Magnitude: an overview

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The idea

For many types of mathematical object, there is a canonical notion of size.

- Sets have cardinality. It satisfies

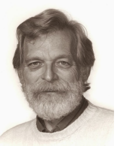
$$|S \cup T| = |S| + |T| - |S \cap T|$$
$$|S \times T| = |S| \times |T|.$$

- Subsets of $\mathbb{R}^n$ have volume. It satisfies

$$\mathrm{vol}(S \cup T) = \mathrm{vol}(S) + \mathrm{vol}(T) - \mathrm{vol}(S \cap T)$$
$$\mathrm{vol}(S \times T) = \mathrm{vol}(S) \times \mathrm{vol}(T).$$

- Topological spaces have Euler characteristic. It satisfies

$$\chi(S \cup T) = \chi(S) + \chi(T) - \chi(S \cap T) \quad \text{(under hypotheses)}$$
$$\chi(S \times T) = \chi(S) \times \chi(T).$$



Stephen Schanuel:
Euler characteristic is the topological analogue of cardinality.

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Challenge Find a general definition of 'size', including these and other examples.

Plan

1. The magnitude of a category

2. The magnitude of an enriched category

   Interlude: (bio)diversity

3. Magnitude homology: a rapid sketch

1. The magnitude of a category

Skipping a categorical background story...

...i.e. here comes a definition that looks unmotivated...

The magnitude of a matrix

Let Z be a matrix.

Definition A weighting on Z is a column vector $\mathbf{w}$ such that $Z\mathbf{w} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

Definition Suppose both Z and Z^{T} admit a weighting.

The magnitude of Z is the total weight

$$|Z| = \sum_i w_i,$$

where $\mathbf{w} = (w_i)$ is any weighting on Z.

(Easy lemma: this is independent of the weighting chosen.)

Important special case If Z is invertible then it has a unique weighting, and

$$|Z| = \sum_{i,j} \left(Z^{-1}\right)_{ij}.$$

The magnitude of a category

Let $\mathbf{A}$ be a finite category. There is a matrix $Z_{\mathbf{A}}$ whose rows and columns are indexed by the objects of $\mathbf{A}$, and whose entries are given by

$$Z_{\mathbf{A}}(a, b) = |\mathrm{Hom}(a, b)|$$

(the number of maps in $\mathbf{A}$ from the object a to the object b).

The magnitude (or Euler characteristic) of $\mathbf{A}$ is

$$|\mathbf{A}| = |Z_{\mathbf{A}}| \in \mathbb{Q}.$$

It is defined as long as $Z_{\mathbf{A}}$ and $Z_{\mathbf{A}}^{\mathsf{T}}$ both admit weightings over $\mathbb{Q}$.

Examples

- If $\mathbf{A}$ contains no maps except for identities then $Z_{\mathbf{A}}$ is the identity matrix and $|\mathbf{A}|$ is just the number of objects of $\mathbf{A}$.
- If $\mathbf{A} = (\bullet \rightrightarrows \bullet)$ then

$$Z_{\mathbf{A}} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad Z_{\mathbf{A}}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix},$$

and $|\mathbf{A}| = 1 + (-2) + 0 + 1 = 0$.

Relation to topological Euler characteristic

Every small category $\mathbf{A}$ has a classifying space $B\mathbf{A}$, a topological space.

Theorem Let $\mathbf{A}$ be a category satisfying a certain finiteness condition. Then

$$\chi(B\mathbf{A}) = |\mathbf{A}|\,.$$

E.g. If $\mathbf{A} = \left(\bullet \rightrightarrows \bullet\right)$ then $B\mathbf{A} = S^1$ and $\chi(S^1) = 0 = |\mathbf{A}|$.

Other theorems connect magnitude of categories to Euler characteristic of manifolds — and more generally, orbifolds (whose Euler characteristics are usually $\notin \mathbb{Z}$).

2. The magnitude of an enriched category

What is an enriched category? The rough idea

Thought 1: In an ordinary category, for any two objects a and b, we have a *set* $\mathrm{Hom}(a, b)$.

But in many contexts, $\mathrm{Hom}(a, b)$ isn't just a set. E.g. it might carry the structure of an abelian group or vector space (as in homological algebra).

