Rips Magnitude

(joint with Richard Hepworth)

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Outline

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Magnitude

Definition (Leinster, 2008)

(X, d) finite metric space. *A weighting* of X is $w : X \to \mathbb{R}$ s.t.

$$\sum_{x \in X} e^{-d(x,y)} w(x) = 1.$$

Given w, define *the magnitude* of X as

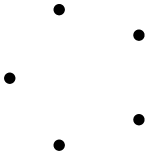
$$|X| = \sum_{x \in X} w(x)$$

and its *magnitude function* as

$$\mathrm{Mag}_X(t) = |tX|.$$

Example

Let $EC_n = \{e^{2\pi i \frac{j}{n}} \mid j = 0, 1, \ldots, n-1\} \subseteq S^1$.



$$\mathrm{Mag}_{EC_5}(t) = \frac{5}{1 + 2e^{-\xi_1 t} + 2e^{-\xi_2 t}}$$

$$\xi_{1,2} = \sqrt{\frac{1}{2}(5 \pm \sqrt{5})}$$

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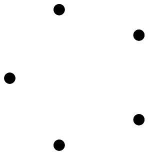

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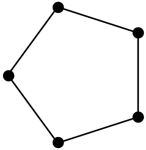
Let C_n be the n-cycle graph. Write $q = e^{-t}$.



$$\begin{aligned}\mathrm{Mag}_{C_5}(t) &= \frac{5}{1 + 2e^{-t} + 2e^{-2t}} \\ &= 5 - 10q + 10q^2 - 20q^4 + 40q^5 + \ldots\end{aligned}$$

Magnitude of a Graph

Graph (G, d) with graph metric. (Can be directed.)

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Theorem (Leinster, 2014)

The magnitude of G is an integer power series in $q = e^{-t}$:

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The coefficients can be expressed explicitly:

$$c_l = \sum_{k=0}^{\infty} (-1)^k \left| \{(x_0, \ldots, x_k) \mid x_{i-1} \neq x_i, \sum_{i=1}^{k} d(x_{i-1}, x_i) = l\} \right| .$$

Magnitude Homology

Definition (Hepworth, Willerton, 2015)

The *magnitude chain complex* of G:

$$\mathrm{MC}_{k,l}(G) = \langle (x_0, \ldots, x_k) \mid x_i \neq x_{i+1}, \ell(x_0, \ldots, x_k) = l \rangle$$

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with boundary operator $\partial_{k,l} : \mathrm{MC}_{k,l}(G) \to \mathrm{MC}_{k-1,l}(G)$

$$\partial_{k,l}(x_0, \dots, x_k) = \sum_{i=0}^{k} (-1)^i \partial_{k,l}^i (x_0, \dots, x_k).$$

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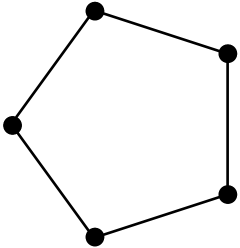
Proposition (Hepworth, Willerton, 2015)

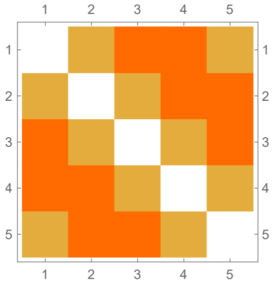
$$\mathrm{Mag}_G(t) = \sum_{l=0}^{\infty} \chi(\mathrm{MH}_{*,l}(G)) q^l \qquad (q = e^{-t})$$

