High-order time domain boundary elements for screen problems

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What's this talk about?

Time domain BEM for wave scattering off a knife's blade (screen problems).

Convergence rates - theory and numerical experiments
(in DOF on a 2d screen, energy norm)

- 0.5: h-version, uniform
- 0.77: h-version, adaptive
- 1.0: p-version, uniform
- $\beta/2$: h-version, β-graded

Theory extends to general polyhedral domains, rates depend on angles.
Examples for p-version

3d wave equation

$$u = u(t,x) \quad \text{sound pressure}$$
$$\partial_t^2 u - \Delta u = 0 \qquad \text{in } \mathbb{R}_t \times \mathbb{R}^3_x \setminus \Omega$$
$$u = 0 \qquad \text{for } t \leq 0 \ .$$

Simple: boundary conditions $u = f$ on $\Gamma = \partial\Omega$.

Realistic: acoustic boundary conditions $\partial_\nu u - \alpha \partial_t u = f$ on $\Gamma = \partial\Omega$.

Some motivations for p-version, h- and hp-adaptive BEM:

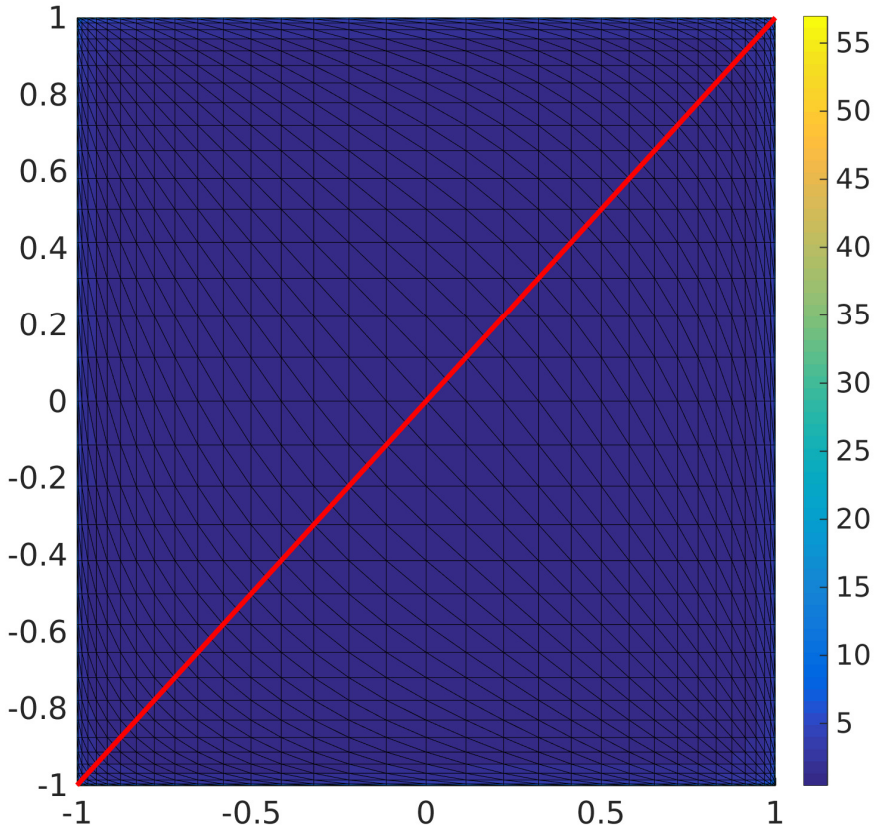
- Edge/corner/geometric singularities.
- Sharp travelling wave crests.

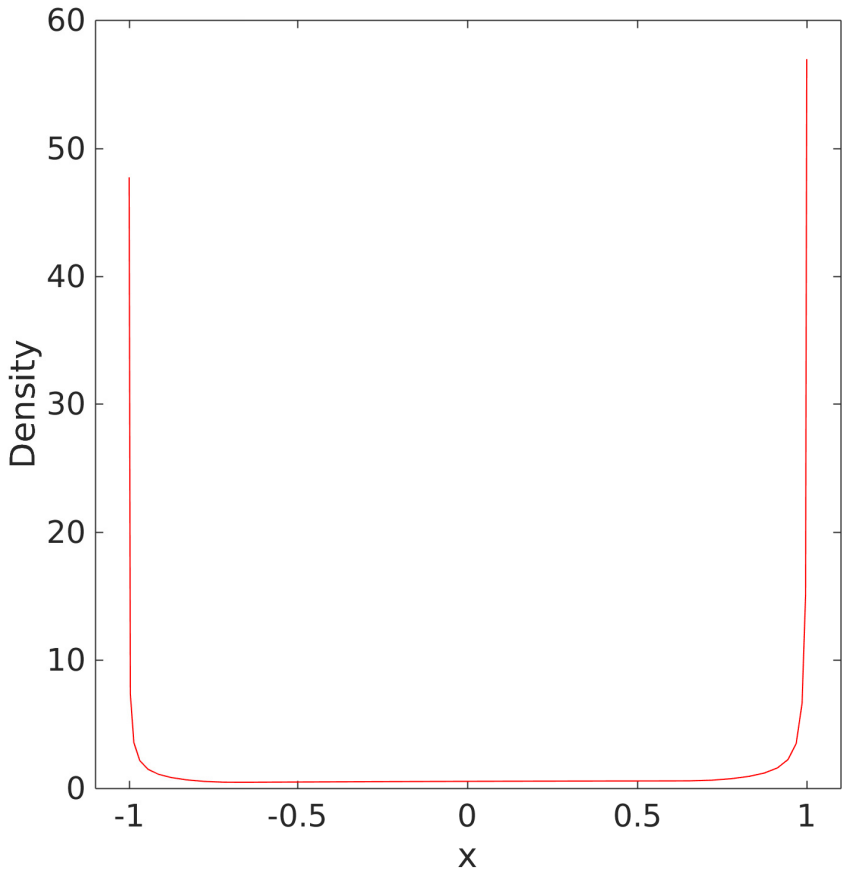
p-FEM: Long history since the 80's with Babuska, Suri, Dorr, Schwab,...
p- and hp-BEM for time-independent problems:
Stephan, Suri, Heuer, Melenk, ...

Screen problems: static graded meshes

$\Omega^c = \mathbb{R}^3 \setminus ([-1,1]^2 \times \{0\})$, $\mathcal{V}\phi(t,x) = \sin(t)^5$ on $[-1,1]^2 \times \{0\}$.

solution near corner $r^{-0.703\ldots}$, near edge $r^{-\frac{1}{2}}$





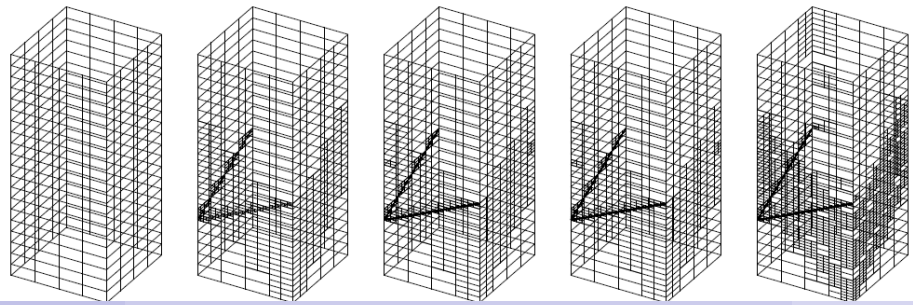
Contents

$$u : \mathbb{R}_t \times \Omega_x \to \mathbb{R} \quad \text{sound pressure}$$

$$\partial_t^2 u - \Delta u = 0 \qquad \text{in } \mathbb{R}_t \times \Omega_x^c, \quad \Omega^c = \mathbb{R}^d \setminus \overline{\Omega}$$

$$u = 0 \qquad \text{for } t \leq 0 \ .$$

- Screen problems:
  singular expansions / approximation on graded meshes for edges and corners
- a posteriori error estimates $\Rightarrow$ space adaptivity / towards space-time adaptivity
- approximation properties of p-version



BEM for the wave equation

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simple: Dirichlet boundary conditions $u = f$ on $\Gamma = \partial\Omega$:

$$\rightsquigarrow \quad f(t,x) = \mathcal{V}\phi(t,x) = \int_\Gamma \frac{\phi(y, t - |x-y|)}{4\pi|x-y|}\, ds_y$$

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Function spaces: space-time Sobolev spaces

$$f = \mathcal{V}\phi(t,x) = \int_\Gamma \frac{\phi(y, t-|x-y|)}{4\pi|x-y|}\, ds_y$$

space–time anisotropic Sobolev spaces $H^r_\sigma(\mathbb{R}^+, H^s(\Gamma))$, $\sigma > 0$:

$H^r_\sigma(\mathbb{R}^+, H^s(\mathbb{R}^2))$ defined using Fourier–Laplace transform

$$\left\{\psi : \text{supp } \psi \subset \overline{\mathbb{R}_+} \times \mathbb{R}^2,\ \int_{\mathbb{R}+i\sigma} d\omega \int_{\mathbb{R}^2} d\xi |\omega|^{2r}(|\omega|^2+|\xi|^2)^s |\mathcal{F}\psi(\omega,\xi)|^2 < \infty \right.$$

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Space-time variational formulation of Dirichlet problem:
Find $\phi \in H^1(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))$ such that $\forall \psi \in H^1(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))$:

$$\langle \mathcal{V}\partial_t \phi, \psi\rangle = \langle \partial_t f, \psi\rangle$$

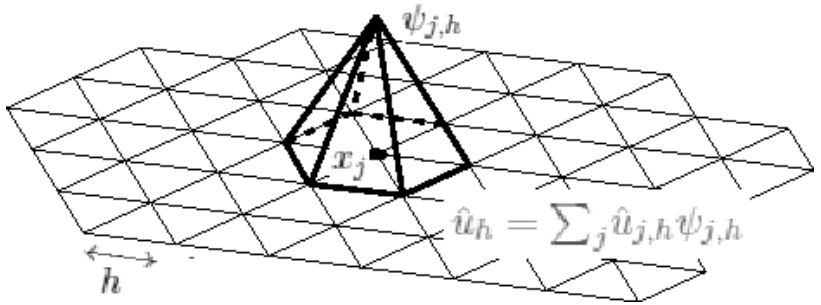
The solution ϕ exists for $f \in H^2(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma))$.