Thought 2: Metric spaces are a bit like categories:

Metric spaces have...	Categories have...
Points $a, b, \ldots$.	Objects $a, b, \ldots$.
For each pair (a, b) of points, a real number $d(a, b) \in [0, \infty]$.	For each pair (a, b) of objects, a set $\mathrm{Hom}(a, b)$.
For each triple (a, b, c) of points, an inequality $d(a, b) + d(b, c) \geq d(a, c)$ (the triangle inequality).	For each triple (a, b, c) of objects, a function $\mathrm{Hom}(a, b) \times \mathrm{Hom}(b, c) \to \mathrm{Hom}(a, c)$ (composition).

Enriched categories are a common generalization.

Enriched categories: semi-formal definition

Roughly, a monoidal category is a category $\mathscr{V}$ equipped with a way of taking products $x \otimes y$ of objects x and y.

E.g. Sets with $\times$; vector spaces with $\otimes$; nonnegative real numbers with $+$.

Let $\mathscr{V} = (\mathscr{V}, \otimes)$ be a monoidal category.

Roughly, a category enriched in $\mathscr{V}$ is like an ordinary category, with a set of objects $a, b, \ldots$, but now $\mathrm{Hom}(a, b)$ is an object of $\mathscr{V}$ (rather than a set).

Examples

- If $\mathscr{V} = \mathbf{Set}$ then a category enriched in $\mathscr{V}$ is an ordinary category.
- If $\mathscr{V} = \mathbf{Vect}$ then a category enriched in $\mathscr{V}$ is a 'linear category': a category where the hom-sets have the structure of vector spaces and composition is bilinear.
- If $\mathscr{V} = \mathbb{R}^+$ then any metric space can be seen as a category enriched in $\mathscr{V}$: objects are points and $\mathrm{Hom}(a, b) = d(a, b)$.

Magnitude of enriched categories: the idea

To define the magnitude of a finite category $\mathbf{A}$, we used the matrix $Z_{\mathbf{A}}$ with entries

$$Z_{\mathbf{A}}(a, b) = |\mathrm{Hom}(a, b)| \, .$$

The right-hand side is the cardinality of a finite set.

So:

starting from the notion of the size of an *object of* **Set**,

we obtained a notion of the size of a *category enriched in* **Set**.

Idea: Do the same with an arbitrary monoidal category in place of Set.

The definition

Let $\mathscr{V}$ be a monoidal category and k a (semi)ring.

Let

$$|\cdot| : \frac{\mathrm{ob}\,\mathscr{V}}{\cong} \to k$$

be a multiplicative function (i.e. $|x \otimes y| = |x|\,|y|$).

Given a $\mathscr{V}$-enriched category $\mathbf{A}$ with finitely many objects, write $Z_{\mathbf{A}}$ for the matrix with rows and columns indexed by the objects of $\mathbf{A}$, and entries

$$Z_{\mathbf{A}}(a, b) = |\mathrm{Hom}(a, b)|\,.$$

The magnitude of $\mathbf{A}$ is $|\mathbf{A}| = |Z_{\mathbf{A}}| \in k$ (if defined).

E.g. Take $\mathscr{V} = \mathbf{Set}$ (or really $\mathbf{FinSet}$), $k = \mathbb{Q}$, and $|\cdot| = \mathrm{card}$: then we recover the definition of the magnitude of a finite category.

The magnitude of a metric space

Let $\mathscr{V} = ([0,\infty], +, 0)$, so that metric spaces are $\mathscr{V}$-enriched categories.

Define $|\cdot| : [0,\infty] \to \mathbb{R}$ by $|x| = e^{-x}$.

(Why? So that $|x+y| = |x|\,|y|$ and $|0| = 1$.)

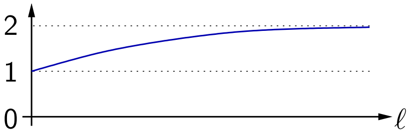
Get notion of the magnitude $|A| \in \mathbb{R}$ of a finite metric space A.

