Magnitude homology of CAT(κ) sp. ①

Univ. of
Yasuhiko Asao (Tokyo)

Q. what does MH measure?

- MH is HH. (Leinster-Shulman)
- HH is deeply related to free loop space LM.
- LM has much information on closed geodesics on M.

Q Is MH related to geodesics?

A. Yes.

As Tomi has given a beautiful talk, we now have a slogan

"the more geodesics are unique,
the more MH vanishes."

We'll give another proof for this slogan.

We consider CAT(κ) space ($\kappa \in \mathbb{R}$)
$\cup$
Simply conn. complete Riem. mfd with sect. curv $\leq \kappa$.

Let $D_\kappa := \begin{cases} \infty & \kappa \leq 0 \\ \frac{\pi}{\sqrt{\kappa}} & \kappa > 0 \end{cases}$.

<u>Thm.</u> X : CAT(κ).
$0 < {}^\forall \ell < D_\kappa$, $\forall n > 0$.

$$MH_n^\ell(X) = 0.$$

<u>Rem.</u>
- For $\kappa \leq 0$, CAT(κ) is uniquely geodesic.
  ("geodesic" is a dist. pres. map $[0, p] \to X$).
- For $\kappa > 0$, length $\ell < D_\kappa$ geodesic is unique.
- We can also show that if $\exists$ closed geod. length $2\pi r$, then $MH_2^{\pi r}(X) \neq 0$.

<u>Example</u> (1) $X = \mathbb{E}^n, \mathbb{H}^n$, its convex subsets, Hilbert sp, Trees $\Rightarrow MH_n^\ell(X) = 0 \quad \forall n, \ell > 0$.
(geom. realization of)

(2) $X = S^n$ of radius 1. (it is CAT(1))
Then $MH_n^\ell(S^n) = 0 \quad 0 < {}^\forall \ell < \pi \quad \forall n > 0$.

Now, we'll sketch the proof of

"CAT(0) $\Rightarrow MH_n^\ell(X) = 0$".

Key fact: By Kaneta–Yoshinaga

$0 < {}^\forall \ell < m_X := \inf\{\text{length of 4-cut}\}$

${}^\forall n > 0$

$$MH_n^\ell(X) \cong \bigoplus \tilde{H}_{n-2m}(I(a_0,a_1) \otimes \cdots \otimes I(a_{m-1},a_m))$$

$I(a,b) = \{c \in X \mid a < c < b\}$

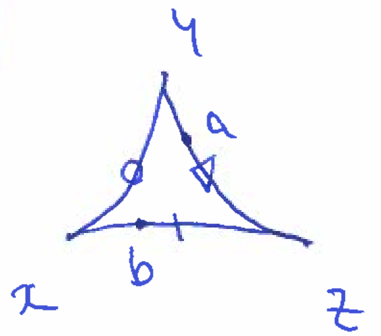
$c \leq c' \overset{\text{def}}{\iff} a < c \leq c'$.

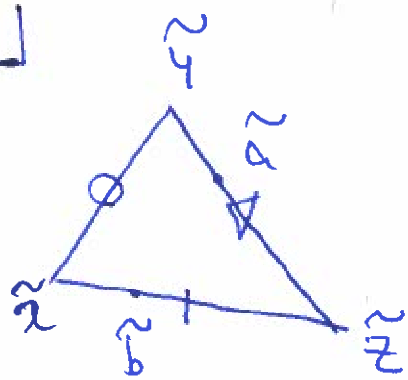
Note If $I(a,b)$ is totally ordered, then $\tilde{H}_*(I(a,b)) = 0$.

Def. of CAT(0) space.

X: geodesic sp.

$X \rfloor$ $\mathbb{E}^2 \rfloor$

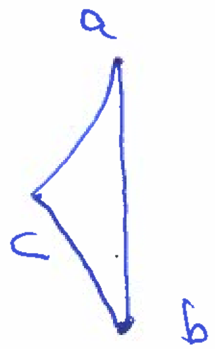




$$X: \mathrm{CAT}(0) \overset{\text{def}}{\iff} d_X(a,b) \leq d_{\mathbb{E}^2}(\tilde{a}, \tilde{b})$$

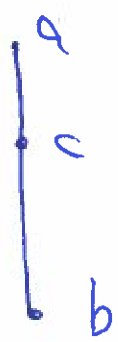
Prop 1 $I(a,b)$ is tot. ord. ${}^\forall a,b \in X$.

(∵) $X \rfloor$



$d(a,c) + d(c,b)$
$= d(a,b)$

$\mathbb{E}^2 \rfloor$



⇝ $c \in$ unique geod. connecting a and b.

⇝ $I(a,b) =$ ——————+

← this is tot. ord. □

Lem. locally geodesic curv. is a geodesic.

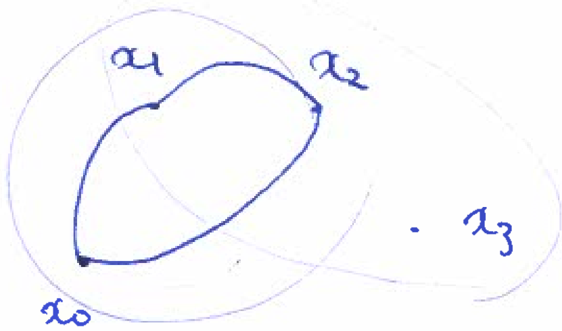
(cf. locally line curv on $\mathbb{E}^2$ is a line).

Prop2 $m_X = \infty$

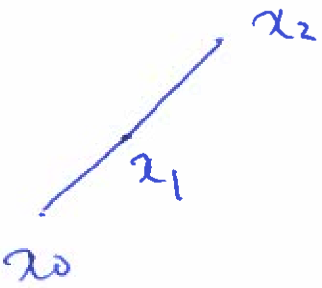
(∵) If $m_X < \infty$, then $\exists$ 4 cut $x = (x_0, x_1, x_2, x_3)$

~~s.t. $m_X \leq |x|$~~

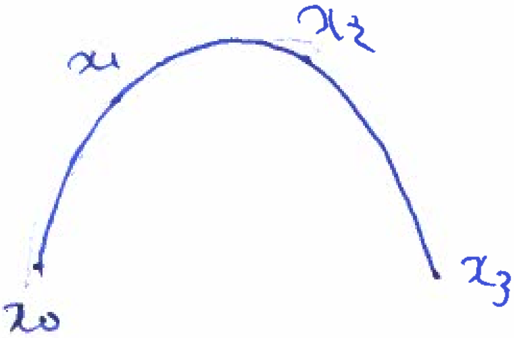
X|



$\mathbb{E}^2$| $d(x_0,x_1) + d(x_1,x_2) = d(x_0,x_2)$



⇝



loc. geod.

⇒ geodesic. ($d(x_0,x_1) + d(x_1,x_2) + d(x_2,x_3) = d(x_0,x_3)$)

⇒ contradicts x is a 4 cut.

□