Key: $\mathcal{V}\partial_t$ coercive with loss (Bamberger – Ha Duong '86)

$$\|\phi\|^2_{1,-\frac{1}{2},\Gamma} \gtrsim \langle \mathcal{V}\partial_t\phi, \phi\rangle \gtrsim \|\phi\|^2_{0,-\frac{1}{2},\Gamma}$$

Discretization

- $\Gamma = \cup_{i=1}^{M} \Gamma_i$ (quasi-uniform) triangulation
- V_h^p piecewise polynomial functions of degree p on $\Gamma = \cup_{i=1}^{M} \Gamma_i$ (continuous if $p \geq 1$)
- $[0, T) = \cup_{n=1}^{L} [t_{n-1}, t_n)$, $t_n = n(\Delta t)$
- $V_{\Delta t}^q$ piecewise polynomial functions of degree q in time (continuous and vanishing at $t = 0$ if $q \geq 1$)
- tensor products in space-time: $V_{h,\Delta t}^{p,q} = V_h^p \otimes V_{\Delta t}^q$



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Time domain BEM: Find $\phi_{h,\Delta t} \in V_{h,\Delta t}^{p,q}$ such that $\forall \psi_{h,\Delta t} \in V_{h,\Delta t}^{p,q}$:

$$\langle \mathcal{V} \partial_t \phi_{h,\Delta t}, \psi_{h,\Delta t} \rangle = \langle \partial_t f, \psi_{h,\Delta t} \rangle$$

Solve by (preconditioned) time stepping scheme (HG – Stark '18).

Screen problems: Edge and corner singularities

$\Omega^c = \mathbb{R}^3 \setminus ([-1,1]^2 \times \{0\})$, $\mathcal{V}\phi = \sin(t)^5$ on $[-1,1]^2 \times \{0\}$.

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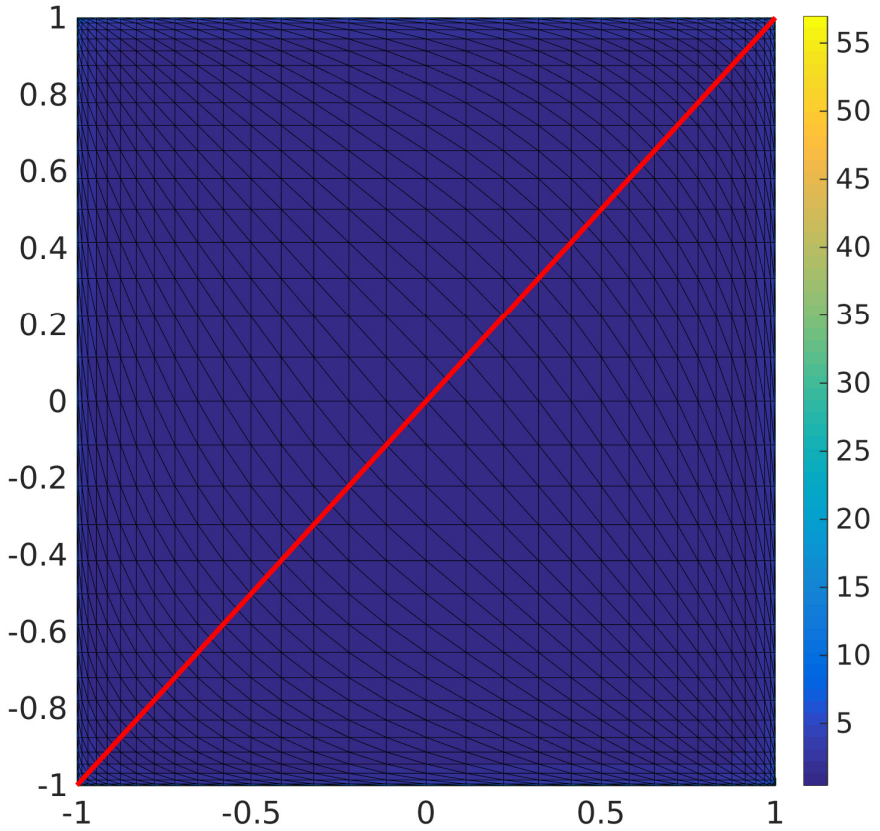

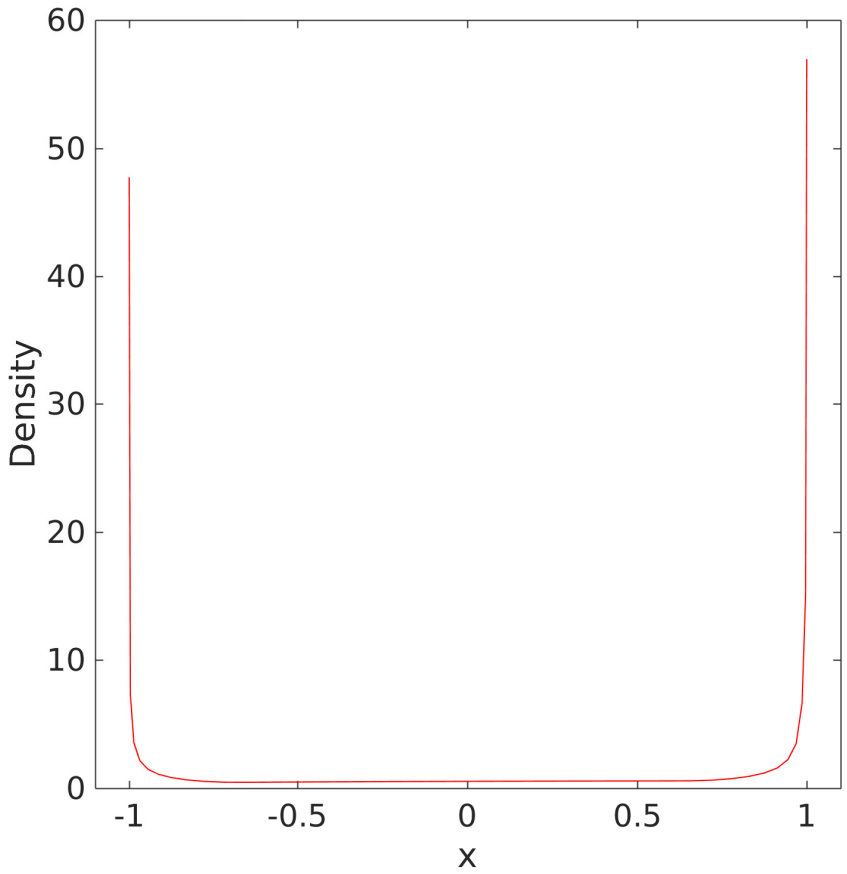

Screen problems: edge and corner singularities

Helmholtz: (Kondratiev, Dauge, ...)
Solution behaves like

- $r^{\gamma-1}$ near corner, γ=0.29 for square screen
- $r^{-\frac{1}{2}}$ near edge.

von Petersdorff '89, +Stephan '90: precise tensor product decomposition, BEM on graded meshes
$\Longrightarrow$ optimal approximation on graded meshes.

Screen problems: edge and corner singularities

Theorem ($r^{\gamma-1}$ in corner, $r^{-\frac{1}{2}}$ at edges, coeffs depend on ω)

Let $\mathcal{V}_\omega\psi_\omega = f_\omega \in H^2(\Gamma)$. Then

$$\psi_\omega = \psi_{0,\omega} + \chi_\omega(r)r^{\gamma-1}\alpha_\omega(\theta) + \tilde{\chi}_\omega(\theta)b_{1,\omega}(r)r^{-1}(\sin(\theta))^{-\frac{1}{2}}$$
$$+ \tilde{\chi}_\omega(\frac{\pi}{2}-\theta)b_{2,\omega}(r)r^{-1}(\cos(\theta))^{-\frac{1}{2}}$$

where $\psi_{0,\omega} \in H^{1-}(\Gamma)$, $\alpha_\omega(\theta) \in H^{1-}[0,\frac{\pi}{2}]$, $b_{i,\omega} = c_{i,\omega}r^\gamma + d_{i,\omega}(r)$, $r^{-\frac{1}{2}}d_{i,\omega}(r) \in H^1(\mathbb{R}^+)$, $r^{-\frac{3}{2}}d_{i,\omega}(r) \in L_2(\mathbb{R}^+)$, $c_{i,\omega} \in \mathbb{R}$.
(r,θ) polar coordinates around (0,0), χ_ω, $\tilde{\chi}_\omega \in C_c^\infty$, $= 1$ near 0.