Explicitly: to compute the magnitude of a metric space $A = \{a_1, \ldots, a_n\}$:

- write down the $n \times n$ matrix with (i,j)-entry $e^{-d(a_i,a_j)}$
- invert it
- add up all n^2 entries.

The magnitude of a finite metric space: first examples

- $|\emptyset| = 0$.
- $|\bullet| = 1$.
- $|\bullet \overset{\leftarrow\ \ell\ \rightarrow}{} \bullet| =$ sum of entries of $\begin{pmatrix} e^{-0} & e^{-\ell} \\ e^{-\ell} & e^{-0} \end{pmatrix}^{-1} = \dfrac{2}{1 + e^{-\ell}}$



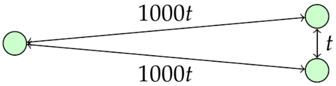
- If $d(a, b) = \infty$ for all $a \neq b$ then $|A| = \text{cardinality}(A)$.

Slogan: Magnitude is the 'effective number of points'

Example: a 3-point space (Simon Willerton)

Take the 3-point space

$A =$



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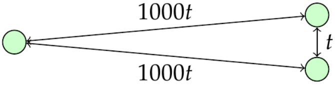
$A =$ 

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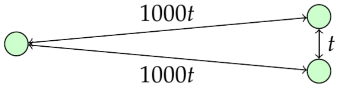
$A =$ 

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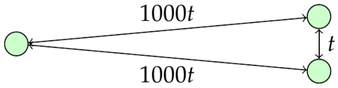
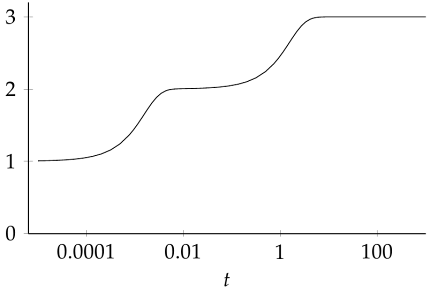
$A =$ 

- When t is small, A looks like a 1-point space.
- When t is moderate, A looks like a 2-point space.
- When t is large, A looks like a 3-point space.

Indeed, the magnitude of A as a function of t is:



Magnitude functions

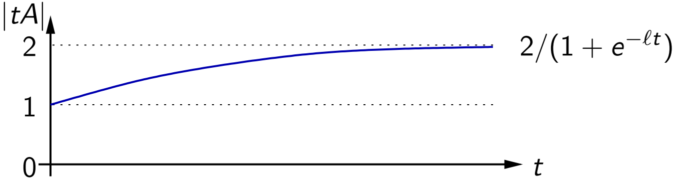
Magnitude assigns to each metric space not just a *number*, but a *function*.

For $t > 0$, write tA for A scaled up by a factor of t.

The magnitude function of a metric space A is the partial function

$$\begin{array}{ccc} (0, \infty) & \to & \mathbb{R} \\ t & \mapsto & |tA| . \end{array}$$

E.g.: the magnitude function of $A = (\bullet \overset{\leftarrow\ \ell\ \rightarrow}{\quad} \bullet)$ is



A magnitude function has only finitely many singularities (none if $A \subseteq \mathbb{R}^n$).

It is increasing for $t \gg 0$, and $\lim_{t \to \infty} |tA| = \text{cardinality}(A)$.

Positive definite metric spaces

Things work best if we insist that our metric spaces A are positive definite, i.e. the matrix Z_A has positive eigenvalues.

E.g. True whenever A is a subset of $\mathbb{R}^N$ with the Euclidean or ℓ^1 metric.

If A is positive definite then A has well-defined magnitude and

$$|A| = \sup_{\mathbf{0} \neq \mathbf{v} \in \mathbb{R}^A} \frac{\left(\sum_a v(a)\right)^2}{\sum_{a,b} v(a) e^{-d(a,b)} v(b)}.$$

It follows that

$$B \subseteq A \implies |B| \leq |A| .$$

In particular, $|A| \geq 1$ for all nonempty A.

The magnitude of a compact metric space

In principle, magnitude is only defined for enriched categories *with finitely many objects* — here, *finite* metric spaces.

Can the definition be extended to, say, compact metric spaces?

Let's restrict to spaces that are positive definite, i.e. every finite subspace is positive definite.

