Magnitudes of compact sets in Euclidean spaces – focusing on the boundary

Juan Antonio Barceló and Anthony Carbery

Universidad Politécnica de Madrid
University of Edinburgh &
Maxwell Institute for Mathematical Sciences

Magnitude 2019: Analysis, Category Theory, Applications
Maxwell Institute for Mathematical Sciences
4th July 2019

Outline

Magnitudes of compact metric spaces

Let (X, d) be a positive-definite compact metric space.

We have already defined the magnitude $|A|$ of a finite metric space A.

Definition. $|X| = \sup\{|A| : A \subseteq X,\ A \text{ finite }\}$.

Proposition (M. Meckes)

If there exists a finite signed Borel measure μ on X such that for all $x \in X$

$$\int_X e^{-d(x,y)} \mathrm{d}\mu(y) = 1$$

then $|X| = \mu(X)$.

Such a measure is called a **weight measure** for X.

Notation: The metric space tX for $t > 0$ is the point-set X with the metric td.

Magnitudes of compact intervals in $\mathbb{R}$

Proposition

Let $X = [-R, R] \subseteq \mathbb{R}$ *with the usual metric. Then*

$$|tX| = tR + 1.$$

This is the only example of a compact convex set in euclidean space whose magnitude was (previously) known.

A bold conjecture

Conjecture (Leinster–Willerton)

Suppose $X \subseteq \mathbb{R}^n$ is compact and convex. Then $t \mapsto |tX|$ is a polynomial of degree n and moreover

$$|tX| = \sum_{i=0}^{n} \frac{1}{i!\omega_i} V_i(X) t^i = \frac{\mathrm{Vol}(X)}{n!\omega_n} t^n + \frac{\mathrm{Surf}(X)}{2(n-1)!\omega_{n-1}} t^{n-1} + \cdots + 1$$

where ω_i is the volume of the unit ball in $\mathbb{R}^i$ and $V_i(X)$ is the i'th intrinsic volume of X.

The i'th intrinsic volume $V_i(X)$ measures the important i-dimensional information concerning a convex body X. And for $i < n$ this relates to its boundary.

Methods based entirely on symmetry do not suffice to resolve the conjecture.

Leinster–Willerton Conjecture for euclidean balls

To fix ideas, the Leinster–Willerton conjecture predicts that for the ball B_R in $\mathbb{R}^n$ we will have $|B_R| =$

$$n = 1 : R + 1$$

$$n = 2 : \frac{R^2}{2!} + \frac{\pi R}{2} + 1$$

$$n = 3 : \frac{R^3}{3!} + R^2 + 2R + 1$$

$$n = 4 : \frac{R^4}{4!} + \frac{\pi R^3}{8} + \frac{3R^2}{2} + \frac{3\pi R}{4} + 1$$

$$n = 5 : \frac{R^5}{5!} + \frac{R^4}{9} + \frac{2R^3}{3} + 2R^2 + \frac{8R}{3} + 1$$

$$n = 6 : \frac{R^6}{6!} + \frac{\pi R^5}{128} + \frac{5R^4}{24} + \frac{5\pi R^3}{16} + \frac{5R^2}{2} + \frac{15\pi R}{16} + 1$$

$$n = 7 : \frac{R^7}{7!} + \frac{R^6}{225} + \frac{R^5}{20} + \frac{R^4}{3} + \frac{4R^3}{3} + 3R^2 + \frac{16R}{5} + 1$$

etc.

Potential theory

There is a strong analogy between what we are addressing and classical potential theory.

Indeed, a variant of our problem is to calculate

$$\sup\{\mu(X) : \int_X e^{-|x-y|} \mathrm{d}\mu(y) \leq 1 \text{ on } X\}$$

where the sup is taken over all finite Borel measures supported on X.