γ eigenvalue: $\gamma \approx 0.2966$ for square

Screen problems: edge and corner singularities

Previous work on wave equation:

- Plamenevskii et al. since '99:
  singular expansions near corners and edges of polygon
- Müller – Schwab '15: $2d$ FEM on graded meshes

Theorem

a) Decomposition of solution in singular / regular parts with same singular exponents $\gamma - 1, -\frac{1}{2}$ as in elliptic case. ($\sim$ Plamenevskii)

b) Error of best approximation in $H^0_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) = \mathcal{O}(h^{\min\{\frac{\beta}{2}, \frac{3}{2}\}-\varepsilon})$.

Screen problems: edge and corner singularities

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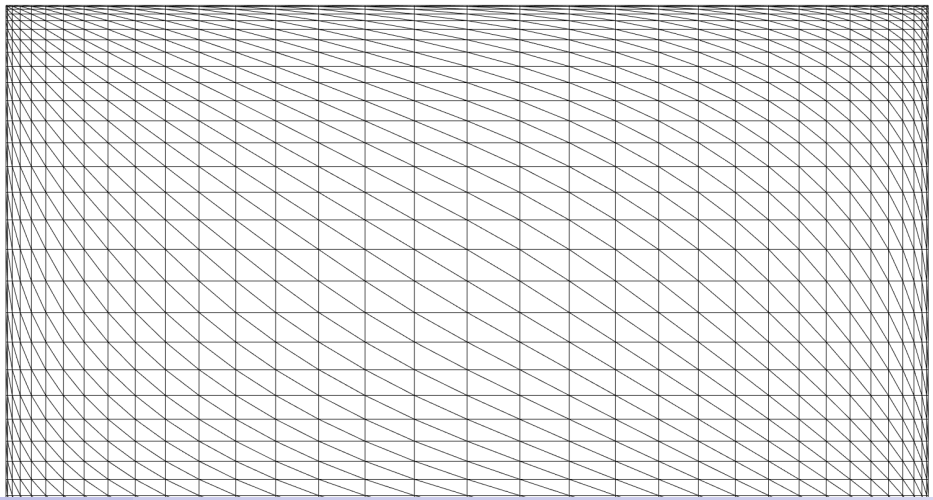
$$x_j = 1 - \left(\frac{j}{N}\right)^\beta, \ j = 1, \ldots, N \ .$$

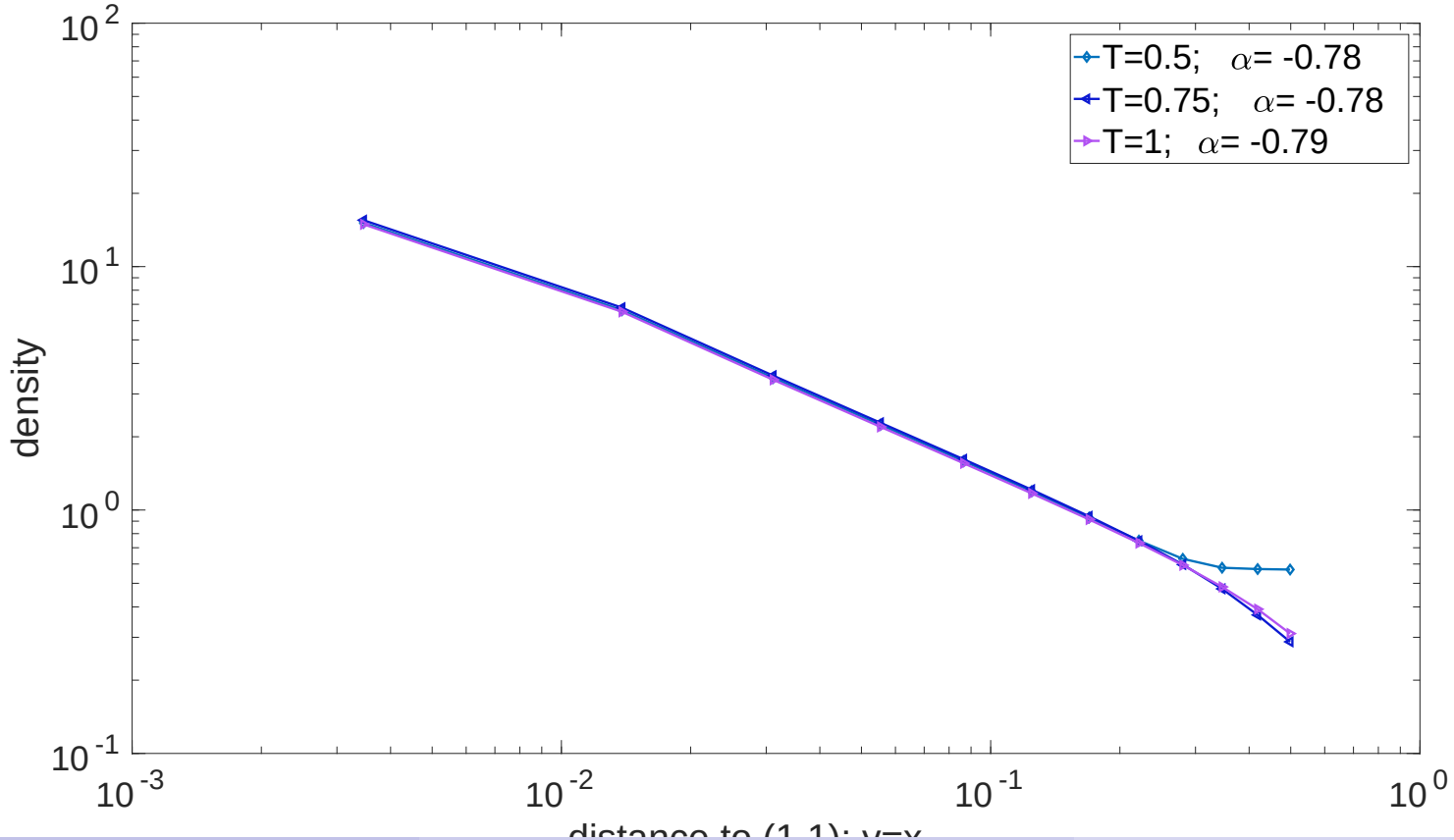


Screen problems: Corner exponents for waves

$\Omega^c = \mathbb{R}^3 \setminus ([-1,1]^2 \times \{0\})$, $\mathcal{V}\phi = \sin(t)^5$ on $[-1,1]^2 \times \{0\}$, $0 < t < 1$.

corner exponent: $-0.78 \sim \gamma - 1 = -0.703$ as in elliptic case

Plot: $\phi(t,r)$ as function of r along $x = y$

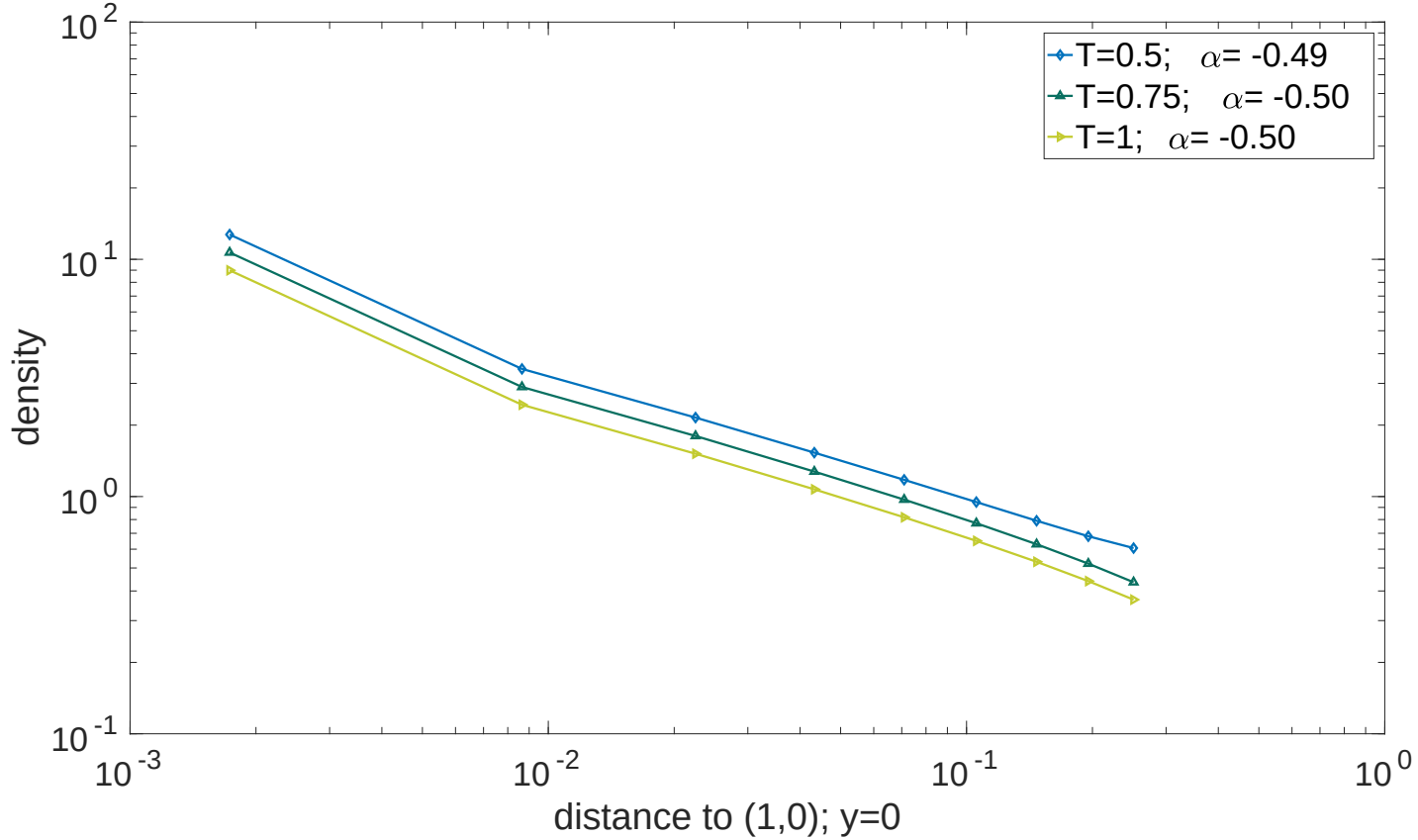


Screen problems: Edge exponents for waves

$\Omega^c = \mathbb{R}^3 \setminus ([-1,1]^2 \times \{0\})$, $\mathcal{V}\phi = \sin(t)^5$ on $[-1,1]^2 \times \{0\}$, $0 < t < 1$.

