On the fractional capacity and fractional relative perimeter

Ritva Hurri-Syrjänen
University of Helsinki, Finland
joint work with
Antti V. Vähäkangas (University of Jyväskylä, Finland)

Magnitude 2019: Analysis, Category Theory, Applications
University of Edinburgh
July 4-5, 2019

The goal

The set up:
G is an open set in $\mathbb{R}^n$, $n \geq 2$,
$u : G \to \mathbb{R}$, $n \geq 2$, is an appropriate function.

The goal is to study the relationship between the fractional capacity and the fractional relative perimeter of a measurable set A with respect to G.

The fractional Sobolev space

Let G be an open set in $\mathbb{R}^n$. Let $0 < p < \infty$ and $0 < \delta < 1$ be given. We write

$$|u|_{W^{\delta,p}(G)} = \left(\int_G \int_G \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} \, dy \, dx \right)^{1/p}$$

for measurable functions $u : G \to \mathbb{R}$. The homogeneous fractional Sobolev space $\dot{W}^{\delta,p}(G)$ consists of all measurable functions $u : G \to \mathbb{R}$ with $|u|_{W^{\delta,p}(G)} < \infty$.

The functions $u \in \dot{W}^{\delta,p}(G)$ are locally L^p-integrable in G, that is, $u \in L^p_{\mathrm{loc}}(G)$:
Let K be a compact set in G. If $u \in \dot{W}^{\delta,p}(G)$, then $u \in L^p(K)$.

The fractional Sobolev space of functions defined in an open set $G \neq \mathbb{R}^n$

Let G be an open set in $\mathbb{R}^n$. Let $0 < p < \infty$ and $0 < \tau, \delta < 1$ be given. We write

$$|u|_{W^{\delta,p}_\tau(G)} = \left(\int_G \int_{B^n(x,\tau\operatorname{dist}(x,\partial G))} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx \right)^{1/p}$$

for appropriate measurable functions u on G. When $G = \mathbb{R}^n$ both of the integrals are taken over the whole space. The homogeneous fractional Sobolev space $\dot{W}^{\delta,p}_\tau(G)$ consists of all measurable functions $u : G \to \mathbb{R}$ with $|u|_{W^{\delta,p}_\tau(G)} < \infty$.

The functions $u \in \dot{W}^{\delta,p}_\tau(G)$ are locally L^p-integrable in G,

The fractional Sobolev inequality

- Let G be an open set in $\mathbb{R}^n$, $n \geq 2$.
- Let $\delta \in (0,1)$ be given.
- Let $1 \leq p < n/\delta$ be given.

If there is a constant C such that the inequality

$$\left(\int_G |u(x)|^{np/(n-\delta p)} \, dx \right)^{(n-\delta p)/np} \leq C \left(\int_G \int_G \frac{|u(x) - u(y)|^p}{|x-y|^{n+\delta p}} \, dy \, dx \right)^{1/p}$$

holds for all measurable functions $u : G \to \mathbb{R}$ with compact support in G, then this inequality is called a fractional Sobolev inequality.

The fractional capacity

Let $0 < p < \infty$ and $0 < \delta < 1$ be given. The fractional (δ, p)-capacity for a compact set K in G is the number

$$\mathrm{cap}_{\delta,p}(K, G) = \inf_u |u|^p_{W^{\delta,p}(G)} = \inf_u \int_G \int_G \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} \, dy \, dx \,,$$

where the infimum is taken over all functions $u \in C_0(G)$ such that $u(x) \geq 1$ for each $x \in K$.

Remarks

- The fractional capacity of a ball $B^n(x,r)$

$$\mathrm{cap}_{\delta,p}(B^n(x,r),\mathbb{R}^n) = \mathrm{cap}_{\delta,p}(B^n(0,1),\mathbb{R}^n) r^{n-\delta p}\,.$$

- $\mathrm{cap}_{\delta,p}(\cdot,\mathbb{R}^n)$ has been studied intensively by David R. Adams (1980's–). Also,P. Silvestre (2014), J. Xiao (2004–). A. Ponce and D. Spector (2018).
- $\mathrm{cap}_{\delta,p}(\cdot, G)$, where G is a Lipschitz domain, have been studied by S. Shi and J. Xiao (2016).
- A. Ponce, and D. Spector (2018) generalized earlier results of N. G. Meyers and W. P. Ziemer (1977) and Adams (1986).

