# The magnitude function of domains

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# Set up

For a compact metric space $X$, and $M(X)$ denoting the finite Borel measures on $X$, we define

$$Z_R : M(X) \to C(X), \quad Z_R\mu(x) := \int_X \mathrm{e}^{-R\mathrm{d}(x,y)}\mathrm{d}\mu(y).$$

A weighting measure is a solution $\mu_R$ to the equation $Z_R\mu_R = 1$. In this case, the magnitude function of $(X, \mathrm{d})$ is defined by

$$\mathcal{M}_X(R) = \mu_R(X).$$

Solutions might not exist. Using results of Meckes, we can however play rather fast and loose with what we mean by a solution...

# The Leinster-Willerton conjecture

The questions we will be interested in stems from a conjecture by Leinster-Willerton claiming a certain asymptotic behaviour of $\mathcal{M}_X(R)$ for a compact convex subset $X \subseteq \mathbb{R}^n$.

## The Leinster-Willerton conjecture

Let $X \subseteq \mathbb{R}^n$ be a compact convex subset. Then

$$\mathcal{M}_X(R) = \sum_{k=0}^{n} \frac{V_k(X)}{k!\omega_k} R^k + o(1), \quad \text{as } R \to \infty,$$

where $V_k(X)$ denotes the $k$:th intrinsic volume of $X$.

For $X$ a smooth convex body: $V_n(X) = \mathrm{vol}_n(X)$, $V_{n-1}(X) = \mathrm{vol}_{n-1}(\partial X)$, $V_{n-2}(X) = \int_{\partial X} H \mathrm{d}S$, ..., $V_0(X) = \chi(X)$.

## Motivating question

Can we analyze the asymptotics for solutions of the equation

$$Z_R \mu_R = 1,$$

in terms of the geometry of $X$?

# Set up, continued

We restrict to the case that $X \subseteq M$ is a compact domain with smooth boundary in a manifold. The distance function $\mathrm{d}$ needs to be (at least) *regular near the diagonal*. We fix a volume density $\mathrm{d}V$ on $M$ and now view $Z_R$ as an operator between "functions"

$$Z_R f(x) := \int_X \mathrm{e}^{-R\mathrm{d}(x,y)} f(y)\mathrm{d}V(y).$$

## Approach for domains

Set $m := (\dim(M) + 1)/2$. We want to show that

$$Z_R : \dot{H}^{-m}(X) \to \bar{H}^m(X) = \dot{H}^{-m}(X)^*,$$

is an isomorphism and positive in form sense ($\langle f, Z_R f\rangle_{L^2} > 0$ for all $f \neq 0$), then

$$\mathcal{M}_X(R) = \langle 1, Z_R^{-1} 1\rangle_{L^2(X)}.$$

1. Semiclassical analysis of $Z_R$ produces algorithms for computing coefficients $c_j(X, \mathrm{d}) \in \mathbb{R}$ and an asymptotic expansion for $R \to \infty$

   $$\mathcal{M}_X(R) = \sum_{j=0}^{\infty} c_j(X, \mathrm{d}) R^{n-j} + O(R^{-\infty}).$$

   General feature: the hard part is proving existence. Computing the terms is a matter of persistance.
2. Soft functional analysis extends $Z_R^{-1}$ meromorphically to $R$ in a right half-plane in $\mathbb{C}$, and $\mathcal{M}_X(R)$ extends meromorphically.

# Fourier transforming $\mathrm{e}^{-|x|}$

On $\mathbb{R}^n$, we have the operator

$$Z_R f(x) := \int_{\mathbb{R}^n} \mathrm{e}^{-R|x-y|} f(y)\mathrm{d}V(y) = g_R * f(x).$$

where $g_R(x) := \mathrm{e}^{-R|x|}$. Under Fourier transformation

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\mathrm{e}^{-ix\xi}\mathrm{d}V(\xi),$$

we have that

$$\mathcal{F}Z_R f = \hat{g}_R \mathcal{F}f.$$

It will later be important to note that

$$\hat{g}_R(\xi) = n!\omega_n R(R^2 + |\xi|^2)^{-(n+1)/2},$$

and in particular that

$$Z_R = n!\omega_n R(R^2 - \Delta)^{-(n+1)/2}.$$

For $n$ odd, $Z_R^{-1}$ is a bafflingly nice differential operator! For $n$ even, $Z_R^{-1}$ is the worst.

# Meckes' reformulation of magnitude

For a positive definite metric space $X_0$, Meckes defined a Hilbert space $\mathcal{H}_R$ of certain functions on $X_0$ from the inner product

$$\langle f, g\rangle = \int_{X_0} f Z_R^{-1}(g)\mathrm{d}V.$$

For $X_0 = \mathbb{R}^n$, $\mathcal{H}_R = H^{(n+1)/2}(\mathbb{R}^n) = (R^2 - \Delta)^{-(n+1)/4} L^2(\mathbb{R}^n)$.

