Magnitude homology of geodesic space

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I will talk about

A complete description of the magnitude homology of a geodesic metric space which satisfies a certain non-branching assumption [arXiv:1902.07044].

- **The homology is described in terms of geodesics.**
- **Examples cover complete and connected Riemannian manifolds as well as uniquely geodesic spaces such as $\mathrm{CAT}(0)$-spaces.**

Plan of my talk:

1. **Review of magnitude homology**
2. **Main theorem**

Review of magnitude homology

- **The notion of magnitude homology is first introduced by Hepworth-Willerton (2015) for simple graphs, and then generalized by Leinster-Schulman (2017) to certain enriched categories, which cover metric spaces and simple graphs.**
- **My talk concerns with only the magnitude homology of metric spaces. So I review here its definition to fix notations.**

Chains

- Let (X, d) be a metric space, namely, a set X equipped with a distance function $d : X \times X \to \mathbb{R}_{\geq 0}$.
- For a non-negative integer n, a (proper) n-chain $\langle x_0, \cdots, x_n \rangle$ is a sequence of $n+1$ points $x_0, \cdots, x_n$ on X such that $x_0 \neq x_1 \neq \cdots \neq x_n$.
- I will call "n" the degree of a chain.
- The length of an n-chain is defined by

$$\ell(\langle x_0, \cdots, x_n \rangle) = d(x_0, x_1) + \cdots + d(x_{n-1}, x_n).$$

- P_n^ℓ : the set of (proper) n-chains of length ℓ.
- $C_n^\ell = \bigoplus_{\gamma \in P_n^\ell} \mathbb{Z}\gamma$: the free abelian group generated by (proper) n-chains of length ℓ.

Some terminology

- A point y is between x and z $(x < y < z)$ when

$$x \neq y \neq z, \qquad d(x, y) + d(y, z) = d(x, z).$$

- In an n-chain $\langle x_0, \cdots, x_n \rangle$, a point x_i $(i \neq 0, n)$ is said to be smooth (straight) if $x_{i-1} < x_i < x_{i+1}$, according to Kaneta-Yoshinaga (or Jubin)
- In other words, x_i is smooth in $\langle x_0, \cdots, x_n \rangle$ iff

$$\ell(\langle x_0, \cdots, x_n \rangle) = \ell(\langle x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n \rangle).$$

- Otherwise, x_i will be called singular (crooked).
- By definition, x_0 and x_n are singular in $\langle x_0, \cdots, x_n \rangle$.

The magnitude homology

- For $\gamma = \langle x_0, \cdots, x_n \rangle \in P_n^\ell$ and $i \neq 0, n$, let us define

$$\partial_i \gamma = \begin{cases} \langle x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n \rangle, & (x_{i-1} < x_i < x_{i+1}) \\ 0. & \text{(otherwise)} \end{cases}$$

$$\partial \gamma = \sum_{i=1}^{n-1} (-1)^i \partial_i \gamma$$

- The linear extension $\partial : C_n^\ell \to C_{n-1}^\ell$ satisfies $\partial\partial = 0$.
  (For $n < 0$, we regard $C_n^\ell = 0$ and $\partial = 0$.)
- Thus, we get a chain complex (C_*^ℓ, ∂).
- Its homology is the magnitude homology $H_n^\ell(X)$.

Complete calculations of magnitude homology

- Generally, magnitude homology is difficult to calculate.
- So far, complete calculations are known for some graphs.
  - The complete graph K_N [Hepworth-Willerton]
  - The path graph P_N (apply Mayer-Vietoris to K_2)
  - The cycle graph C_N [Yuzhou Gu]
  - $\cdots$
- Other complete calculations appear to be obtained for Menger-convex geodetic spaces without 4-cut [Jubin]:
  - convex subsets in $\mathbb{R}^N$ [Kaneta-Yoshinaga],
  - connected, complete and geodetic Riemannian manifolds [Jubin].

  In these cases, $H_n^\ell(X) = 0$ for $n > 0$.
- The main theorem adds non-trivial examples of complete calculations.

Plan of my talk

1. Review of magnitude homology
2. Main theorem

Main theorem

- To state the main theorem, let us recall the notion of geodesics on a metric space (X, d). (This is different from that in differential geometry.)
- A geodesic joining x to y is a map $f : [0, d(x, y)] \to X$,

$$d(f(t), f(t')) = |t - t'|. \quad (t, t' \in [0, d(x, y)])$$

- It follows that a geodesic is a continuous map.
- A metric space is said to be geodesic if, for any points $x, y \in X$, there exists a geodesic joining x to y.
- Typical examples are connected and complete Riemannian manifolds, whereas graphs are not.
- $\mathrm{Geod}(x, y)$ denotes the set of geodesics joining x to y.

