# Capacities: What are they and what are they good for?

## Every problem has its own capacity

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# Physics: electrostatic capacity

**Condenser (capacitor): $K(\text{cpt}) \subset \Omega$**

Distribute charge $\mu(K)$ on $K$ so that the energy $\int_\Omega |\nabla u|^2\, dx$ of the induced electric field $\vec{E} = \nabla u$ in $\Omega \setminus K$ is minimal

When $\Omega = \mathbf{R}^n$, $n > 2$: potential $u(x) = U^\mu(x) = \int \frac{d\mu(y)}{|x-y|^{n-2}}$

$n = 2$: logarithmic potentials

Other $\Omega$: use Green function for $\Omega$

**Capacity of $K \subset \mathbf{R}^n$**

$$\operatorname{cap}_2(K) = \inf\left\{ \int_{\mathbf{R}^n} |\nabla u|^2\, dx : u \in C_0^\infty(\mathbf{R}^n) \text{ and } u \geq 1 \text{ on } K \right\}$$

**Dual formulation**

$$\operatorname{cap}_2(K) = \sup\{\mu(K) : \operatorname{supp}\mu \subset K \text{ and } U^\mu \leq 1 \text{ (on } K)\}$$

Extremal function and capacitary measure: $u \in W^{1,2}(\mathbf{R}^n)$ superharmonic in $\mathbf{R}^n$, harmonic in $\mathbf{R}^n \setminus K$ and $\mu = -\Delta u$

# Sobolev space $W^{k,p}$, $k \in \mathbf{N}_{\geq 1}$, $1 < p < \infty$

Sobolev space $W^{k,p}(\mathbf{R}^n)$ is normed by

$$\|u\|_{W^{k,p}} = \left( \sum_{0 \leq |\alpha| \leq k} \int_{\mathbf{R}^n} |D^\alpha u|^p \, dx \right)^{1/p}, \quad \text{sum over multiindices } \alpha.$$

$C_0^\infty(\mathbf{R}^n)$ is dense in $W^{k,p}(\mathbf{R}^n)$.

Calderón showed that $u \in W^{k,p}(\mathbf{R}^n)$ iff $\exists g \in L^p(\mathbf{R}^n)$ s.t.

$$u = G_k * g, \quad \text{where } G_k = \mathcal{F}^{-1}((1 + |\xi|^2)^{-k/2}),$$

(so $g = (I - \Delta)^{k/2} f$), and that

$$\|u\|_{k,p} := \|g\|_{L^p} \simeq \|u\|_{W^{k,p}}.$$

Makes sense also for noninteger $k$.

# Sobolev capacity

## Definition

*Sobolev capacity* is for cpt $K$ defined by

$$C_{k,p}(K) = \inf\{\|\varphi\|^p_{W^{k,p}} : \varphi \in C_0^\infty(\mathbf{R}^n),\ \varphi \geq 1 \text{ on } K\}.$$

Extended to arbitrary sets by letting

$$C_{k,p}(G) = \sup\{C_{k,p}(K) : K \subset G,\ K \text{ cpt}\}, \quad G \text{ open},$$
$$C_{k,p}(E) = \inf\{C_{k,p}(G) : G \supset E,\ G \text{ open}\}, \quad E \text{ arbitrary}.$$

For $k = 1$ minimizers are solutions of

$$-\Delta_p u + u|u|^{p-2} = 0, \quad \text{where } \Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

which are very difficult to handle if $p \neq 2$.

# Properties of the Sobolev capacity

Monotonicity: $C_{k,p}(E_1) \leq C_{k,p}(E_2)$ if $E_1 \subset E_2$.

Subadditivity: $C_{k,p}\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} C_{k,p}(E_j)$.

Outer measure, but lacks reasonable measurable sets.

Choquet capacity: for all Borel (even Suslin) sets $E$:

$$\begin{aligned} C_{k,p}(E) &= \sup\{C_{k,p}(K) : K \subset E,\ K \text{ cpt}\} \\ &= \inf\{C_{k,p}(G) : G \supset E,\ G \text{ open}\}. \end{aligned}$$

Finer notion than measure:

- $C_{k,p}(E) = 0 \quad \Rightarrow \quad \dim_H(E) \leq n - kp$,
- $\mathcal{H}_{n-kp}(E) < \infty \quad \Rightarrow \quad C_{k,p}(E) = 0$,
- Singletons have positive capacity if $kp > n$.

Depends on the underlying $\mathbf{R}^n$!