The fractional relative δ-perimeter of a set A in G with respect to G

- Let G be an open set in $\mathbb{R}^n$, $n \geq 2$, and let A be a measurable set in G.
- Let $\delta \in (0,1)$ be given.

The fractional relative δ-perimeter of A with respect to G is defined as

$$P_\delta(A, G) = \int_A \int_{G \setminus A} \frac{1}{|x-y|^{n+\delta}} \, dy \, dx \, .$$

The fractional relative δ-perimeter of a set A in G with respect to G

- L. Caffarelli, J.-M. Roquejoffre, O. Savin (2010).
- N. Fusco, V. Millot, M. Morrini (2011).
- L. Caffarelli, O. Savin, E. Valdinoci (2015).

The fractional perimeter of A in G with respect to G and the characteristic function of A

The fractional δ-perimeter of a measurable set A in G with respect to G is defined as

$$P_\delta(A, G) = \int_A \int_{G \setminus A} \frac{1}{|x-y|^{n+\delta}} \, dy \, dx \, .$$

We note that

$$P_\delta(A, G) = \frac{1}{2} |\chi_A|_{W^{\delta,1}(G)} = \frac{1}{2} \int_G \int_G \frac{|\chi_A(x) - \chi_A(y)|}{|x-y|^{n+\delta}} \, dy \, dx \, .$$

Properties of $P_\delta(A, \mathbb{R}^n)$

$P_\delta(A, \mathbb{R}^n)$ is called the fractional perimeter or non-local δ-perimeter. It gives one notion of an intermediate object between the classical perimeter and the Lebesgue measure.

$P_\delta(A, \mathbb{R}^n)$ satisfies an isoperimetric inequality

$$\frac{P_\delta(B(0,1), \mathbb{R}^n)}{|B(0,1)|^{(n-\delta)/n}} \leq \frac{P_\delta(A, \mathbb{R}^n)}{|A|^{(n-\delta)/n}} .$$

F. J. Almgren, E. Lieb (1989), A. Figalli, N. Fusco, F. Maggi, V. Millot, M. Morini (2015).

The s-dimensional Hausdorff measure

For any set $A \subset \mathbb{R}^n$

$$\mathcal{H}^s_\eta(A) := \inf \left\{ \sum_{i=0}^{\infty} \omega_s r_i^s : \cup_{i=0}^{\infty} B(x_i, r_i) \supset A, r_i \leq \eta \right\}.$$

The s-dimensional Hausdorff measure of A is the limit

$$\mathcal{H}^s(A) = \lim_{\eta \to 0} \mathcal{H}^s_\eta(A).$$

Here $\omega_s = \pi^s / \Gamma(s/2 + 1)$.

The Hausdorff content of dimension $n - \delta$

For any set $A \subset \mathbb{R}^n$ the Hausdorff content of dimension $n - \delta$ is given by

$$\mathcal{H}_\infty^{n-\delta}(A) := \inf \Big\{ \sum_{i=0}^{\infty} \omega_{n-\delta} r_i^{n-\delta} : \cup_{i=0}^{\infty} B(x_i, r_i) \supset A \Big\} ,$$

where $\omega_{n-\delta} = \pi^{n-\delta} / \Gamma((n-\delta)/2 + 1)$.

Asymptotics

There are the asymptotics

$$\lim_{\delta\to 0} \delta P_\delta(A,\mathbb{R}^n) = C_1|A|$$
$$\lim_{\delta\to 1}(1-\delta)P_\delta(A,\mathbb{R}^n) = C_2 Per(A)\,,$$

where $Per(A)$ denotes the perimeter of A, that is, integration of the $(n-1)$-dimensional measure over the topological boundary of ∂A.