Example: 5-Cycle Graph C_5

$MC_{k,l}$	0	1	2	3	4
0	5				
1		10			
2		10	20		
3			40	40	
4			20	120	80

$MH_{k,l}$	0	1	2	3	4
0	5				
1		10			
2			10		
3			10	10	
4				30	10

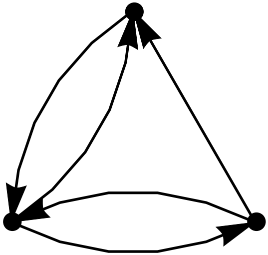


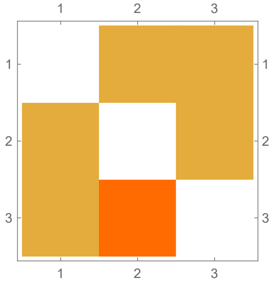


Example: A Directed Graph

$MC_{k,l}$	0	1	2	3	4
0	3				
1		5			
2		1	8		
3			4	13	
4				10	21

$MH_{k,l}$	0	1	2	3	4
0	3				
1		5			
2			7		
3				9	
4					11

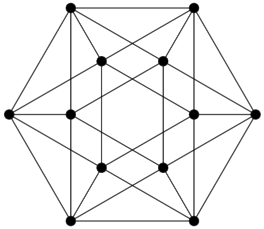


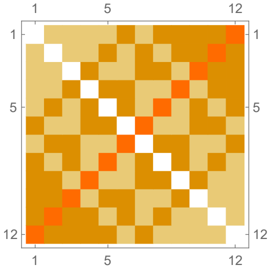


Example: Icosahedral Graph

$MC_{k,l}$	0	1	2	3	4
0	12				
1		60			
2		60	300		
3		12	600	1500	
4			420	4500	7500

$MH_{k,l}$	0	1	2	3	4
0	12				
1		60			
2			240		
3				912	
4					3420





Magnitude of a Compact Metric Space

Definition (Meckes, 2010)

(X, d) compact positive definite metric space. *The magnitude of X is:*

$$|X| = \sup\{|W| \mid W \subseteq X, W \text{ finite.}\}$$

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- $|t \cdot (S^1, \text{euclidean})| = \pi t + O(t^{-1})$ as $t \to \infty$
- $|t \cdot (S^1, \text{geodesic})| = \frac{\pi t}{1 - e^{-\pi t}}$

Outline

Persistent Homology

Definition (various authors)

Given a *filtration functor* $S : (I, \leq) \to \mathbf{SCx}$ ($I \subseteq \mathbb{R}$ locally finite),

$$F = H_k \circ S$$

is known as the *k-th persistent homology* of S.

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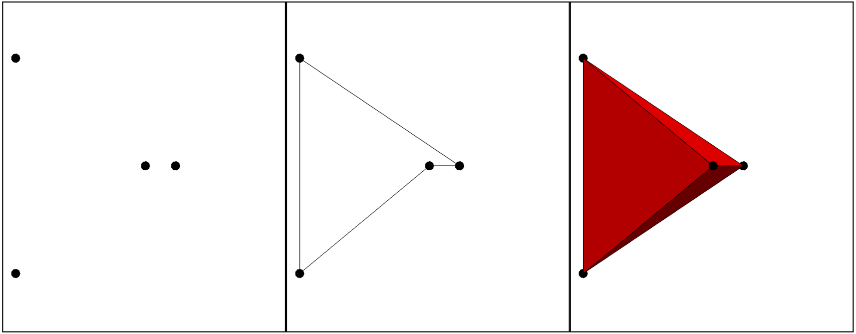
Definition

Given (X, d) and $A \subseteq X$ finite, define the *Rips filtration*:

$$A \in \mathcal{R}_r(X) \iff \operatorname{diam} A \leq r.$$

Example: Rips Filtration of the 4-Cycle Graph C_4



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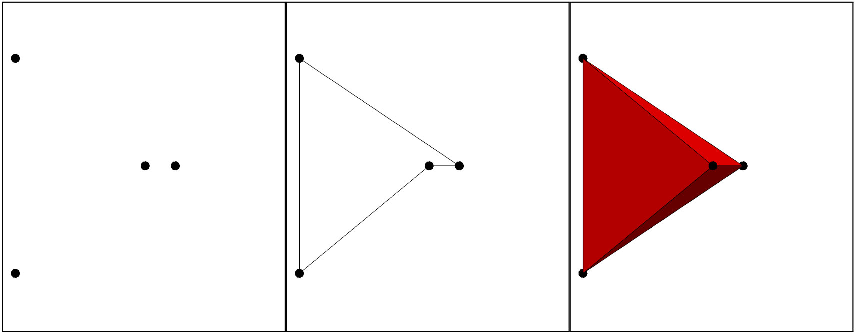

$$\mathrm{PH}_0 \cong \frac{k[t]^4}{(t \cdot k[t])^3}$$

$$\mathrm{PH}_1 \cong \frac{t \cdot k[t]}{t^2 \cdot k[t]}$$

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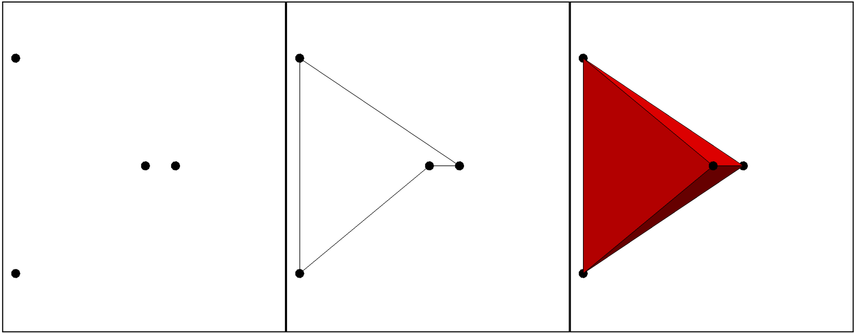