Compare this with the classical Newtonian capacity ($n \geq 3$ case)

$$\mathrm{Cap}(X) = \sup\{\mu(X) : \int_X \frac{\mathrm{d}\mu(y)}{|x-y|^{n-2}} \leq 1 \text{ on } X\}$$

– which can also be calculated as the energy integral

$$C_n \inf \left\{ \int_{\mathbb{R}^n} |\nabla h|^2 : h \in \dot{H}^1(\mathbb{R}^n),\ h \equiv 1 \text{ on } X \right\}.$$

[Formally, if $h = I_2\mu = |x|^{-(n-2)} * \mu = 1$ on X then $\mu(X) = \int_{X\times X} \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^{n-2}} = \int h\mathrm{d}\mu = -C_n \int h\Delta h = C_n \int |\nabla h|^2$.]

Variational formulation of magnitude

Similarly, the Fourier transform of $e^{-|x|}$ is $n!\omega_n(1+4\pi^2|\xi|^2)^{-(n+1)/2}$, so

$$e^{-|\cdot|} * f = n!\omega_n(I-\Delta)^{-(n+1)/2} f,$$

and this leads to:

Theorem (M. Meckes)

If $X \subseteq \mathbb{R}^n$ is compact, then

$$|X| = \frac{1}{n!\omega_n} \inf\left\{ \|h\|^2_{H^{(n+1)/2}(\mathbb{R}^n)} \ : \ h \in H^{(n+1)/2}(\mathbb{R}^n),\ h \equiv 1 \text{ on } X \right\},$$

and moreover there exists a unique extremiser.

Note. It's important that the $H^s(\mathbb{R}^n)$ norm is given *precisely* by

$$\|h\|^2_{H^s(\mathbb{R}^n)} = \|(I-\Delta)^{s/2} h\|_2^2 = \int_{\mathbb{R}^n} |\widehat{h}(\xi)|^2 (1+4\pi^2|\xi|^2)^s \mathrm{d}\xi.$$

Classical Bessel capacity

One should compare Meckes' theorem

$$|X| = \frac{1}{n!\omega_n} \inf \left\{ \|h\|^2_{H^{(n+1)/2}(\mathbb{R}^n)} \ : \ h \in H^{(n+1)/2}(\mathbb{R}^n),\ h \equiv 1 \text{ on } X \right\}$$

with the classical Bessel capacity of order $n+1$ (a.k.a. maximum diversity) given by

$$\mathrm{Cap}_{n+1}(X) = \frac{1}{n!\omega_n} \inf \left\{ \|h\|^2_{H^{(n+1)/2}(\mathbb{R}^n)} \ : \ h \in H^{(n+1)/2}(\mathbb{R}^n),\ h \geq 1 \text{ on } X \right\}.$$

The capacity version featuring "$\geq$" is certainly a more stable quantity analytically. On the other hand the associated PDE corresponds to an obstacle problem, rather than a BVP.

Explicit computations of capacities, even for balls?

Does capacity of convex bodies have geometric characteristics – say in terms of intrinsic volumes?

Asymptotics

Leinster (also Meckes): for nonempty compact sets in $\mathbb{R}^n$

$$|X| \geq \frac{\mathrm{Vol}(X)}{n!\omega_n}.$$

Theorem (JAB and AC)

Let X be a nonempty compact set in $\mathbb{R}^n$. Then (continuity at 0)

$$|RX| \to 1 \text{ as } R \to 0$$

and

$$R^{-n}|RX| \to \frac{\mathrm{Vol}(X)}{n!\omega_n} \text{ as } R \to \infty.$$

So the Leinster–Willerton magnitude conjecture is true for *general* compact sets, at least as far as leading terms, in both **asymptotic** regimes $R \to \infty$ and $R \to 0$.