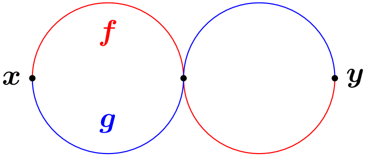
A non-branching assumption

- **The non-branching assumption on a geodesic space in the main theorem is as follows:**

Assumption

For any $x, y \in X$, if $f, g \in \mathrm{Geod}(x, y)$ admit $s \in (0, d(x, y))$ such that $f(s) = g(s)$, then $f = g$.

- **A branching example:**



- **An example is a connected and complete Riemannian manifold.** **(A geodesic is locally characterized by ODE.)**

Main theorem

Theorem [G, arXiv:1902.07044]

Let X be a geodesic space which satisfies Assumption.

(a) **If n is odd, then $H_n^\ell(X) = 0$ for any ℓ.**

(b) **If $n = 2q$ is even, then**

$$H_n^\ell(X) \cong \bigoplus_{\ell_i} \bigoplus_{\varphi_i} \bigoplus_{f_i} \mathbb{Z}(f_1, \ldots, f_q).$$

- $\ell_1, \ldots, \ell_q > 0$ **are such that** $\ell_1 + \cdots + \ell_q = \ell$,
- $\varphi_0, \ldots, \varphi_q \in X$ **are such that** $d(\varphi_{i-1}, \varphi_i) = \ell_i$,
- $f_i \in \mathrm{Geod}(\varphi_{i-1}, \varphi_i)$ **are such that** $f_i \neq \overline{f}_i$,

where $\overline{f}_i \in \mathrm{Geod}(\varphi_{i-1}, \varphi_i)$ **are arbitrary references.**

- **In particular, $H_n^\ell(X)$ is torsion free.**

Details about the even homology

$$H^{\ell}_{2q}(X) \cong \bigoplus_{\ell_i} \bigoplus_{\varphi_i} \bigoplus_{f_i} \mathbb{Z}(f_1, \ldots, f_q)$$

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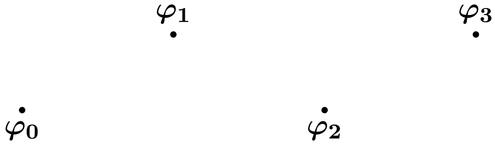

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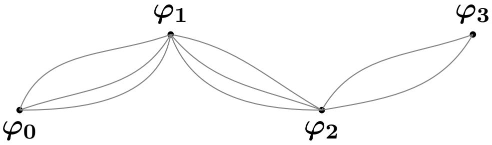

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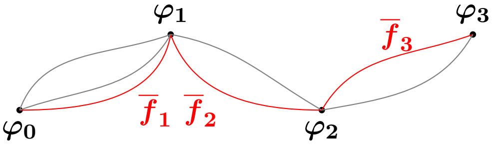

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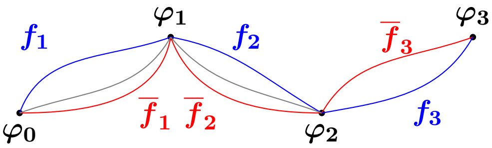

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where $\overline{f}_i \in \mathrm{Geod}(\varphi_{i-1}, \varphi_i)$ are arbitrary references.

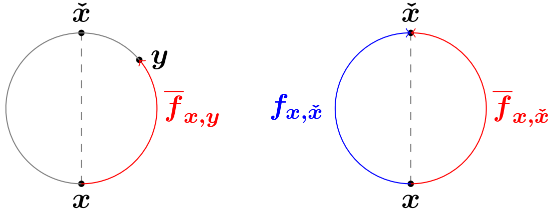
- For a non-trivial contribution to $H_{2q}^{\ell}(X)$, there must be more than two geodesics joining φ_{i-1} to φ_i.
- In particular, $H_n^{\ell}(X) = 0$, $(n > 0)$ for uniquely geodesic spaces X, such as $\mathrm{CAT}(k)$ with $k \leq 0$.

