \,\mathrm{vol}_i(B^i) V_i(A)$.

  Lower semicontinuity again.

A last observation

If $n \geq 2$, the coefficient of $V_n(tA)$ in the theorem doesn't match the asymptotics of $|tA|$ as $t \to \infty$.
So the upper bound in the theorem is strict for large t.

Therefore in at least one of the places where semicontinuity is used in the proof, actual continuity does not hold.

The end

Thank you!