# Bessel capacity

**Definition**

*Bessel capacity* is for cpt $K$ defined by

$$B_{k,p}(K) = \inf\{\|\varphi\|_{k,p}^p : \varphi \in C_0^\infty(\mathbf{R}^n),\ \varphi \geq 1 \text{ on } K\}.$$

$\exists$ *capacitary measure* $\mu^K$ s.t.

$$B_{k,p}(K) = \|(G_k * \mu^K)^{p'-1}\|_{L^p}^p$$

The extremal function for $B_{k,p}(K)$ is

$$\varphi^K = G_k * (G_k * \mu^K)^{p'-1}$$

and $\varphi^K \geq 1$ on $K$, except for a set of cap zero.
Leads to the *dual definition*

$$B_{k,p}(K)^{1/p} = \sup\{\mu(K) : \|G_k * \mu\|_{L^{p'}} \leq 1\}.$$

Good for upper/lower estimates of capacity.

# Alternative capacity $N_{k,p}$

**Definition**

Define for cpt $K$,

$$N_{k,p}(K) = \inf\{\|\varphi\|_{k,p}^{p} : \varphi \in C_0^\infty(\mathbf{R}^n),\ \varphi = 1 \text{ on } K\}.$$

Formulas for the extremal elements also in this case, but they are not measures, only distributions when $k > 1$.
Inconvenient drawback!

**Theorem**

*$C_{k,p}$, $B_{k,p}$ and $N_{k,p}$ are comparable.*

$k = 1$: Truncating is possible in $W^{1,p}$ $\implies$ $N_{1,p} = B_{1,p}$.

Capacities in general impossible to calculate!
Estimates usually enough.

# Use of capacities: Removable singularities

**Theorem**

*$\mathcal{L}$ elliptic linear PDO of order $k < n$ with constant coefficients, $K$ cpt.*
*Then $C_{k,p}(K) = 0$ iff $K$ is removable for $\mathcal{L}$ in $L^p$, i.e. if $\Omega \supset K$ is bdd and open, and $u \in L^p(\Omega \setminus K)$ solves*

$$\mathcal{L}u = 0 \quad \text{in } \Omega \setminus K,$$

*then $\exists \tilde{u} \in L^p(\Omega)$ s.t. $\mathcal{L}\tilde{u} = 0$ in $\Omega$, and $\tilde{u} = u$ in $\Omega \setminus K$.*

Similarly $C_{1,p}(K) = 0$ characterizes removability for bdd $p$-harmonic functions, i.e. solutions of

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0,$$

and for bdd $p$-superharmonic functions (i.e. $-\Delta_p u \geq 0$).

# Capacity and Sobolev spaces

Sets of zero capacity are not seen by Sobolev functions:

**Theorem**

Let $E \subset \Omega$ be relatively closed.

- If $C_{k,p}(E) = 0$, then $E$ is removable for $W^{k,p}$, i.e.

$$W^{k,p}(\Omega \setminus E) = W^{k,p}(\Omega)$$

- $W_0^{k,p}(\Omega \setminus E) = W_0^{k,p}(\Omega)$ iff $C_{k,p}(E) = 0$.

$W_0^{k,p}(\Omega) =$ completion of $C_0^\infty(\Omega)$ in $W^{1,p}$-norm.

$W_0^{k,p}$ can be used instead of $C_0^\infty$ to define capacity.

# Lebesgue points and quasicontinuity

**Theorem**

$u \in W^{k,p}(\mathbf{R}^n) \implies$ *u has Lebesgue pts outside E with* $C_{k,p}(E) = 0$

A function $u : \mathbf{R}^n \to \mathbf{R}$ is $C_{k,p}$-*quasicontinuous* if
$\forall \varepsilon > 0 \ \exists G$ open : $C_{k,p}(G) < \varepsilon$ and $u|_{\mathbf{R}^n \setminus G}$ is continuous.

**Theorem**

- $u \in W^{k,p}(\mathbf{R}^n) \quad \implies \quad \exists$ *quasicontinuous representative of* $u$.
- *If* $u_1$, $u_2$ *quasicontinuous then*

$$u_1 = u_2 \text{ a.e.} \quad \implies \quad u_1 = u_2 \text{ q.e. (quasieverywhere)},$$

*i.e. outside a set of capacity zero.*

We can form a refined Sobolev space by only considering quasicontinuous representatives (automatic later in metric spaces).

**Variational capacity (of condensor $(K, \Omega)$)**

$$\operatorname{cap}_p(K,\Omega) = \inf\left\{ \int_G |\nabla u|^p \, dx : u \in C_0^\infty(\Omega) \text{ and } u \geq 1 \text{ on } K \right\}$$

**Definition**

$G$ *finely open* if for all $x \in G$,

$$\int_0^1 \left( \frac{\operatorname{cap}_p(B(x,r) \setminus G, B(x,2r))}{\operatorname{cap}_p(B(x,r), B(x,2r))} \right)^{1/(p-1)} \frac{dr}{r} < \infty,$$

i.e. $G^c$ is *thin* at every $x \in G$

"Wiener integral"

**Theorem**

The fine topology is the coarsest topology making $p$-superharmonic functions continuous.