J. Davila (2002), Vl. Mazya, T. Shaposhnikova (2003), A. Ponce, D. Spector (2017).

Very recent results of A. Ponce and D. Spector, 2018

Theorem

For every $\delta \in (0,1)$ and every bounded set $A \subset \mathbb{R}^n$

$$\mathcal{H}^{n-\delta}_{\infty}(A) \leq C\delta(1-\delta)P_{\delta}(A,\mathbb{R}^n)\,.$$

Theorem

For every $\delta \in (0,1)$ and every $A \subset \mathbb{R}^n$

$$\mathrm{cap}_{\delta,1}(A,\mathbb{R}^n) \sim \mathcal{H}^{n-\delta}_{\infty}(A)\,.$$

N. G. Meyers, W. P. Ziemer (1977), D. R. Adams (1977), A. Ponce, D. Spector (2018).

The coarea formula

We recall an extension of the classical coarea formula

$$\int_{\mathbb{R}^n} |\nabla u(x)|\, dx = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{u = t\})\, dt\,,$$

which is valid for every real-valued Lipschitz function u on $\mathbb{R}^n$.

The fractional coarea formula

The following fractional coarea formula is due to A. Visintin (1990).

The fractional coarea formula

Suppose that G is an open set in $\mathbb{R}^n$. Let $0 < \delta < 1$ be given. Then

$$\frac{1}{2}|u|_{W^{\delta,1}(G)} = \int_0^\infty P_\delta(\{u > t\}, G)\, dt$$

for every $u : G \to [0, \infty)$ with $u \in \dot{W}^{\delta,1}(G)$.

Proof

We note that

$$|u(x) - u(y)| = \int_0^\infty |\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)|\, dt$$

for every $x, y \in G$. On the other hand,

$$|\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)| = \chi_{\{u>t\}}(x)\chi_{G\setminus\{u>t\}}(y) + \chi_{\{u>t\}}(y)\chi_{G\setminus\{u>t\}}(x)\,.$$

Hence, by Fubini's theorem

$$\begin{aligned} |u|_{W^{\delta,1}(G)} &= \int_G \int_G \int_0^\infty \frac{|\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)|}{|x-y|^{n+\delta}}\, dt\, dy\, dx \\ &= 2\int_0^\infty \int_{\{u>t\}} \int_{G\setminus\{u>t\}} \frac{1}{|x-y|^{n+\delta}}\, dx\, dy\, dt \\ &= 2\int_0^\infty P_\delta(\{u>t\}, G)\, dt\,. \end{aligned}$$

□

An approximation lemma

Let $\varphi \in C_0^\infty(B^n(0,1))$ be a non-negative bump function such that

$$\int_{\mathbb{R}^n} \varphi(x)\, dx = 1\,.$$

Let $j \in \mathbf{N}$ be given. We define $\varphi_j(x) = 2^{jn}\varphi(2^j x)$ for all $x \in \mathbb{R}^n$. If $u \in L^p(\mathbb{R}^n)$ and $1 \le p < \infty$, it is well known that $u * \varphi_j \to u$ in $L^p(\mathbb{R}^n)$ when $j \to \infty$. We use this fact in the proof of the following lemma which tells that the standard mollification converges to u in the fractional seminorm $|\cdot|_{W^{\delta,1}(G)}$.

An approximation lemma

An approximation lemma
Suppose that G is an open set in $\mathbb{R}^n$. Let $0 < \delta < 1$ be given. Let $u : G \to \mathbb{R}$ be a function in $\dot{W}^{\delta,1}(G)$ with compact support in G. Then,

$$|u - u * \varphi_j|_{W^{\delta,1}(G)} \xrightarrow{j\to\infty} 0 .$$

Characterization. Suppose that G is an open set in $\mathbb{R}^n$. Let $\delta \in (0,1)$, and let a constant $C > 0$ be given.

Then the following conditions are equivalent.

(1) The fractional inequality

$$\left(\int_G |u(x)|^{n/(n-\delta)}\, dx \right)^{(n-\delta)/n} \le C \int_G \int_G \frac{|u(x)-u(y)|}{|x-y|^{n+\delta}}\, dy\, dx$$

holds for all measurable functions $u : G \to \mathbb{R}$ with compact support in G.

(2) The fractional isocapacitary inequality

$$|K|^{(n-\delta)/n} \le C\, \mathrm{cap}_{\delta,1}(K,G)$$

holds for every compact set K in G.

(3) The fractional isoperimetric inequality

$$|D|^{(n-\delta)/n} \le 2C\, P_\delta(D,G)$$

holds for every open set $D \subset\subset G$ whose boundary ∂D is an $(n-1)$-dimensional C^∞-manifold in G.

Remarks

- The constant C is the same in each step (1), (2), and (3).
- Vl. Mazya has characterized the fractional Sobolev inequality in terms of fhe fractional capacity.