**Meckes' Hilbert space formulation for positive definite spaces**

If $X \subseteq X_0$ is a compact subset of the positive definite metric space $X_0$,

$$\mathcal{M}_X(R) = \inf\{\|h\|^2_{\mathcal{H}_R} : h = 1 \text{ on X}\}.$$

In particular, if $X_0 = \mathbb{R}^n$, then $\mathcal{M}_X(R) = \frac{1}{n!\omega_n R}\|(R^2 - \Delta)^{(n+1)/4} h_R\|^2_{L^2(\mathbb{R}^n)}$, where $h_R \in H^{(n+1)/2}(\mathbb{R}^n)$ solves

$$\begin{cases} (R^2 - \Delta)^{(n+1)/2} h_R & = 0 \text{ weakly in } \mathbb{R}^n \setminus X, \\ h_R & = 1 \text{ on } X. \end{cases}$$

# The boundary value problem of Barcelo-Carbery

Barcelo and Carbery refined Meckes' results into a boundary value problem and provided an interesting formula for the magnitude of a compact convex body.
Let $X \subseteq \mathbb{R}^n$ be a compact smooth domain and set $\Omega := \mathbb{R}^n \setminus X$. For $h \in C^\infty(\overline{\Omega})$, we use the notation

$$\mathcal{D}_R^j h := \begin{cases} (R^2 - \Delta)^{j/2} h|_{\partial\Omega}, \; j \text{ even}, \\ \partial_\nu (R^2 - \Delta)^{(j-1)/2} h|_{\partial\Omega}, \; j \text{ odd}, \end{cases}$$

## The BC-boundary value problem

Let $n = 2m - 1$ be odd. The boundary value problem

$$\begin{cases} (R^2 - \Delta)^m h_R & = 0 \text{ in } \Omega, \\ \mathcal{D}_R^j h_R & = R^j \text{ on } \partial\Omega \text{ for even } j = 1, \dots, m-1, \\ \mathcal{D}_R^j h_R & = 0 \text{ on } \partial\Omega \text{ for odd } j = 1, \dots, m-1, \end{cases}$$

has a unique solution $h_R \in H^{2m}(\Omega)$. Moreover,

$$\mathcal{M}_X(R) = \frac{\mathrm{vol}_n(X)}{n!\omega_n} R^n - \frac{1}{n!\omega_n} \sum_{\frac{m}{2} < j \leq m} R^{n-2j} \int_{\partial X} \mathcal{D}_R^{2j-1} h_R \, \mathrm{d}S$$

# Reformulation to a problem on the boundary

To find the general structure of the magnitude, we will go further into the BC-BVP. For $h \in C^\infty(\Omega)$, write

$$\mathcal{D}_R^+ h := (\mathcal{D}_R^j h)_{j=0}^{m-1} \quad \text{and} \quad \mathcal{D}_R^- h := (\mathcal{D}_R^j h)_{j=m}^{2m-1}.$$

We also write $\mathcal{H}_+ := \bigoplus_{j=0}^{m-1} H^{2m-1/2-j}(\partial\Omega)$ and $\mathcal{H}_- := \bigoplus_{j=m}^{2m-1} H^{2m-1/2-j}(\partial\Omega)$. Note that $\mathcal{D}_R^+ : H^{2m}(\Omega) \to \mathcal{H}_+$ and $\mathcal{D}_R^- : H^{2m}(\Omega) \to \mathcal{H}_-$.

## The Dirichlet-Neumann operator

Let $n = 2m - 1$ be odd. Define the operator $\Lambda(R) : \mathcal{H}_+ \to \mathcal{H}_-$ by

$$\Lambda(R)\vec{g} := \mathcal{D}_R^- h_R, \quad \text{where } h_R \in H^{2m}(\Omega) \text{ solves } \begin{cases} (R^2 - \Delta)^m h_R & = 0 \text{ in } \Omega, \\ \mathcal{D}_R^+ h = \vec{g}. \end{cases}$$

Then $\Lambda(R)$ is a classical parameter dependent pseudo-differential operator which is parameter elliptic in an Agmon-Douglis-Nirenberg sense. Finally,

$$\mathcal{M}_X(R) = \frac{\mathrm{vol}_n(X)}{n!\omega_n} R^n - (w(R), \Lambda(R)v(R))_{L^2(\partial\Omega)},$$

for vectors $v$ and $w$ that are polynomial in $R$ and constant on $\partial\Omega$.

# More on the Dirichlet-Neumann operator

The operator $\Lambda(R) = (\Lambda_{ij})_{i,j=0}^{m-1}$ is a pseudodifferential operator with parameter with $\Lambda_{ij}$ being of order $m + j - i$. In other words we can in local coordinates write

$$\Lambda_{ij} f(x) = \int \mathrm{e}^{i\xi\cdot x)} a_{ij}(x, \xi, R)\hat{f}(\xi)\mathrm{d}\xi = \iint \mathrm{e}^{i\xi\cdot(x-y)} a_{ij}(x, \xi, R) f(y)\mathrm{d}y\mathrm{d}\xi,$$

for an $a_{ij}$ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta \partial_R^k a_{ij}(x, \xi, R)| \lesssim (1 + |R| + |\xi|)^{m+j-i-|\beta|-k}. \tag{1}$$