$$\mathrm{PH}_0 \cong [0, \infty) \oplus [0, 1)^3$$
$$\mathrm{PH}_1 \cong [1, 2)$$

Outline

Rips Magnitude

Definition

The *Rips magnitude* of (X, d) is the filtered Euler characteristic of the Rips complex $\mathcal{R}_r(X)$:

$$\begin{aligned}
\mathrm{RMag}_X(t) &= \sum_{\emptyset \neq A \subseteq X} (-1)^{|A|-1} e^{-t\,\mathrm{diam}(A)} \\
&= \sum_{i=0}^{n} \chi(\mathcal{R}_{r_i}(X))(e^{-r_i t} - e^{-r_{i+1} t}) \\
&= \sum_{j=1}^{m} (-1)^{|\beta_i|}(e^{-a_j t} - e^{-b_j t})
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Remark

This can be generalized beyond Rips filtrations.

Properties of Rips Magnitude

Observation

The Rips magnitude function of a finite metric space (X, d) satisfies

$$\lim_{t \downarrow 0} \mathrm{RMag}_X(t) = 1, \qquad \lim_{t \to \infty} \mathrm{RMag}_X(t) = |X|.$$

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Question

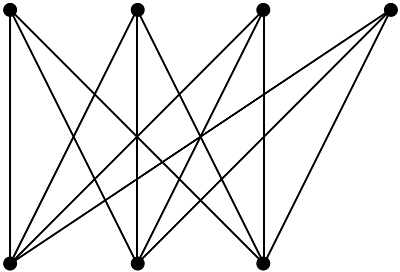
What other properties does it have?

- *Is it increasing and convex?*
- *Do its values always lie in* $[1, |X|]$*?*
- *Does it make sense for infinite subsets* $X \subseteq \mathbb{R}^n$*?*
- *For instance, does* $\lim_{n\to\infty} \mathrm{RMag}(EC_n)(t)$ *exist?*

Example: Bipartite Graphs

Interesting examples among graphs:

- $\mathrm{RMag}_{K_{2,3}}(t) = 5 - 6e^{-t} + 2e^{-2t}$ non-convex,



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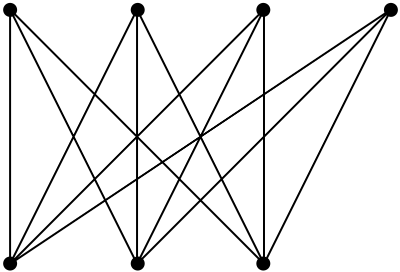

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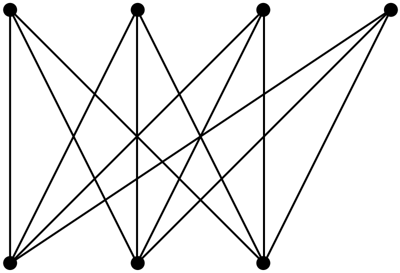
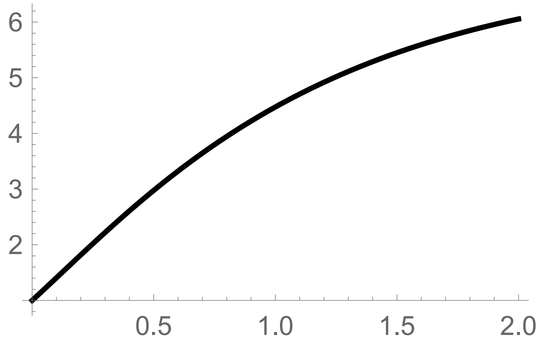
- $\mathrm{RMag}_{K_{5,6}}(t) = 11 - 30e^{-t} + 20e^{-2t}$ non-positive.