edge exponent: $-0.49 \sim -\frac{1}{2}$ as in elliptic case

Plot: $\phi(t,x,y)$ as function of y, at $x = 0.8754$



Screen problems: Convergence rates

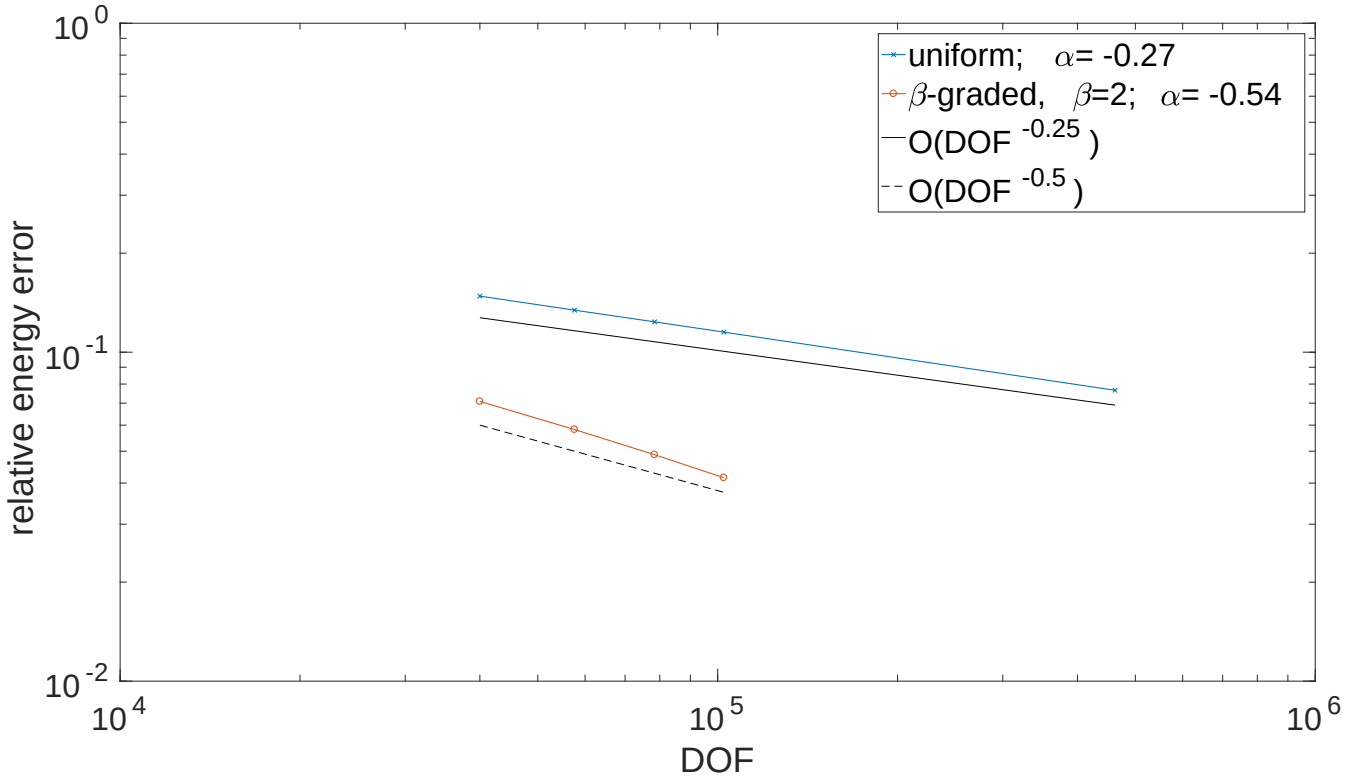
$\Omega^c = \mathbb{R}^3 \setminus ([-1,1]^2 \times \{0\})$, $\mathcal{V}\phi = \sin(t)^5$ on $[-1,1]^2 \times \{0\}$, $0 < t < 1$.

Convergence for fixed $\Delta t = 0.01$:

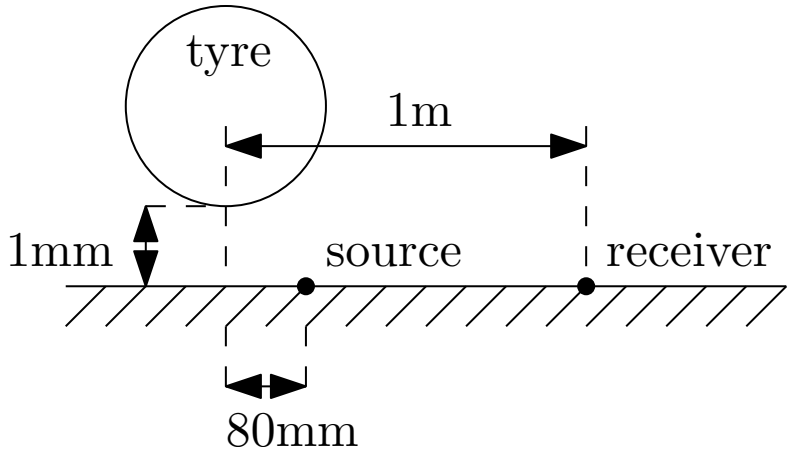
Energy norm^2 $= \langle \mathcal{V}\partial_t(\phi_h - \phi), \phi_h - \phi \rangle \sim h^2 \simeq DOF(\Gamma)^{-1}$ (2-graded)

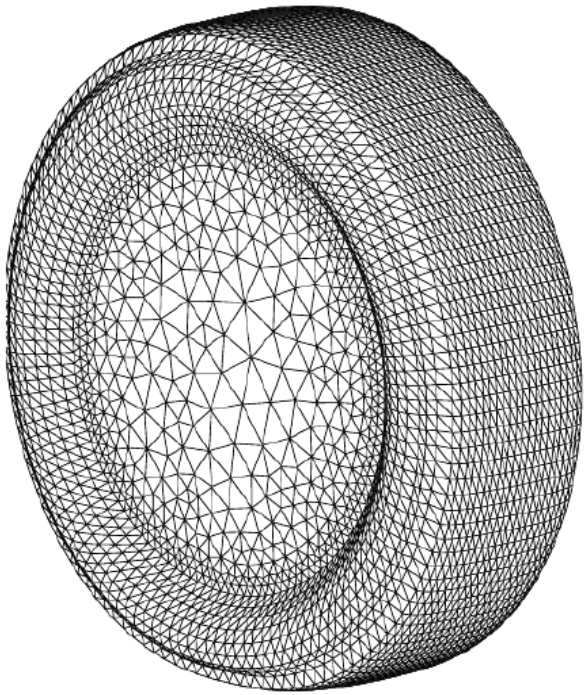
$\sim h \simeq DOF(\Gamma)^{-1/2}$ (uniform)

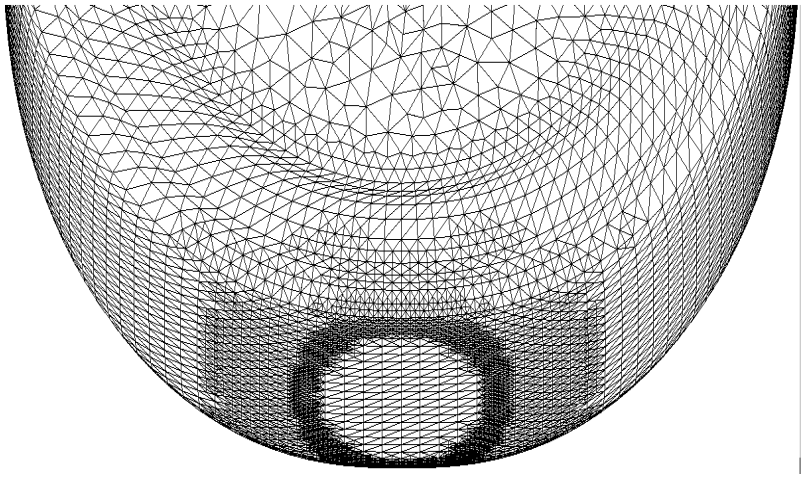
similar results for W and for Dirichlet-to-Neumann operator



Real World: Graded meshes for tyres







Error Indicators

Theorem

Let $\phi_{h,\Delta t} \in H^0_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))$ such that
$\mathcal{R} = \partial_t\ f - \mathcal{V}\partial_t\phi_{h,\Delta t} \in H^0_\sigma(\mathbb{R}^+, H^1(\Gamma)) \Longrightarrow$

$$\|\phi - \phi_{h,\Delta t}\|^2_{0,-\frac{1}{2}} \lesssim \sum_{i,\Delta} \max\{\Delta t, h_\Delta\}\ \|\mathcal{R}\|^2_{0,1,[t_i,t_{i+1})\times\Delta}$$

$$\max\{\Delta t, h\}\|\mathcal{R}\|^2_{0,1-\epsilon} \lesssim \|\phi - \phi_{h,\Delta t}\|^2_{2,-\frac{1}{2}}$$

The upper bound is independent of the approximation method: TDBEM, convolution quadrature, no assumption on mesh.