Theorem (Mark Meckes)
All sensible ways of extending the definition of magnitude from finite metric spaces to compact positive definite spaces are equivalent.

Proof Uses functional analysis.

The magnitude of a compact positive definite space A is

$$|A| = \sup\{|B| \ : \ \text{finite } B \subseteq A\}.$$

Magnitude of a compact space: examples

E.g. Line segment: $|t[0, L]| = 1 + \frac{1}{2}L \cdot t$.

Sample theorem Let $A \subseteq \mathbb{R}^2$ be a convex body with the ℓ^1 (taxicab) metric. Then

$$|tA| = \chi(A) + \tfrac{1}{4}\text{perimeter}(A) \cdot t + \tfrac{1}{4}\text{area}(A) \cdot t^2.$$

There's a similar theorem in higher dimensions.

E.g. Magnitude is multiplicative with respect to the ℓ^1 product. So in the case of boxes

$$A = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

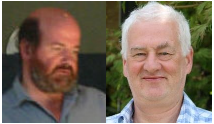
with the ℓ^1 metric, the theorem and its higher-dimensional analogues are easy.

Magnitude encodes geometric information

Let A be a compact subset of $\mathbb{R}^n$, with Euclidean metric.

Theorem (Meckes) From the magnitude function of A, you can recover the Minkowski dimension of A.

Proof Uses a deep theorem from potential analysis, plus the notion of maximum diversity.



Theorem (Barceló and Carbery) From the magnitude function of A, you can recover the volume of A.

Proof Uses PDEs and Fourier analysis.

Theorem (Gimperlein and Goffeng) From the magnitude function of A, you can recover the surface area of A.

(Needs n odd and some regularity hypotheses.)

Proof Uses heat trace asymptotics (techniques related to the heat equation proof of the Atiyah–Singer index theorem).

Inclusion-exclusion for magnitude

Theorem (Gimperlein and Goffeng) Let $A, B \subseteq \mathbb{R}^n$, subject to hypotheses. Then

$$|t(A \cup B)| + |t(A \cap B)| - |tA| - |tB| \to 0$$

as $t \to \infty$.

Magnitude of metric spaces doesn't *literally* obey inclusion-exclusion, as that would make it trivial.

But it *asymptotically* does.

Exact formulas for magnitude

We have more asymptotic results than exact results.

Many very ordinary spaces have unknown magnitude.

E.g. we don't even know the magnitude of a 2-dimensional disc!

But we know a few things...

Theorem (Willerton) The n-sphere of radius t, with the geodesic metric, has magnitude

$$\frac{\text{polynomial in } t}{1 + (-1)^n e^{-\pi t}}.$$

This polynomial is known and fairly simple (but omitted here).

Theorem (Barceló and Carbery) The magnitude function of the n-dimensional Euclidean ball is a rational function, assuming n is odd.

It is a polynomial when n is 1 or 3, but not for $n \geq 5$.

The magnitude of a graph

Any graph A can be viewed as a metric space:

- points are vertices
- distances are shortest path-lengths (which are integers!).

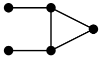
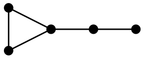
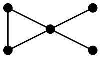
The magnitude of the graph A is the magnitude of this metric space.

Fact The magnitude function $t \mapsto |tA|$ is a *rational function* over $\mathbb{Z}$ of the formal variable $x = e^{-t}$.

It can also be expanded as a *power series* in x over $\mathbb{Z}$.

The magnitude of a graph: examples and theorems

Examples

$\Big|$  $\Big| = \Big|$  $\Big| = \Big|$  $\Big| = \dfrac{5+5x-4x^2}{(1+x)(1+2x)}$

$$= 5 - 10x + 16x^2 - 28x^3 + \cdots$$

Sample theorems:

- $|A \otimes B| = |A| \cdot |B|$, where $\otimes$ is a certain graph product
- $|A \cup B| = |A| + |B| - |A \cap B|$, under quite strict hypotheses
- Graph magnitude has other invariance properties shared with the Tutte polynomial.

Interlude: (bio)diversity

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Conceptual question Given an ecological community, consisting of individuals grouped into species, how can we reasonably quantify its ‘diversity’?