Suppose that G is an open set in $\mathbb{R}^n$. Let $1 \leq q < \infty$, $\delta \in (0,1)$, and let a constant $C > 0$ be given.

The following conditions are equivalent.

(I) The fractional inequality

$$\left(\int_G |u(x)|^q \, dx \right)^{1/q} \leq C \int_G \int_G \frac{|u(x) - u(y)|}{|x-y|^{n+\delta}} \, dy \, dx$$

holds for all measurable functions $u : G \to \mathbb{R}$ with compact support in G.

(II) The fractional isocapacitary inequality

$$|K|^{1/q} \leq C \, \mathrm{cap}_{\delta,1}(K, G)$$

holds for every compact set K in G.

(III) The fractional isoperimetric inequality

$$|D|^{1/q} \leq 2C \, P_\delta(D, G)$$

holds for every open set $D \subset\subset G$ whose boundary ∂D is an $(n-1)$-dimensional C^∞-manifold in G.

A characterization of the Hardy inequality

Let G be an unbounded convex domain $\mathbb{R}^n$, $G \neq \mathbb{R}^n$, such that G is a union of bounded convex domains D_i, $\overline{D_i} \subset D_{i+1}$, $|D_1| > 0$,and the ratio of the outer radius and inner radius of D_i is bounded by the same constant for all D_i.

A characterization of the Hardy inequality

Let $\delta \in (0,1)$ and $1 < p, q < \infty$ be given such that $p < n/\delta$ and $0 \le 1/p - 1/q \le \delta/n$. Then the fractional (δ, q, p)-Hardy inequality holds in G, that is,

$$\int_G \frac{|u(x)|^q}{\operatorname{dist}(x,\partial G)^{q(\delta+n(1/q-1/p))}}\,dx \le C\bigg(\int_G\int_G \frac{|u(x)-u(y)|^p}{|x-y|^{n+\delta p}}\,dy\,dx\bigg)^{q/p},$$

for all $u \in C_0(G)$ if and only if, there exists a positive constant $N > 0$ such that inequality

$$\bigg(\sum_{Q\in\mathcal{W}(G)} \mathrm{cap}_{\delta,p}(K\cap Q, D)^{q/p}\bigg)^{p/q} \le N\,\mathrm{cap}_{\delta,p}(K,D)$$

holds for every compact set K in D. Here $\mathcal{W}(G)$ is a Whitney decomposition of G.

A Mazya-type characterization for the Hardy inequality

Let $0 < \delta < 1$ and $1 < p \leq q < \infty$. Then the (δ, q, p)-Hardy inequality

$$\int_G \frac{|u(x)|^q}{\operatorname{dist}(x, \partial G)^{q(\delta + n(1/q - 1/p))}}\, dx \leq C \left(\int_G \int_G \frac{|u(x) - u(y)|^p}{|x - y|^{n + \delta p}}\, dy\, dx \right)^{q/p}$$

holds in any proper open set G in $\mathbb{R}^n$ for all $u \in C_0(G)$ if and only if there is a constant $C > 0$ such that the inequality

$$\int_K \operatorname{dist}(x, \partial G)^{-q(\delta + n(1/q - 1/p))}\, dx \leq C \operatorname{cap}_{\delta, p}(K, G)^{q/p}$$

holds for every compact set K in G.
The special case $p = q$ was considered by B. Dyda and A. V. Vähäkangas earlier.

A result for any measurable set $A \subset\subset G$

Let G be an open set in $\mathbb{R}^n$. If $1 \leq q < \infty$ and $\delta \in (0,1)$ are given such that the general fractional Sobolev inequality holds with a constant $C > 0$, then the inequality

$$|A|^{1/q} \leq 2C\, P_\delta(A, G)$$

holds for every measurable set $A \subset\subset G$. This follows from the fractional Sobolev inequality when $u = \chi_A$.

A sufficient condition for an inequality between the fractional capacity and perimeter

Theorem

Suppose that G is an open set in $\mathbb{R}^n$ and $0 < \delta < 1$. If $D \subset\subset G$ is an open set such that $\mathcal{H}^{n-\delta}(\partial D) = 0$, then inequality

$$\mathrm{cap}_{\delta,1}(\overline{D}, G) \leq 2P_\delta(D, G)\,.$$

holds with respect to G.