The lower bound holds on quasi-uniform meshes.

Error Indicators

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$$\max\{\Delta t, h\}\|\mathcal{R}\|^2_{0,1-\epsilon} \lesssim \|\phi - \phi_{h,\Delta t}\|^2_{2,-\frac{1}{2}}$$

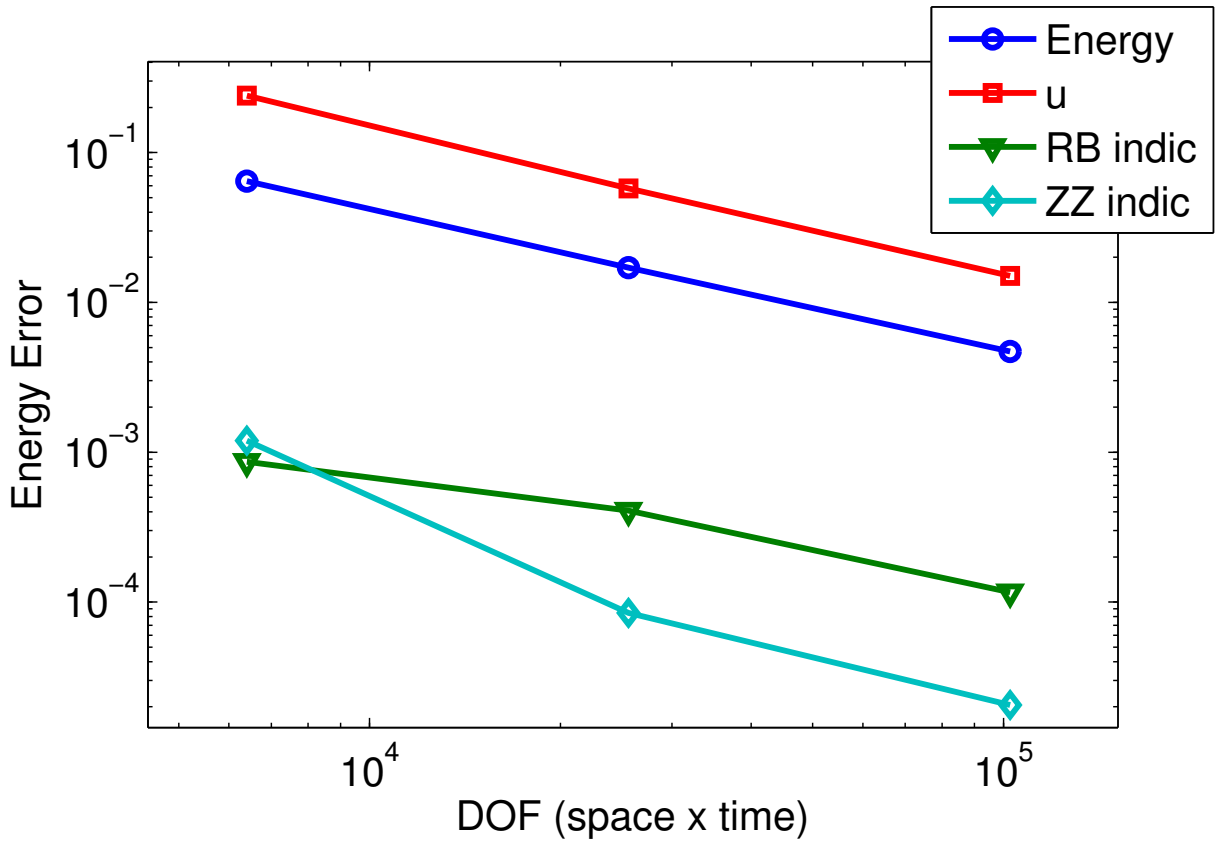
Residual error indicators:

$$\eta^2(\Delta, i) = \max\{\Delta t, h_\Delta\}\ \|\mathcal{R}\|^2_{0,1,[t_i,t_{i+1})\times\Delta}$$

Adaptivity: Spherical Harmonic on the Uniform Sphere

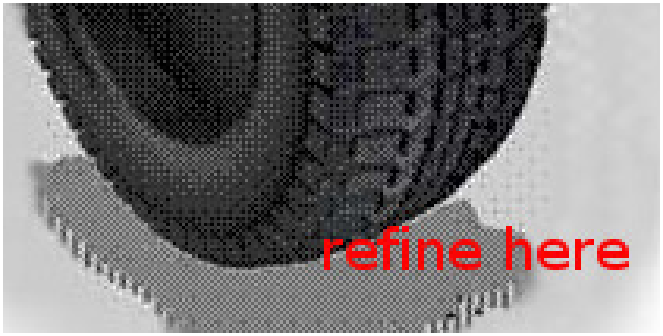
$f(t, \mathbf{x}) = \sin^5(t) z^2$ on $\Gamma = \{x, y, z \mid x^2 + y^2 + z^2 = 1\}$, $0 < t < 2.5$.

We consider residual and ZZ indicators on a uniform series of meshes. Compare to error in energy norm and sound pressure (with respect to benchmark).



- Efficient: Indicators scale like error in energy norm.

A First Adaptive Method: Singular Geometry

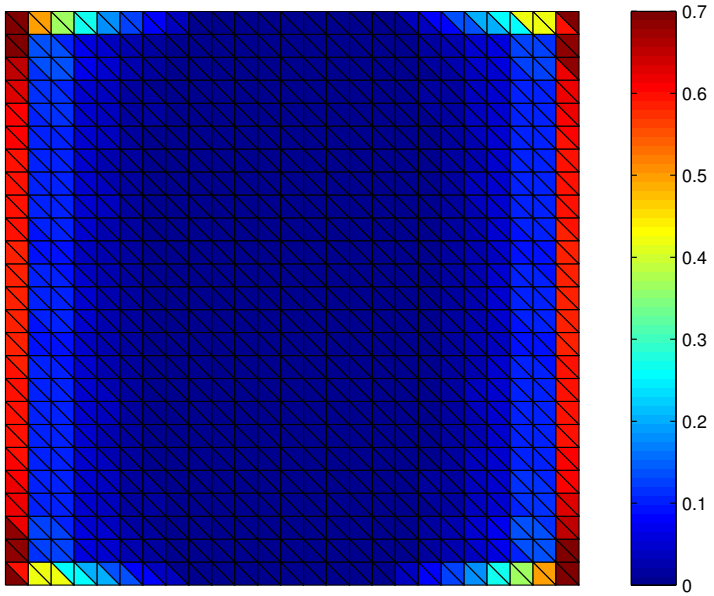


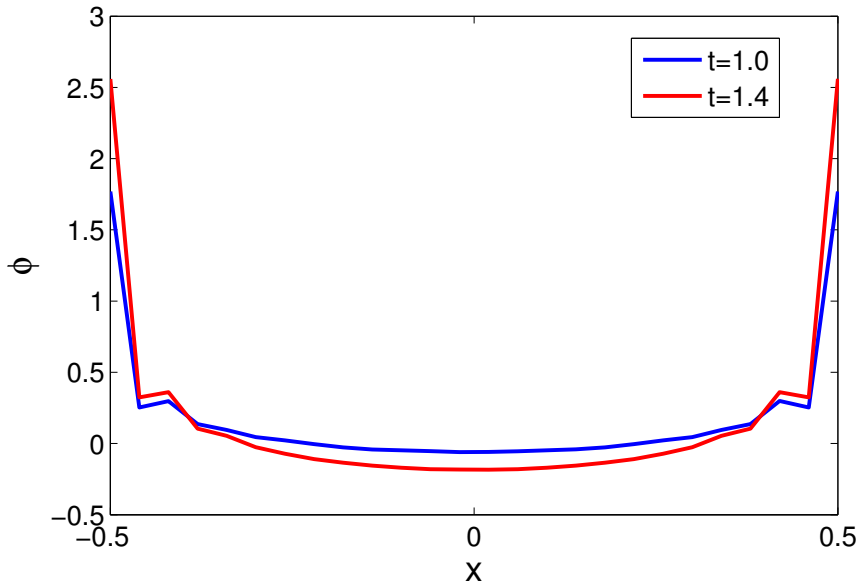
1. Start with coarse space-time grid: $(\Delta t)_i \simeq (\Delta x)_i \simeq h_0 \ \forall \Delta_i$
2. Solve discretisation of $\mathcal{V}\dot{\phi} = \dot{f}$.
3. Compute time-integrated error indicator $\eta(\Delta_i)$
4. $\sum_i \eta(\Delta_i) < \varepsilon \implies$ STOP
5. $\eta(\Delta_i) > \delta\eta_{max} \implies \Delta_i \to \Delta/4,\ (\Delta t)_i \to \frac{(\Delta t)_i}{2}$
6. GO TO 2.