Asymptotics: $R \to \infty$

If $f \in H^{(n+1)/2}(\mathbb{R}^n)$ satisfies $f \equiv 1$ on a compact set X, it must manifestly satisfy $\|f\|^2_{H^{(n+1)/2}(\mathbb{R}^n)} \geq \|f\|^2_{L^2(\mathbb{R}^n)} \geq \mathrm{Vol}(X)$, so that

$$|X| \geq \frac{\mathrm{Vol}(X)}{n!\omega_n} \tag{1}$$

(Leinster, Meckes). But also

$$\|f(R^{-1}\cdot)\|^2_{H^{(n+1)/2}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |R^n \widehat{f}(R\xi)|^2 (1 + 4\pi^2|\xi|^2)^{(n+1)/2} \mathrm{d}\xi$$

$$= R^n \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1 + 4\pi^2(|\xi|/R)^2)^{(n+1)/2} \mathrm{d}\xi$$

so that, by the monotone (or dominated) convergence theorem,

$$R^{-n}\|f(R^{-1}\cdot)\|^2_{H^{(n+1)/2}(\mathbb{R}^n)} \to \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \mathrm{d}\xi = \int_{\mathbb{R}^n} |f(x)|^2 \mathrm{d}x \tag{2}$$

as $R \to \infty$.

Asymptotics: $R \to \infty$, cont'd

So we have

$$\frac{\mathrm{Vol}(X)}{n!\omega_n} \leq R^{-n}|RX| \leq \frac{R^{-n}\|f(R^{-1}\cdot)\|^2_{H^{(n+1)/2}(\mathbb{R}^n)}}{n!\omega_n} \to \frac{\int_{\mathbb{R}^n}|f(x)|^2 \mathrm{d}x}{n!\omega_n}$$

using (1), the extremal characterisation of magnitude, and (2) successively.

But we can find $f \in H^{(n+1)/2}(\mathbb{R}^n)$ with $f \equiv 1$ on X such that $\|f\|_2^2$ is as close as we like to $\mathrm{Vol}(X)$.

Therefore

$$R^{-n}|RX| \to \frac{\mathrm{Vol}(X)}{n!\omega_n}$$

as $R \to \infty$.

The argument for $R \to 0$ is at a similar level of difficulty.

PDE formulation of variational problem

There is a unique extremiser (Meckes) to the problem

$$|X| = \inf \left\{ \|h\|^2_{H^{(n+1)/2}(\mathbb{R}^n)} \ : \ h \in H^{(n+1)/2}(\mathbb{R}^n),\ h \equiv 1 \text{ on } X \right\}.$$

Its Euler–Lagrange equation is, unsurprisingly,

$$(I - \Delta)^{(n+1)/2} h = 0 \text{ weakly on } X^c.$$

So the related PDE problem is

$$(I - \Delta)^{(n+1)/2} h = 0 \text{ on } X^c$$

$$h \in H^{(n+1)/2}(\mathbb{R}^n),\ h \equiv 1 \text{ on } X$$

Existence of solutions is guaranteed by existence of solution to extremal problem.
Note that this is really a PDE problem only when n is odd.

Analysis of the PDE

$$(I-\Delta)^{(n+1)/2}h = 0 \text{ weakly on } X^c$$
$$h \in H^{(n+1)/2}(\mathbb{R}^n),\ h \equiv 1 \text{ on } X$$

- Not a standard BVP – more a mixed BVP/extension problem
- Higher-order
- Exterior domain
- Results in literature tend to deal with $(-\Delta)^m$ for $m = 1, 2, \ldots$
- (More recently for $(-\Delta)^s$ for $s > 0$)
- Existence of solutions guaranteed by existence of extremisers

No “off the shelf” theory we found to handle this equation...

But...