Simplest answer Count the number n of species present.

(Mathematically: cardinality of a finite set.)

Better answer Use the relative abundance distribution $\mathbf{p} = (p_1, \ldots, p_n)$ of species.

For any choice of parameter $q \in \mathbb{R}^+$, can quantify diversity as

$$D_q(\mathbf{p}) = \left(\sum_i p_i^q \right)^{1/(1-q)}.$$

(E.g. if $\mathbf{p} = (1/n, \ldots, 1/n)$ then $D_q(\mathbf{p}) = n$.)

(Mathematically: $\sim$entropy of a probability distribution on a finite set.)

Interlude: (bio)diversity

Even better answer Also use the matrix Z of similarities between species.

For any choice of parameter $q \in \mathbb{R}^+$, can quantify diversity as

$$D_q^Z(\mathbf{p}) = \left(\sum_i p_i (Z\mathbf{p})_i^{q-1} \right)^{1/(1-q)}.$$

The formula is not important here. But...

Discovery (with Christina Cobbold) Most of the biodiversity measures most commonly used in ecology are special cases of D_q^Z.

(Mathematically: $\sim$entropy of a probability distribution on a finite metric space.)

Interlude: (bio)diversity

The maximization problem

Fix a list of species, with known similarity matrix Z.

What is the maximum diversity that can be achieved by varying the species abundances? I.e., what is $\sup_{\mathbf{p}} D_q^Z(\mathbf{p})$?

In principle, the answer depends on the parameter q.

Theorem (with Mark Meckes) The answer is independent of q.

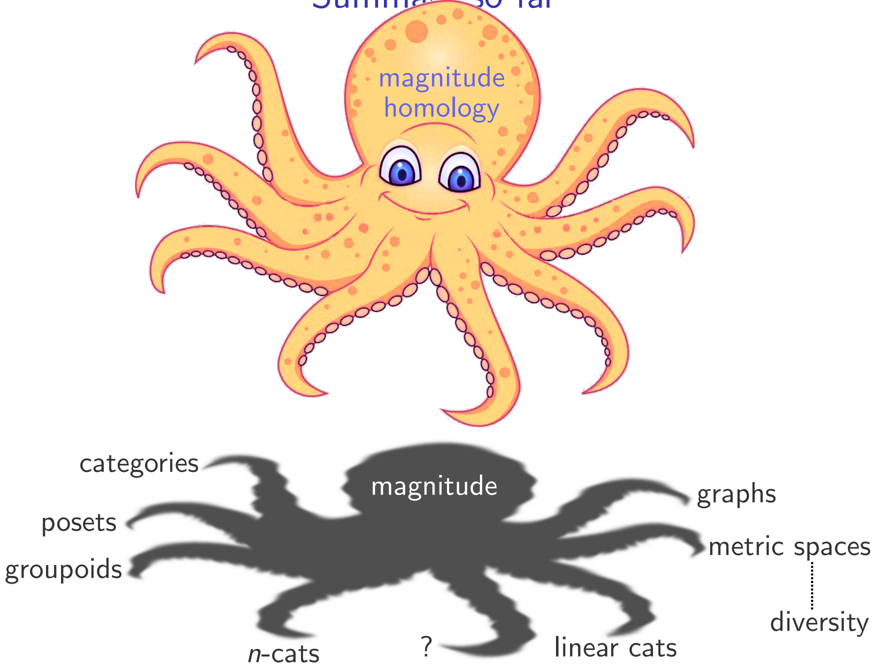
So, $\sup_{\mathbf{p}} D_q^Z(\mathbf{p})$ is a canonical number associated with the matrix Z
— the maximum diversity $D_{\max}(Z)$ of Z.

Fact $D_{\max}(Z)$ is the magnitude of some submatrix of Z.

> Conclusion: Magnitude is closely related to maximum diversity.

3. Magnitude homology: a rapid sketch

Summary so far



Two perspectives on Euler characteristic

So far: Euler characteristic has been treated as an analogue of cardinality.

Alternatively: Given any homology theory H_* of any kind of object A, can define

$$\chi(A) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rank} H_n(A).$$

Note:

- $\chi(A)$ is a *number*
- $H_*(A)$ is an *algebraic structure*, and functorial in A.