Adaptivity: Wave Scattering on the Screen

$\mathcal{V}\phi = \sin^5(t)x^2$ on $\Gamma = [-0.5, 0.5]^2 \times \{z = 0\}$, $0 < t < 2.5$, $\Delta t = 0.1$.

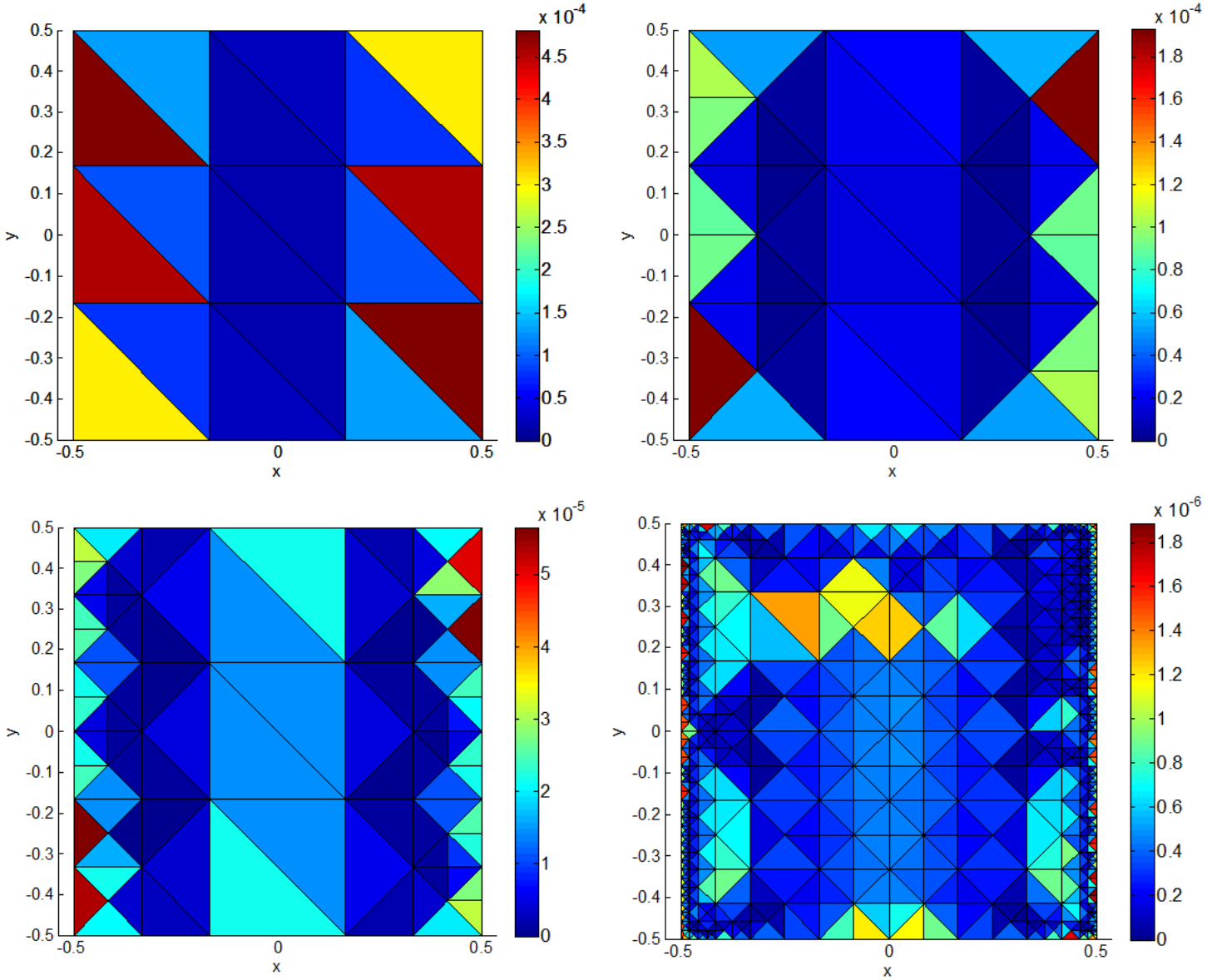
Compare residual indicators, energy, and sound pressure for uniform / adaptive mesh refinements.





- Uniform method: Density ϕ at $t = 1.0, 1.4$

Adaptivity: Wave Scattering on the Screen



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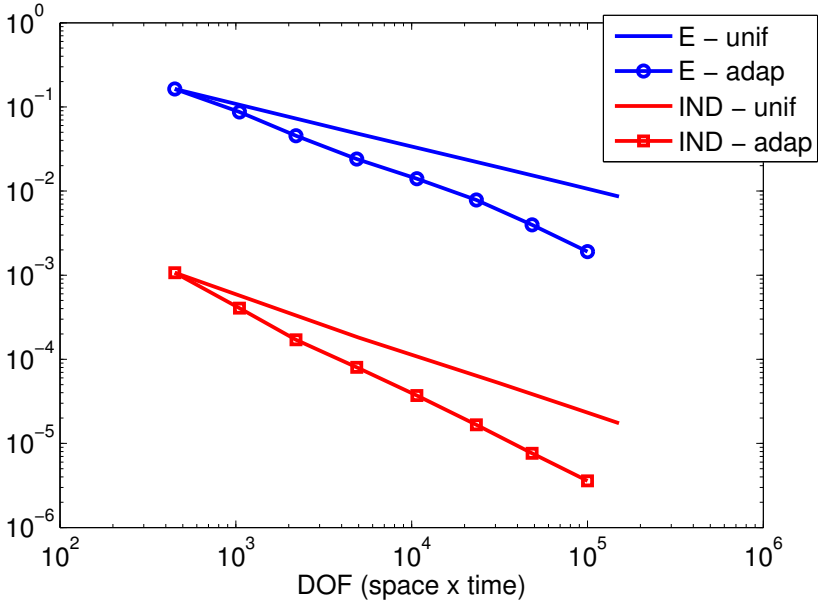
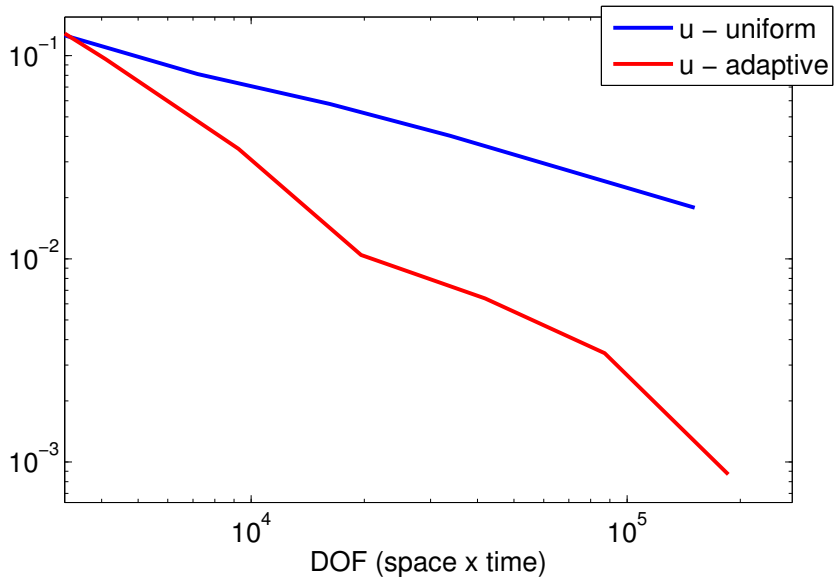