A formula for extremiser

$$(I-\Delta)^m h = 0 \text{ on } X^c$$

$$h \in H^m(\mathbb{R}^n),\ h \equiv 1 \text{ on } X$$

Theorem (JAB and AC)

If $m \in \mathbb{N}$ and $X \subseteq \mathbb{R}^n$ is convex and compact with nonempty interior, then there is a unique solution h to this problem which satisfies

$$\|h\|^2_{H^m(\mathbb{R}^n)} = \mathrm{Vol}(X) + \sum_{\frac{m}{2}<j\leq m} (-1)^j \binom{m}{j} \int_{\partial X_+} \frac{\partial}{\partial \nu} \Delta^{j-1} h \, \mathrm{d}S.$$

Here, ν is the unit normal pointing out of X and $\int_{\partial X_+}$ is a limit of integrals taken over $\partial(rX)$ as $r \downarrow 1$. (N.B. $\|h\|^2_{H^m(\mathbb{R}^n)}$ also equals $\int_{\mathbb{R}^n} h$, but the Leinster–Willerton conjectures inspired us to look for a formula emphasising the geometrical characteristics of X reflected in ∂X.)

Inner product

We use the inner product

$$\langle f, g\rangle_{H^m(\Omega)} := \sum_{j=0}^{m} \binom{m}{j} \int_\Omega D^j f \cdot D^j g$$

which gives rise to the norm $\|\cdot\|_{H^m}$ on $H^m(\mathbb{R}^n)$. Here,

$$D^j f = \Delta^{j/2} f \quad \text{for } j \text{ even}$$

and

$$D^j f = \nabla \Delta^{(j-1)/2} f \quad \text{for } j \text{ odd} .$$

It's convenient to work with *general* $g \in H^m(\mathbb{R}^n)$ and then substitute $g = h$ later.

So, for $g \in H^m(\mathbb{R}^n)$ with compact support we have, by the dominated convergence theorem,

$$\langle g, h\rangle_{H^m(\mathbb{R}^n)} = \int_X g + \sum_{j=0}^{m} \binom{m}{j} \lim_{r\downarrow 0} \int_{\{d(x,X)\geq r\}} D^j g \cdot D^j h.$$

Integration by parts

Now h satisfies an elliptic equation on $\{d(x, X) \geq r\}$ for each $r > 0$, and so, by elliptic regularity, h is smooth here. So we can study

$$\int_{\{d(x,X)\geq r\}} D^j g \cdot D^j h$$

by integrating by parts, systematically using Green's formulae

$$\int_\Omega \nabla\phi \cdot \nabla\psi = -\int_\Omega \phi\Delta\psi - \int_{\partial\Omega} \phi \frac{\partial\psi}{\partial\nu}\,\mathrm{d}S$$

and $$\int_\Omega (\Delta\phi)\psi = -\int_\Omega \nabla\phi \cdot \nabla\psi - \int_{\partial\Omega} \frac{\partial\phi}{\partial\nu}\psi\,\mathrm{d}S.$$

We are always trying to increase the differentiability of ψ, which will be a function of h, and decrease that of ϕ, which will be a function of g. Since g has compact support, there are no boundary terms at ∞.

Eventually we'll be able to use the equation satisfied by h, do a limiting argument to extend the validity of the expressions obtained to all g, plug in $g = h$ and arrive at our desired formula.

Back to magnitude

So the game is now to find the unique solution to

$$(I-\Delta)^{(n+1)/2}h = 0 \text{ on } X^c$$

$$h \in H^{(n+1)/2}(\mathbb{R}^n),\ \ h \equiv 1 \text{ on } X$$

and then calculate the quantities

$$\mathcal{A}_j(X) = \int_{\partial X_+} \frac{\partial}{\partial \nu} \Delta^{j-1} h \mathrm{d}S.$$

The magnitude of the compact convex X for n odd will then be given by

$$|X| = \frac{1}{n!\omega_n}\left(\mathrm{Vol}(X) + \sum_{\frac{n+1}{4}<j\leq\frac{n+1}{2}} (-1)^j \binom{\frac{n+1}{2}}{j} \mathcal{A}_j(X)\right).$$

(It will also be given by $(n!\omega_n)^{-1}\int_{\mathbb{R}^n} h$.)