The second homology of the circle

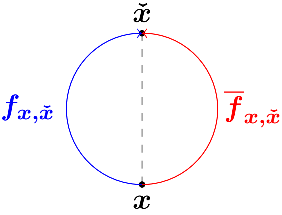
$$H_2^\ell(X) \cong \bigoplus_{\langle \varphi_0, \varphi_1 \rangle \in P_1^\ell} \bigoplus_{\substack{f \in \mathrm{Geod}(\varphi_0, \varphi_1) \\ f \neq \overline{f}}} \mathbb{Z}(f)$$

- **Let S^1 be the circle of radius $r > 0$ with geodesic metric. For any $x \in S^1$, let $\check{x} \in S^1$ be the antipodal point (the unique point such that $d(x, \check{x}) = \pi r$).**

$$\Rightarrow \ \mathrm{Geod}(x, y) = \begin{cases} \{\overline{f}_{x,y}\}, & (y \neq \check{x}) \\ \{f_{x,\check{x}}, \overline{f}_{x,\check{x}}\}. & (y = \check{x}) \end{cases}$$



The second homology of the circle



- Thus, if $\ell \neq \pi r$, then $H_2^\ell(S^1) = 0$, and

$$H_2^{\pi r}(S^1) \cong \bigoplus_{\langle x,y\rangle \in P_1^{\pi r}} \bigoplus_{\substack{f \in \mathrm{Geod}(x,y) \\ f \neq \overline{f}}} \mathbb{Z}(f)$$

$$\cong \bigoplus_{x \in S^1} \mathbb{Z}(f_{x,\check{x}})$$

$$\cong \mathbb{Z}[S^1],$$

as known by Leinster-Schulman and Kaneta-Yoshinaga.

The fourth homology

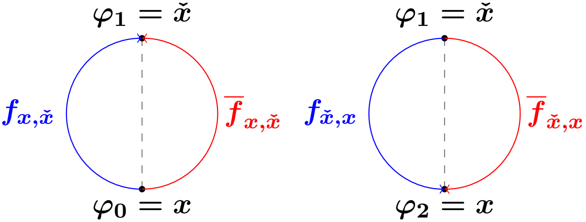
- Similarly, the fourth homology is described as follows.

$$H_4^\ell(X) \cong \bigoplus_{\substack{\ell_1,\ell_2>0 \\ \ell_1+\ell_2=\ell}} \bigoplus_{\substack{\langle\varphi_0,\varphi_1,\varphi_2\rangle\in P_2^\ell \\ d(\varphi_{i-1},\varphi_i)=\ell_i}} \bigoplus_{\substack{f_i\in \mathrm{Geod}(\varphi_{i-1},\varphi_i) \\ f_i\neq \overline{f}_i}} \mathbb{Z}(f_1, f_2)$$

- We have $H_4^\ell(S^1) = 0$ if $\ell \neq 2\pi r$.
- In the case of $\ell = 2\pi r$, the $4th$ magnitude homology $H_4^\ell(S^1)$ is non-trivial: If $\ell_1 = \ell_2 = \pi r$, $\varphi_0 = x \in S^1$, $\varphi_1 = \check{x}$, and $\varphi_2 = \check{\check{x}} = x$, then

$$\mathrm{Geod}(\varphi_0, \varphi_1) = \{f_{x,\check{x}}, \overline{f}_{x,\check{x}}\},$$
$$\mathrm{Geod}(\varphi_1, \varphi_2) = \{f_{\check{x},x}, \overline{f}_{\check{x},x}\}.$$

The fourth homology of the circle



$$H_4^{2\pi r}(S^1) \cong \bigoplus_{\substack{\ell_1,\ell_2>0 \\ \ell_1+\ell_2=2\pi r}} \bigoplus_{\substack{\langle\varphi_0,\varphi_1,\varphi_2\rangle\in P_2^{2\pi r} \\ d(\varphi_{i-1},\varphi_i)=\ell_i}} \bigoplus_{\substack{f_i\in \mathrm{Geod}(\varphi_{i-1},\varphi_i) \\ f_i\neq \overline{f}_i}} \mathbb{Z}(f_1, f_2)$$

$$\cong \bigoplus_{x\in S^1} \mathbb{Z}(f_{x,\check{x}}, f_{\check{x},x}) \cong \mathbb{Z}[S^1],$$

$$H_{2q}^{\ell}(S^1) \cong \begin{cases} \mathbb{Z}[S^1], & (\ell = q\pi r \geq 0) \\ 0. & (\text{otherwise}) \end{cases}$$

Representative of the homology class

- So far, I explained the detail of the description of the generators of the homology group.
- Each generator $(f_1, \ldots, f_q) \in H_{2q}^{\ell}(X)$ has an explicit representative.
- For example, a representative of $(f_1, f_2) \in H_4^{2\pi r}(S^1)$ is

$$\langle \varphi_0, x^{f_1} - x^{\overline{f}_1}, \varphi_1, x^{f_2} - x^{\overline{f}_2}, \varphi_2 \rangle$$
$$:= \langle \varphi_0, x^{f_1}, \varphi_1, x^{f_2}, \varphi_2 \rangle - \langle \varphi_0, x^{f_1}, \varphi_1, x^{\overline{f}_2}, \varphi_2 \rangle$$
$$- \langle \varphi_0, x^{\overline{f}_1}, \varphi_1, x^{f_2}, \varphi_2 \rangle + \langle \varphi_0, x^{\overline{f}_1}, \varphi_1, x^{\overline{f}_2}, \varphi_2 \rangle,$$

where x^{f_i} and $x^{\overline{f}_i}$ lie on $f_i, \overline{f}_i \in \mathrm{Geod}(\varphi_{i-1}, \varphi_i)$ so that φ_1 is singular in each chain. (x^{f_i} and $x^{\overline{f}_i}$ are automatically smooth.)