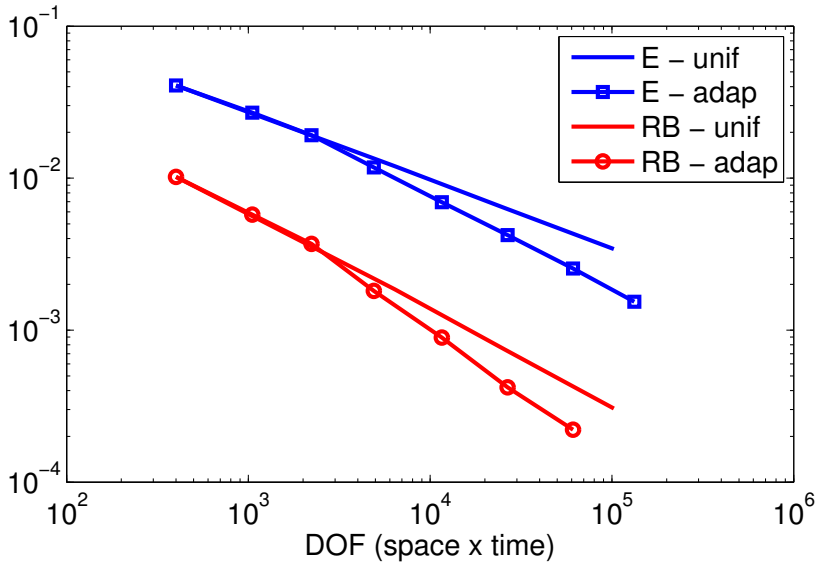


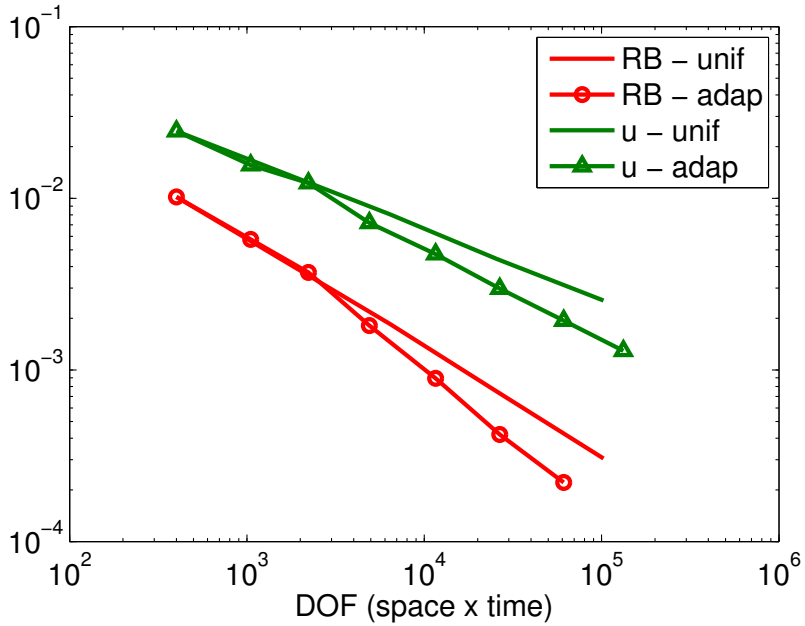
- Convergence rate 0.5 (uniform), 0.77 adaptive reproduces rates for time-independent BEM.

Adaptivity: Wave Scattering on Triangular Screen

$\mathcal{V}\phi = \sin^5(t)$ on $\Gamma = 30 - 60 - 90$ Triangle, $0 < t < 2.5$.

Compare residual indicators, energy, and sound pressure for uniform / adaptive mesh refinements.





- Convergence rate 0.45 (uniform), 0.65 adaptive.

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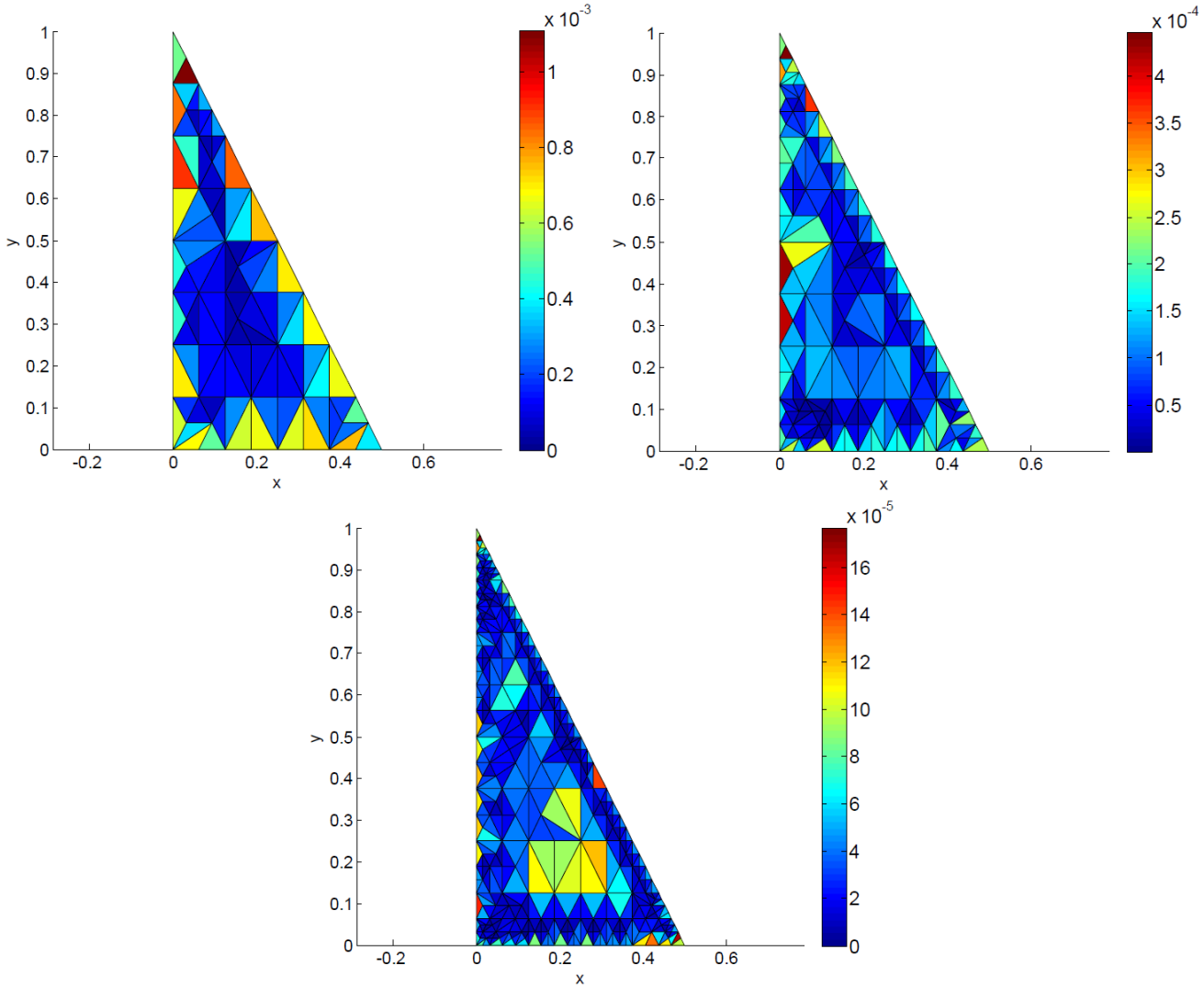
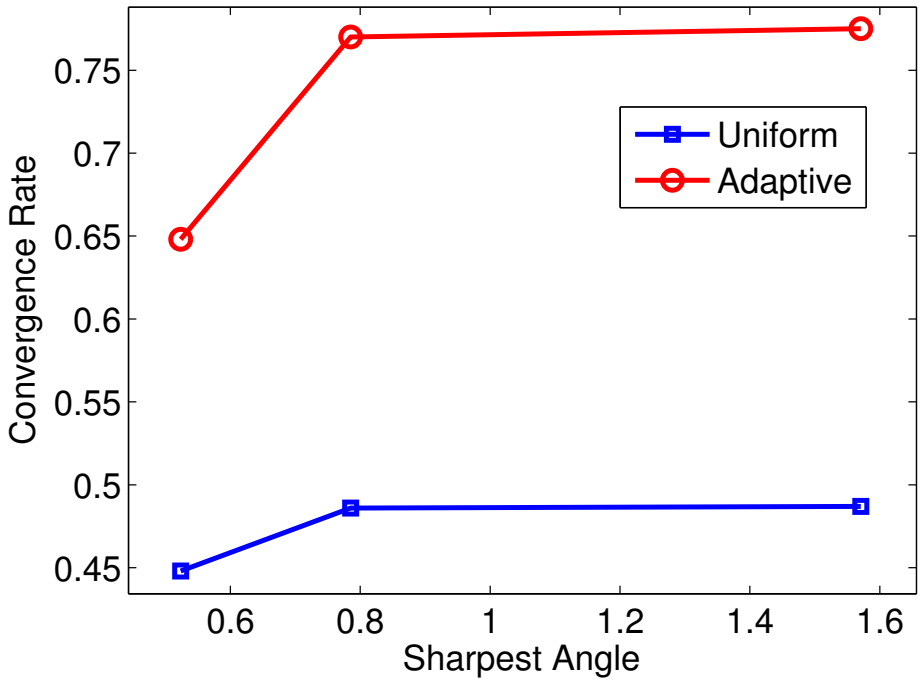


Adaptivity on Screens

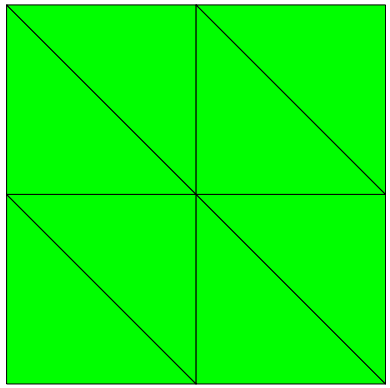
$\mathcal{V}\phi = F$ on $\Gamma =$ Polygonal Screen, $0 < t < 2.5$.