In this sense, homology is a categorification of Euler characteristic.

The magnitude homology of a graph

Richard Hepworth and Simon Willerton defined the magnitude homology of a graph A.

(Definition omitted here.)

Features:

- It's a *graded* homology theory, i.e. each $H_n(A)$ is a *graded* abelian group.
- Hence $\chi(A) = \sum(-1)^n \operatorname{rank} H_n(A)$ is a *sequence* of integers.
- Viewing this sequence as a power series over $\mathbb{Z}$, it is exactly the magnitude of A.
  So: magnitude homology categorifies magnitude.
- The formulas for $|A \otimes B|$ and $|A \cup B|$ can be categorified to give Künneth and Mayer–Vietoris theorems.
- Magnitude homology can distinguish between graphs that mere magnitude cannot.

The magnitude homology of an enriched category

Let $\mathscr{V}$ be a monoidal category.

Mike Shulman gave a general definition of the magnitude homology $H_*(\mathbf{A})$ of a $\mathscr{V}$-enriched category $\mathbf{A}$.

(Definition omitted here.)

Features:

- It generalizes both homology of ordinary categories and magnitude homology of graphs.
- The Euler characteristic of the magnitude homology $H_*(\mathbf{A})$ is the magnitude $|\mathbf{A}|$ (in a suitably formal sense).
  So: magnitude homology categorifies magnitude.
- The general definition is a kind of Hochschild homology.
- There's an accompanying cohomology theory.

The magnitude homology of a metric space

In particular, the general definition gives a homology theory of metric spaces.

It's a genuinely *metric* homology theory — not just topological.

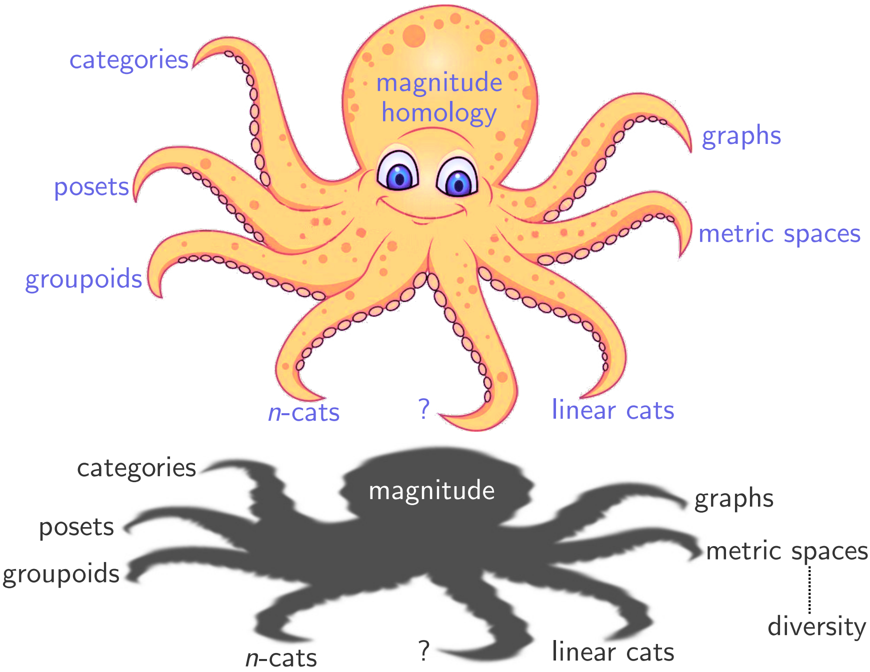
Sample theorem For compact $A \subseteq \mathbb{R}^n$,

$$H_1(A) = 0 \iff A \text{ is convex.}$$

Result of Nina Otter (2018):

magnitude homology is related to (but different from!) persistent homology.

Summary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Thanks

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 Christina Cobbold

 Sonia Mitchell

BBSRC bioscience for the future

 Heiko Gimperlein

 Nina Otter

EPSRC Engineering and Physical Sciences Research Council

 Magnus Goffeng

 Richard Reeve

You