A representative of $(f_1, f_2) \in H_4^{2\pi r}$

$$\langle \varphi_0, x^{f_1}, \varphi_1, x^{f_2}, \varphi_2 \rangle - \langle \varphi_0, x^{f_1}, \varphi_1, x^{\overline{f}_2}, \varphi_2 \rangle$$
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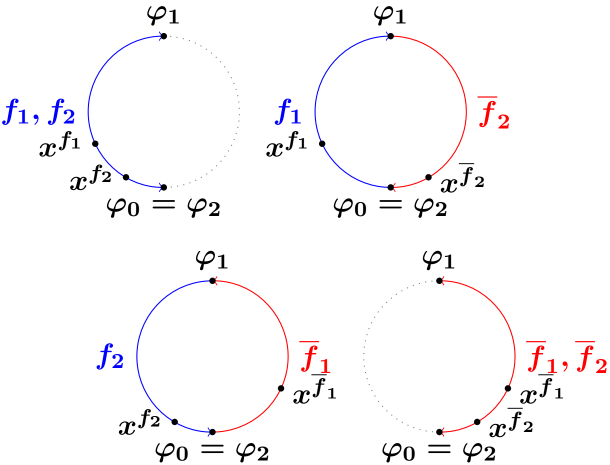

Representatives in general

- The general case is similar, and a representative of $(f_1, \ldots, f_q) \in H^{\ell}_{2q}(X)$ is

$$\begin{aligned}&\langle \varphi_0, x^{f_1} - x^{\overline{f}_1}, \varphi_1, \cdots, \varphi_{q-1}, x^{f_q} - x^{\overline{f}_q}, \varphi_q \rangle \\ &\quad = \langle \varphi_0, x^{f_1}, \varphi_1, \cdots, \varphi_{q-1}, x^{f_q}, \varphi_q \rangle + \cdots \\ &\qquad \cdots + (-1)^q \langle \varphi_0, x^{\overline{f}_1}, \varphi_1, \cdots, \varphi_{q-1}, x^{\overline{f}_q}, \varphi_q \rangle,\end{aligned}$$

  where $x^{f_i}, x^{\overline{f}_i}$ lie on $f_i, \overline{f}_i \in \mathrm{Geod}(\varphi_{i-1}, \varphi_i)$ so that φ_j are singular in each chain.
- Such a choice of the points x^{f_i} and $x^{\overline{f}_i}$ is possible, because of the non-branching assumption.

Outline of the proof

- The proof is a direct calculation based on the smoothness spectral sequence [G, arXiv:1809.06593].
- This is associated to a filtration of the magnitude complex given by the number of smooth points in chains.
- In the setup of the main theorem, the spectral sequence turns out to degenerate at E^2.
- The calculation of E^2 (namely, the homology of E^1) is based on constructions of homotopy operators which make a given element in E^1 into the form of the representatives of generators of $H^{\ell}_{\text{even}}(X)$ step by step.
- The concrete constructions of the homotopies are intricate, and I will not explain it moreover.

A generalization

Proposition

For a dense subspace $X \subset \overline{X}$ in a geodesic space $\overline{X}$ which satisfies Assumption, the following holds true.

(a) **If n is odd, then $H_n^\ell(X) = 0$ for any ℓ.**

(b) **If $n = 2q$ is even, then**

$$H_n^\ell(X) \cong \bigoplus_{\ell_i} \bigoplus_{\varphi_i} \bigoplus_{f_i} \mathbb{Z}(f_1, \ldots, f_q).$$

- $\ell_1, \ldots, \ell_q > 0$ **are such that** $\ell_1 + \cdots + \ell_q = \ell$,
- $\varphi_0, \ldots, \varphi_q \in X$ **are such that** $d(\varphi_{i-1}, \varphi_i) = \ell_i$,
- $f_i \in \mathrm{Geod}_{\overline{X}}(\varphi_{i-1}, \varphi_i)$ **are such that** $f_i \neq \overline{f}_i$,

where $\overline{f}_i \in \mathrm{Geod}_{\overline{X}}(\varphi_{i-1}, \varphi_i)$ are arbitrary references.

Thank you!