Example: Cycle Graphs C_n

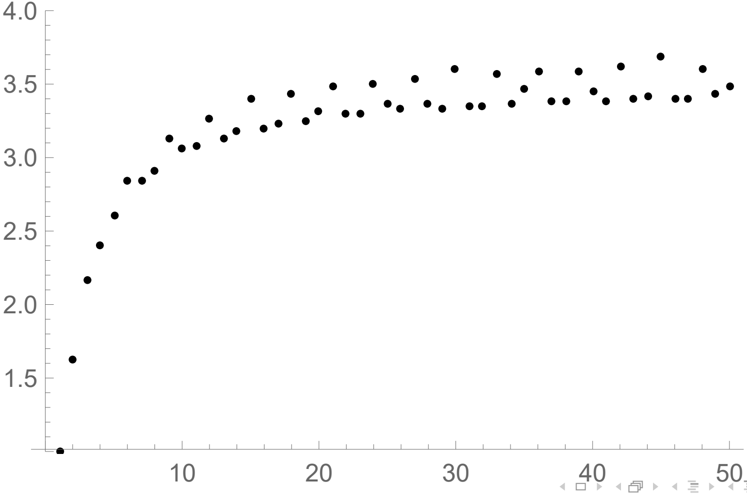
The Rips magnitude RMag_{C_n} of cycle graphs is:

- convex for $n = 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, ...$
- non-convex for $n = 7, 8, 11, 13, 14, 16, 17, 19, 20, ...$



Example: Does RMag_{EC_n} converge?

The asymptotic behaviour of $\mathrm{RMag}_{EC_n}(\frac{1}{2})$ seems strange:



Outline

Rips Filtrations of Cycles

Theorem (Adamaszek, 2011)

Let $0 \leq r < \frac{n}{2}$ and let $\mathcal{R}_r(C_n)$ be the r-th stage of the Rips filtration corresponding to the n-cycle graph. Then

$$\mathcal{R}_r(C_n) \simeq \begin{cases} \bigvee_{n-2r-1} S^{2l}; & r = \frac{l}{2l+1}n, \\ S^{2l+1}; & \frac{l}{2l+1}n < r < \frac{l+1}{2l+3}n. \end{cases}$$

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Remark

We have computational evidence that the number of i-simplices in $\mathcal{R}_r(C_n)$ for $r < \operatorname{diam} C_n$ is given by

$$N_{n,r,i} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{2k+1} \binom{(2k+1)r - kn + 2k}{2k} \binom{(2k+1)r - kn}{i - 2k}$$

Outline

Rips Magnitude of Cycle Graphs

Corollary

$$\chi(\mathcal{R}_r(C_n)) = \begin{cases} n - 2r; & \frac{n}{n-2r} \text{ odd integer}, \\ 1; & n = 2r, \\ 0; & \text{otherwise}. \end{cases}$$

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Theorem

Writing $q = e^{-t}$, we have:

$$\mathrm{RMag}_{C_n}(t) = \sum_{\substack{\text{odd } r|n \\ r \neq n}} \frac{n}{r} q^{\frac{n}{r}\frac{r-1}{2}}(1-q) + q^{\lfloor \frac{n}{2} \rfloor}.$$

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Rips Magnitude of Euclidean Cycles

Theorem

The Rips magnitude of Euclidean cycles is given by:

$$\mathrm{RMag}_{EC_n}(t) = \sum_{\substack{\text{odd } r|n \\ r\neq n}} \frac{n}{r}(e^{-\delta_r t} - e^{-\delta_{r,n} t}) + e^{-\delta_n t},$$

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where

$$\delta_r = \mathrm{diam}(EC_r) = 2\sin\pi\frac{\lfloor\frac{r}{2}\rfloor}{r} \quad \textit{and} \quad \delta_{r,n} = 2\sin\pi\left(\frac{1}{n} + \frac{\lfloor\frac{r}{2}\rfloor}{r}\right).$$

Outline

Interval

Proposition

Let $I = [0, 1]$. Given a finite subset $A \subseteq I$, we have

$$\mathrm{RMag}_A(t) = n - \sum_{j=1}^{n-1} e^{-t(a_j - a_{j-1})}$$

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Corollary

The Rips magnitude function of the closed interval I is given by

$$\mathrm{RMag}(I)(t) = 1 + t.$$

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Circle (Euclidean)

Theorem

$$\lim_{\substack{p\to\infty \\ p \text{ prime}}} \operatorname{RMag}_{EC_p}(t) = e^{-2t} + 2\pi t$$

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$$\lim_{\substack{p\to\infty \\ p \text{ prime}}} \mathrm{RMag}_{EC_{3p}}(t) = e^{-2t} + 2\pi t + \frac{\pi t}{3} e^{-\sqrt{3}t}$$