Convergence rate dominated by edge singularity.
Apparent rate decay for small angles due to preasymptotic regime?

p-version TDBEM

Fixed coarse mesh, increase polynomial degree p:



Theorem

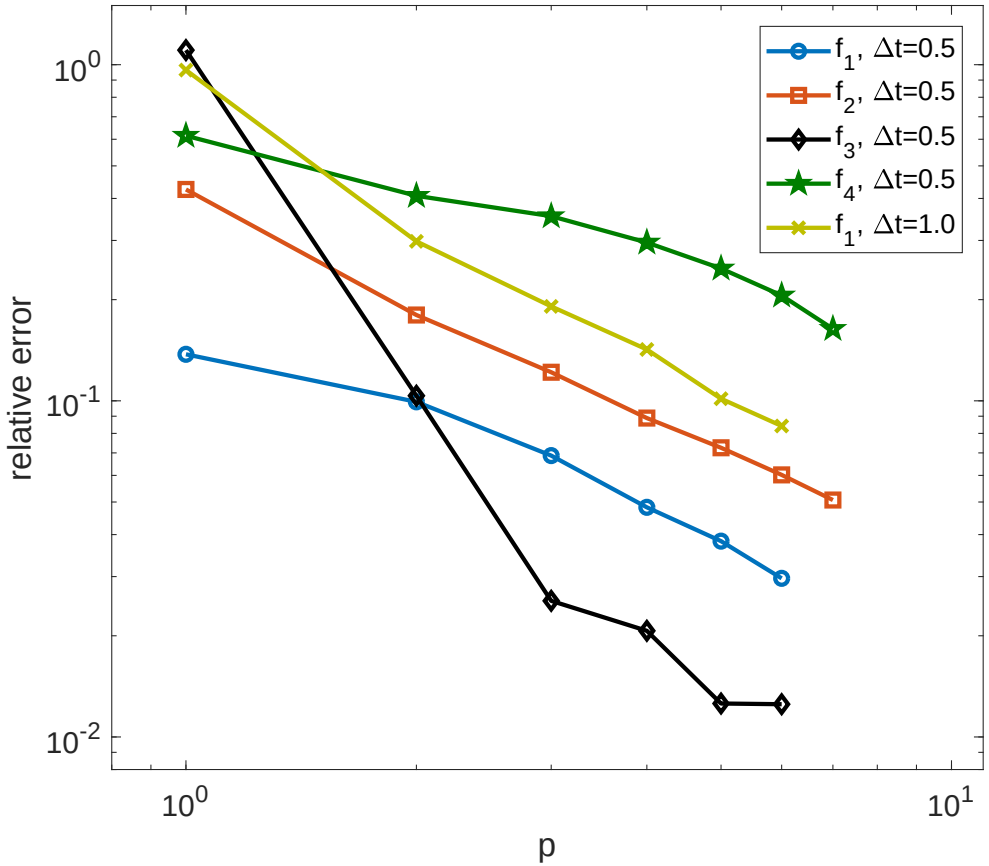
- ϕ solution of $V\phi = f$
- $\phi_{h,\Delta t} \in V_{h,\Delta t}^{p,p}$ best approximation $H_\sigma^r(\mathbb{R}^+, \widetilde{H}^{-\frac{1}{2}}(\Gamma))$ norm to ϕ

Then $\forall \varepsilon > 0$

$$\|\phi - \phi_{h,\Delta t}\|_{r,-\frac{1}{2},\Gamma,*} \leq C_\varepsilon h^{1/2-\varepsilon} p^{-1+\varepsilon} \ .$$

time-independent problems: Babuska-Suri, Dorr, Stephan-Suri, Bespalov-Heuer, ...

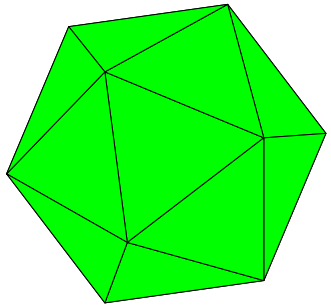
p-version TDBEM



$\Gamma = \{(x, y, 0) : -\frac{1}{2} \leq x, y \leq -\frac{1}{2}\}$, $\Delta t = 0.5$, different right hand sides

Convergence rate $= 1$, twice the rate of h-version.

p-version TDBEM in polyhedral domains



Theorem ($r^{\gamma-1}$ in corner, $r^{\frac{\pi}{\alpha}-1}$ at edges, coeffs depend on ω)

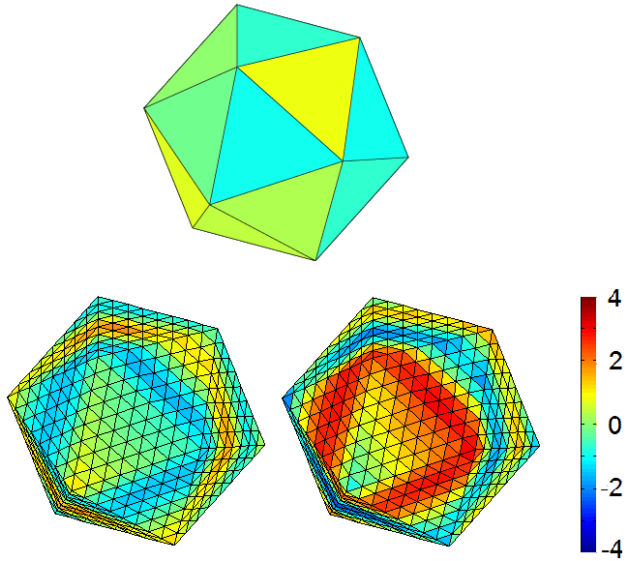
Let $\mathcal{V}_\omega \psi_\omega = f_\omega \in H^2(\Gamma)$. Then

$$\psi_\omega = \psi_{0,\omega} + \chi_\omega(r) r^{\gamma-1} \alpha_\omega(\theta) + \tilde{\chi}_\omega(\theta) b_{1,\omega}(r) r^{-1} (\sin(\theta))^{\frac{\pi}{\alpha}-1}$$
$$+ \tilde{\chi}_\omega(\frac{\pi}{2} - \theta) b_{2,\omega}(r) r^{-1} (\cos(\theta))^{\frac{\pi}{\alpha}-1}$$

(r, θ) polar coordinates around (0,0), $\chi_\omega,\ \tilde{\chi}_\omega \in C_c^\infty$, $= 1$ near 0.
α opening angle of wedge

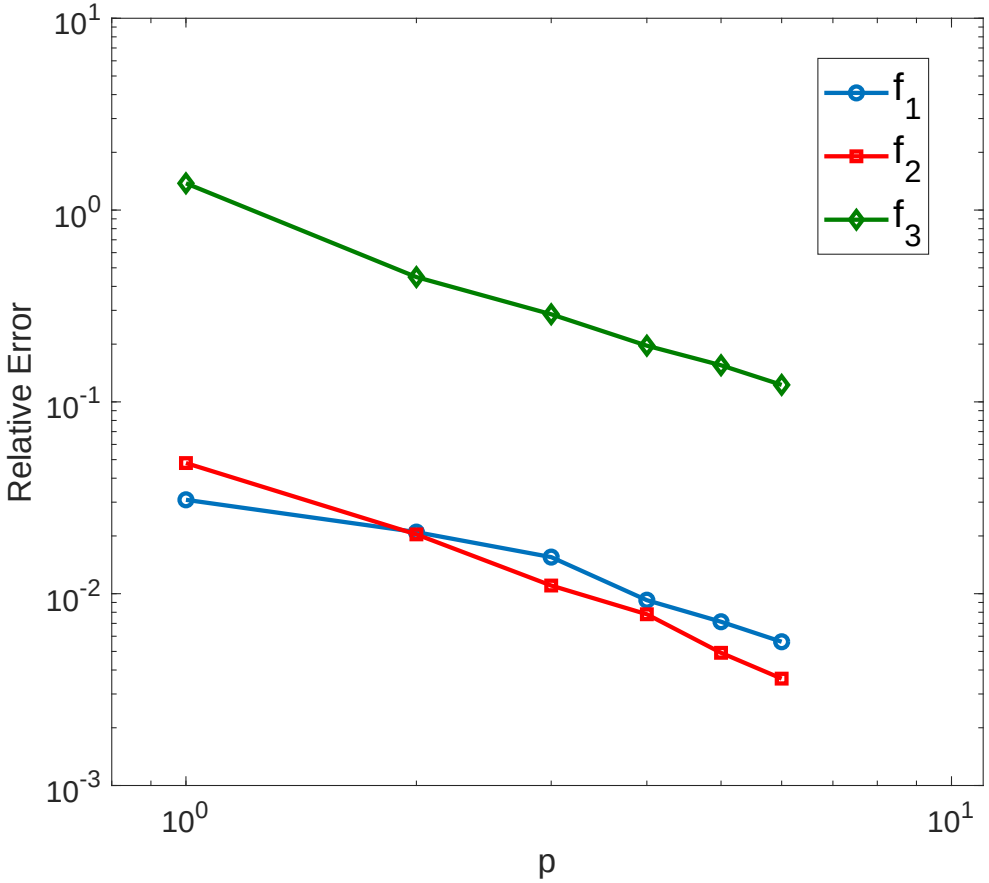
TD asymptotics + smoothness estimates for remainder: convergence rates

p-version TDBEM in polyhedral domains



$\Gamma =$ icosahedron, $\Delta t = 0.5$, different right hand sides

p-version TDBEM in polyhedral domains



$\Gamma =$ icosahedron, $\Delta t = 0.5$, different right hand sides

Convergence rate ~ 1.62 predicted by theory

Conclusions

Analysis & numerics: convergence rates for screen problems
(in DOF on a 2d screen, energy norm)

- 0.5: h-version, uniform
- 0.77: h-version, adaptive (as for Δ)
- 1.0: p-version, uniform
- $\beta/2$: h-version, β-graded

- Singular expansions & (optimal) graded meshes for edges and corners
- A posteriori estimate for numerical approximations without assumptions on mesh (TDBEM, CQ, ...)
- Analysis of p-version on polyhedra

HG, D. Stark, E. P. Stephan, hp-version time-domain boundary elements for the wave equation on quasiuniform meshes, preprint.

HG, C. Özdemir, D. Stark, E. P. Stephan, A residual a posteriori error estimate for the time domain boundary element method, preprint.

HG, F. Meyer, C. Özdemir, D. Stark, E. P. Stephan, Boundary elements with mesh refinements for the wave equation, Numer. Math. 2018.