## Examples of finely open sets

- $G$ open: integrand $= 0$ for small $r \Rightarrow G$ finely open
- $G =$ open set $\setminus$ set of capacity zero, e.g. $G = (0,1)^2 \setminus \mathbf{Q}^2$:
  - $\mathbf{Q}^2$ has no influence on the capacity (if $p \leq 2$)
  - $G$ finely open but int $G = \varnothing$
- Lebesgue spine $\left\{ x \in \mathbf{R}^3 : x_1 > 0 \text{ and } \sqrt{x_2^2 + x_3^2} \leq e^{-1/x_1} \right\}$ is thin at the origin when $p \leq 2$

Well suited for potential theory and Sobolev functions

## Theorem

Quasicontinuous (hence Sobolev) functions are *finely continuous* at all pts outside a set of zero capacity.

# Wiener integral and boundary regularity

Dirichlet problem on bdd open $\Omega \subset \mathbf{R}^n$: Find $u$ such that

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

**Theorem (Wiener criterion)**

A boundary point $x \in \partial\Omega$ is *regular*, i.e.

$$\lim_{\Omega \ni y \to x} u(y) = f(x) \quad \text{for all continuous } f,$$

if and only if $\mathbf{R}^n \setminus \Omega$ is not thin at $x$, i.e.

$$\int_0^1 \left( \frac{\operatorname{cap}_p(B(x,r) \setminus \Omega, B(x,2r))}{\operatorname{cap}_p(B(x,r), B(x,2r))} \right)^{1/(p-1)} \frac{dr}{r} = \infty.$$

Some partial results also for higher order (polyharmonic) operators

# Capacities must be seen within a space!

**Examples**

- 2-capacity of a 1-dimensional segment is zero in $\mathbf{R}^n$, $n > 2$, but positive in $\mathbf{R}^2$
- Singletons have positive capacity when $kp > n$.

Capacity in metric spaces?

**Need:**

- "gradient"
- measure
- space of test functions (e.g. Lipschitz, Sobolev)

Let $X = (X, d, \mu)$ metric space with a metric $d$ and a Borel regular measure $\mu$ s.t.

$$0 < \mu(B) < \infty \quad \forall \text{ balls } B \subset X$$

(Can think of $\mathbf{R}^n$ and its subsets, e.g. nice open sets, also fractals)

# Upper gradients

**Definition (Heinonen, Koskela, 1998)**

$g \geq 0$ is an *upper gradient* of $u : X \to \mathbf{R}$ if

$$|u(x) - u(y)| \leq \int_\gamma g\, ds \qquad (*)$$

for all rectifiable curves $\gamma$ in $X$. ($x, y =$ endpoints of $\gamma$)

**Koskela, MacManus, Shanmugalingam, 1998:**

$\exists$ minimal $p$-weak upper gradient $g_u$:

- minimal in $L^p$ and pointwise a.e.
- $(*)$ holds for $p$-almost all curves ($\mathrm{Mod}_p = 0$)

Difficulties:

- scalar, not vector
- $g_{u+v} \neq g_u + g_v$ in general

# Sobolev (Newtonian) spaces on metric spaces, k=1

**Definition (Shanmugalingam, 2000)**

Newtonian space

$$N^{1,p}(X) = \left\{ u : \int_X (|u|^p + g_u^p)\, d\mu < \infty \right\}$$

$(E, d|_E, \mu|_E)$ as metric space gives $N^{1,p}(E)$ for measurable $E$

Examples:

- In $\mathbf{R}^n$: $g_u = |\nabla u|$ a.e.
- $N^{1,p}(\Omega) = W^{1,p}(\Omega) \quad \forall$ open $\Omega \subset \mathbf{R}^n$,
  but $N^{1,p}$ has only quasicontinuous representatives
  (defined up to sets of zero capacity)
- X discrete or von Koch's snowflake curve:
  no rectifiable curves $\Rightarrow g_u \equiv 0\ \forall u$ and $N^{1,p}(X) = L^p(X)$
  (also if "bad" measure)

# Capacity (Depends on the underlying space $X$!)

Newtonian functions better defined than usual Sobolev functions (i.e. up to sets of zero capacity) and are automatically quasicontinuous under standard assumptions.