Lemma for a proof of a sufficient condition

Suppose that G is an open set in $\mathbb{R}^n$ and let $0 < \delta < 1$ be given. Let $D \subset\subset G$ be an open set such that $\mathcal{H}^{n-\delta}(\partial D) = 0$. Let $\varepsilon > 0$ be given. Then, there exists a function u in $C_0(G)$ such that $0 \le u \le 1$ and $u(x) = 1$ for every $x \in \overline{D}$. Moreover,

$$\int_{G \setminus D} \int_{G \setminus D} \frac{|u(x) - u(y)|}{|x - y|^{n+\delta}} \, dy \, dx < \varepsilon \, .$$

Proof for the theorem of a sufficient condition

Suppose that $D \subset\subset G$ is an open set and $\mathcal{H}^{n-\delta}(\partial D) = 0$. Let $\varepsilon > 0$ and let $u = u_\varepsilon$ be the $C_0(G)$ function given by previous lemma. Then, we obtain

$$\begin{aligned}
&\operatorname{cap}_{\delta,1}(\overline{D}, G) \le \int_G \int_G \frac{|u(x) - u(y)|}{|x-y|^{n+\delta}}\, dy\, dx \\
&\le 2 \int_D \int_{G\setminus D} \frac{1}{|x-y|^{n+\delta}}\, dy\, dx + \int_{G\setminus D} \int_{G\setminus D} \frac{|u(x) - u(y)|}{|x-y|^{n+\delta}}\, dy\, dx \\
&< 2P_\delta(D, G) + \varepsilon\,.
\end{aligned}$$

The theorem is proved by taking $\varepsilon \to 0$. □

An example

As a corollary we obtain the following result.
Suppose that $u \in C_0^\infty(G)$. Let $0 < \delta < 1$ be given. Then the set

$$D_t := \{x \in G \,:\, u(x) > t\}$$

satisfies inequality

$$\mathrm{cap}_{\delta,1}(\overline{D_t}, G) \leq 2P_\delta(D_t, G)$$

with respect to G for almost every $t > 0$.

A question on the characteristic function of a set $D \subset\subset G$ belonging to $\dot{W}^{\delta,1}(G)$

We note that the left hand side of inequality

$$\mathrm{cap}_{\delta,1}(\overline{D}, G) \leq 2P_\delta(D, G)\,.$$

may be viewed as a lower bound for

$$|\chi_D|_{W^{\delta,1}(G)} = 2P_\delta(D, G)\,.$$

Hence, inequality

$$\mathrm{cap}_{\delta,1}(\overline{D}, G) \leq 2P_\delta(D, G)\,.$$

is related to the question if the characteristic function χ_D of a set $D \subset\subset G$ belongs to the fractional homogeneous Sobolev space $\dot{W}^{\delta,1}(G)$. The question is studied by D. Faraco and K. M. Rogers, 2013.

Quasiballs

All quasiballs satisfy inequality

$$\mathrm{cap}_{\delta,1}(\overline{D},\mathbb{R}^n) \leq 2P_\delta(D,\mathbb{R}^n)\,.$$

A homeomorphism $f:\mathbb{R}^n \to \mathbb{R}^n$ is K-quasiconformal, $K \geq 1$, if f belongs to the Sobolev class $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ and $|Df(x)|^n \leq KJ_f(x)$ for almost every $x \in \mathbb{R}^n$. Here $J_f = \det(Df)$. If $f:\mathbb{R}^n \to \mathbb{R}^n$ is a K-quasiconformal mapping, then the image of the unit ball in $\mathbb{R}^n$ under the mapping f is called a quasiball. Examples of quasiballs are snowflakes in the plane and snowballs in $\mathbb{R}^3$, which are 3-dimensional analogues of snowflakes.

This talk is based on the following papers

Ritva Hurri-Syrjänen and Antti V. Vähäkangas:
Fractional Sobolev-Poincaré inequalities and fractional Hardy inequalities in unbounded John domains.
Mathematika 61, 2015.

Ritva Hurri-Syrjänen and Antti V. Vähäkangas:
Characterizations to the fractional Sobolev inequality.
Complex Analysis and Dynamical Systems VII, Contemp. Math., 699, Amer. Math. Soc., Providence, RI, 2017.

Thank you for your attention.