Classical= built in asymptotic expansions in terms of functions homogeneous in $(R, \xi)$

## Crucial feature

$$(1, \Lambda_{ij}(R)1)_{L^2(\partial\Omega)} = \int_{\partial\Omega} a_{ij}(x, 0, R)\mathrm{d}S(x) + O(R^{-\infty}).$$

## Exercise (not related to $Z_R$)

For $f \in C_c^\infty(\mathbb{R})$, define $g \in C^\infty(\mathbb{R})$ by $g(x) := -\partial_y u(x, 0)$, where $u \in L^2$ solves

$$\begin{cases} (R^2 - \partial_x^2 - \partial_y^2)u(x, y) = 0, & x \in \mathbb{R}, y > 0, \\ u(x, 0) = f(x), & x \in \mathbb{R}. \end{cases}$$

Show that $\hat{g}(\xi) = \sqrt{R^2 + \xi^2}\hat{f}(\xi)$. In other words, show that $\sqrt{R^2 - \partial_x^2}$ is the D-N operator for $R^2 - \Delta$ on the upper half-plane. *(Extra credit: show that* $\sqrt{R^2 + \xi^2}$ *satisfies* (1) *for* $R >> 0$ *with* $m = i = 0, j = 1$.*)*

# Asymptotic expansions

## Asymptotic expansions

Let $X \subseteq \mathbb{R}^n$ be compact with smooth boundary and $n = 2m - 1$. There are constants $(c_k(X))_{k\in\mathbb{N}}$ such that for all $N \in \mathbb{N}$

$$\mathcal{M}_X(R) = \frac{1}{n!\omega_n} \sum_{k=0}^{n+N} c_k(X) R^{n-k} + O(R^{-N}).$$

The first four coefficients are given by

$$c_0(X) = \mathrm{vol}_n(X), \; c_1(X) = m\mathrm{vol}_{n-1}(\partial X),$$
$$c_2(X) = \frac{m^2}{2}\,(n-1)\int_{\partial X} H \,\mathrm{d}S,$$
$$c_3(X) = \alpha_n \int_{\partial X} H^2 \,\mathrm{d}S$$

Here $H$ is the mean curvature of $\partial X$. For $j \geq 4$, the coefficient $c_j(X)$ is an integral over $\partial X$ of a universal polynomial in covariant derivatives of the fundamental form of total order $j-2$ and total degree $j-1$.

# Consequences for the Leinster-Willerton conjecture

## Asymptotic inclusion/exclusion

Let $n = 2m - 1$. If $A, B \subseteq \mathbb{R}^n$, as well as $A \cup B$ and $A \cap B$, are smooth compact domains then

$$\mathcal{M}_{A\cup B}(R) + \mathcal{M}_{A\cap B}(R) = \mathcal{M}_A(R) + \mathcal{M}_B(R) + O(R^{-\infty}).$$

In particular, if $k$ is such that

$$X \mapsto (c_j(X))_{j=0}^k \quad \text{and} \quad X \mapsto \mathcal{M}_X(R) \mod o(R^{n-k}),$$

are Hausdorff continuous then there are constants $\beta_0, \ldots, \beta_k$ such that

$$\mathcal{M}_X(R) = \sum_{j=0}^{k} \beta_j V_{n-j}(X) R^{n-j} + o(R^{n-k}),$$

for all convex compact subsets $X \subseteq \mathbb{R}^n$.

# On the merry Christmas of 2018

This makes it look feasible that the Leinster-Willerton conjecture could hold, it is “just” a question of Hausdorff continuity. However, it fails:

**Failure of Hausdorff continuity and definite failure of the Leinster-Willerton conjecture**

The coefficient $c_3(X) = \alpha_n \int_{\partial X} H^2 \mathrm{d}S$ is proportional to the Willmore energy which is not Hausdorff continuous and not an intrinsic volume. Indeed, if that would have been the case then $c_3(X)$ would need to be proportional to the Euler characteristic if $n = 3$ which it can not be since $c_3$ can be made arbitrarily large on surfaces of genus zero.

# Meromorphic extension

**The magnitude function extends meromorphically**

Let $X \subseteq \mathbb{R}^n$ be compact with smooth boundary and $n = 2m - 1$ is odd. Then $\mathcal{M}_X$ extends to a meromorphic function of $R \in \mathbb{C}$. Its poles are located outside a sector in the right half-plane.

The proof is based on the identity

$$\mathcal{M}_X(R) = \frac{\mathrm{vol}_n(X)}{n!\omega_n} R^n - (w(R), \Lambda(R)v(R))_{L^2(\partial\Omega)},$$

for vectors $v$ and $w$ that are polynomial in $R$ and constant on $\partial\Omega$. Now $\Lambda(R)$ is constructed from inverting and multiplying holomorphic Fredholm operator valued functions on $\mathbb{C}$ defined from the fundamental solution $\mathrm{e}^{-R|x-y|}$.

# Thanks

Thank you for your attention!

More details in: arXiv:1706.06839