Circle (Euclidean)

Theorem

For q prime:

$$\lim_{\substack{p\to\infty \\ p\ \mathit{prime}}} \mathrm{RMag}_{EC_{qp}}(t) = e^{-2t} + 2\pi t + \frac{2\pi t}{q} e^{-2t\cos\frac{\pi}{2q}} \sin\frac{\pi}{2q}$$

Circle (Euclidean)

Theorem

For any $n \in \mathbb{N}$:

$$\lim_{\substack{p\to\infty \\ p \text{ prime}}} \operatorname{RMag}_{EC_{np}}(t) = e^{-2t} + 2\pi t \sum_{\text{odd } r|n} \frac{1}{r} e^{-2t\cos\frac{\pi}{2r}} \sin\frac{\pi}{2r}$$

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Theorem

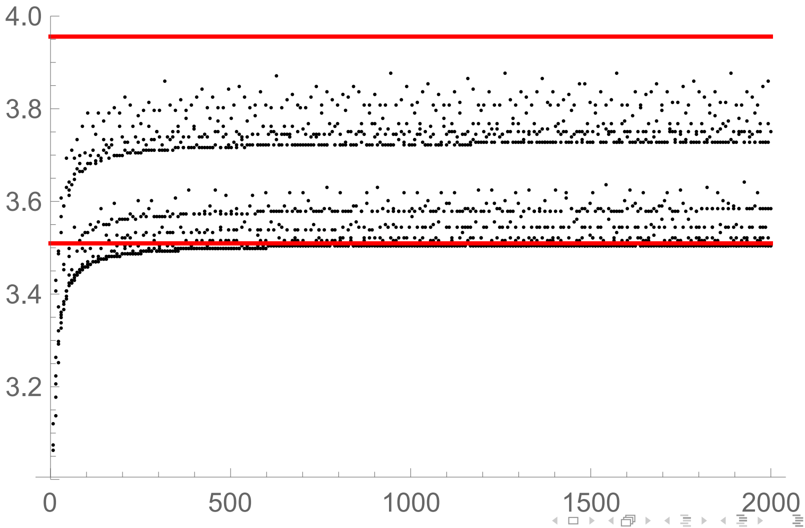
The Rips magnitudes of Euclidean cycles EC_n satisfy

$$\liminf_{n\to\infty} \mathrm{RMag}_{EC_n}(t) = e^{-2t} + 2\pi t$$

and

$$\limsup_{n\to\infty} \mathrm{RMag}_{EC_n}(t) = e^{-2t} + 2\pi t \sum_{r\ \text{odd}} \frac{1}{r} e^{-2t\cos\frac{\pi}{2r}} \sin\frac{\pi}{2r}.$$

Asymptotic Behaviour of $\mathrm{RMag}_{EC_n}(\frac{1}{2})$



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Normalize the cycle graph C_n so that the length is 2π.
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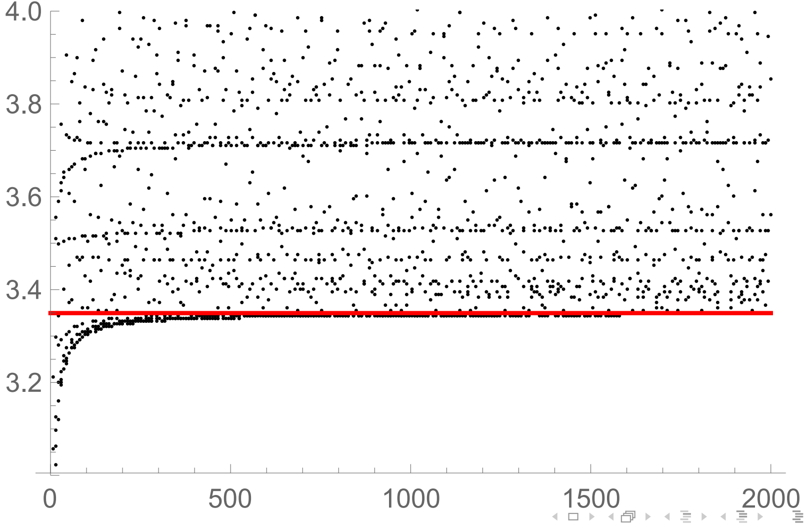

Future Work

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- Čech magnitude,
- Connections to number theory?

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Thank you for your attention!