**Definition (Sobolev capacity, $k = 1$)**

$$C_p(E) = \inf \int_X (|u|^p + g_u^p)\, d\mu$$

with infimum taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on $E$.

**Definition (Variational capacity, $k = 1$)**

$$\mathrm{cap}_p(E, \Omega) = \inf \int_X g_u^p\, d\mu$$

with infimum taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on $E$ and $u = 0$ outside $\Omega$.

# Nice metric spaces

Much of the $\mathbf{R}^n$ theory for Sobolev spaces, capacity, $p$-harmonic functions extended to metric spaces under standard assumptions:

- $\mu$ doubling: $\mu(2B) \le C\mu(B) \quad \forall$ balls $B \subset X$
- *p-Poincaré inequality*: $\forall$ balls $B \subset X$ and $\forall u$

$$\fint_B |u - u_B| \, d\mu \le C \operatorname{diam} B \biggl( \fint_{\lambda B} g_u^p \, d\mu \biggr)^{1/p},$$

where $u_B = \fint_B u \, d\mu = \dfrac{1}{\mu(B)} \int_B u \, d\mu$

- ($X$ complete)

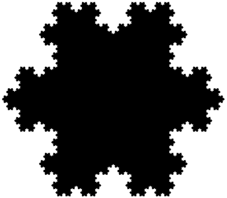
OK if $X$ has many well-behaved curves, nice geometry

HIGHER ORDER SOBOLEV SPACES AND CAPACITIES NOT DEVELOPED IN METRIC SPACES.
PROBLEM: HIGHER ORDER GRADIENTS.

# Spaces satisfying assumptions:

- Weighted $\mathbf{R}^n$, e.g. with Muckenhoupt $A_p$ weights (or $|x|^\alpha$)
- Manifolds and their Gromov–Hausdorff limits
- Graphs
- Reasonable closed subsets of $\mathbf{R}^n$: disc, half-space,

  interior of von Koch snowflake curve,

  and closures of other uniform domains
- Glueing spaces together

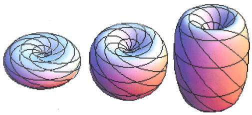
# Heisenberg group $\mathbf{H}^1 = \mathbf{C} \times \mathbf{R}$ (popular by physicists)

Metric = shortest path tangential to span$(X; Y)$

$$X = \frac{\partial}{\partial x} + 2y\,\frac{\partial}{\partial t} \quad \text{and} \quad Y = \frac{\partial}{\partial y} - 2x\,\frac{\partial}{\partial t}$$

Comparable to the norm

$$\|(x, y, t)\| = \left((x^2 + y^2)^2 + t^2\right)^{1/4}$$



**Three different dimensions:**

- Topologically $\mathbf{R}^3$
- "gradient" $(Xu, Yu)$ is 2-dimensional
- $\mathrm{Vol}(B_r) \simeq r^4$ (measure theoretically)

# Sierpiński carpet

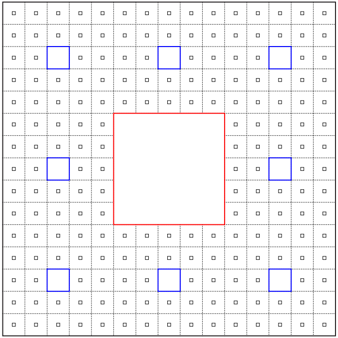
Scale factors

$$a_n = \frac{1}{\text{odd number}}$$

with

$$\sum_{n=1}^{\infty} a_n < \infty$$

Here $a_1 = \frac{1}{3}, \quad a_2 = \frac{1}{5}, \quad a_3 = \frac{1}{7}.$



Doubling + Poincaré inequality OK:
Mackay–Tyson–Wildrick 2013

# Other "gradients" and Sobolev type spaces

**Definition (Hajłasz space and gradient)**

$u \in M^{1,p}(X)$ if $u \in L^p(X)$ and $\exists$ *Hajłasz gradient* $g \in L^p(X)$ s.t.

$$|u(x) - u(y)| \leq d(x,y)(g(x) + g(y)) \quad \text{for a.e. } x, y \in X.$$

In $\mathbf{R}^n$: $g$ corresponds to maximal function $M|\nabla u|$.
Minimal Hajłasz gradient exists, but impossible to calculate and thus also capacity.

- Cheeger (vector-valued) gradient $\longrightarrow$ Sobolev type space
- Gradients and Sobolev spaces based on Poincaré inequalities

Under standard assumptions, all such spaces on $X$ coincide.
(But not on all open subsets.)
Gradients (and thus capacities) need not be comparable.

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THANK YOU FOR YOUR ATTENTION!