Spherical symmetry

When X is a ball we can work in polar coordinates to reduce matters to ODEs, and make explicit calculations.

First, we identify the solutions belonging to $H^{(n+1)/2}(\mathbb{R}^n)$ of

$$(I-\Delta)^{(n+1)/2}h = 0 \text{ on } \mathbb{R}^n \setminus \{0\}$$

as linear combinations

$$\sum_{j=0}^{(n-1)/2} \alpha_j \psi_j$$

where

$$\psi_j(x) = e^{-|x|} \sum_{k=j}^{2j-1} c_k^j |x|^{-k}$$

and where the c_k^j are determined by combinatorial considerations.

Pascal's Triangle

Indeed, c_k^j is given by

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 3 \quad 3 \\ 1 \quad 6 \quad 15 \quad 15 \\ 1 \quad 10 \quad 45 \quad 105 \quad 105 \\ 1 \quad 15 \quad 105 \quad 420 \quad 945 \quad 945 \\ \vdots \end{array}$$

In fact, $c_j^j = 1$ and an explicit representation for c_k^j for $j < k \leq 2j - 1$ is given by

$$c_k^j := \frac{(k-1)(k-2)\cdots(2j-k)}{2^{k-j}(k-j)!}.$$

Boundary conditions

Next we have to find the α_j such that $h = \sum_{j=0}^{(n-1)/2} \alpha_j \psi_j$ satisfies the $(n+1)/2$ boundary conditions

$$
\begin{aligned}
h(R) &= 1 \\
h'(R) &= 0 \\
\Delta h(R) &= 0 \\
(\Delta h)'(R) &= 0 \\
&\vdots \\
(\Delta^{(n-1)/4} h)(R) &= 0 \text{ or } (\Delta^{(n-3)/4} h)'(R) = 0
\end{aligned}
$$

(depending on whether $(n-1)$ or $(n-3)$ is a multiple of 4). In either case, this gives us $(n+1)/2$ linear conditions on the $(n+1)/2$ unknowns $\alpha_0, \ldots, \alpha_{(n-1)/2}$ which therefore determine them. $(**)$

Thus our h is (in principle) determined, and we can plug it into the previous formula to obtain the magnitude of B_R in $\mathbb{R}^n$ for n odd.

Magnitudes of balls

To cut a long story short, what we obtain is:

Theorem (JAB and AC)

The magnitude of the closed ball of radius R in $\mathbb{R}^n$ is:

$$n = 1 : R + 1$$

$$n = 3 : \frac{R^3}{3!} + R^2 + 2R + 1$$

$$n = 5 : \frac{R^5}{5!} + \frac{72 + 216R + 216R^2 + 105R^3 + 27R^4 + 3R^5}{24(R+3)}$$

$$n = 7 : \frac{R^7}{7!} + \frac{60 + 240R + 360R^2 + \frac{1165}{4}R^3 + 145R^4 + \frac{189}{4}R^5 + \frac{31}{3}R^6 + \cdots}{60 + 48R + 12R^2 + R^3}$$

etc.

Final remarks

In general, the magnitude of the ball B_R is a **rational** function of R, with rational coefficients, but **not** a polynomial. What, if any, is the significance of its poles ($R = -3$ in the case $n = 5$)?

In the algorithmic procedure for deriving $|B_R|$ in $\mathbb{R}^n$ for n odd, the formula for $|B_R|$ in $\mathbb{R}^{n-2}$ makes a mysterious appearance.

Relation with capacity? Capacities of balls?

Magnitudes of balls in even dimensions? The "calculus" of fractional Bessel potentials such as $(I - \Delta)^{1/2}$? (The calculus for $(-\Delta)^{1/2}$ is much better established...) Possibly use superposition but perhaps not so helpful for a precise calculus? Lift the problem to a space of one greater dimension?