Lecture 1:

Aims of course: Differentiation + Integration as operators on (spaces of) functions

Basic tools for analysis and PDE

Examples: ODE $\rightsquigarrow$ integral operators

$x \in (0,\infty)$: $u'(x) = f(x) \rightsquigarrow u(x) = \underbrace{\int_0^x f(s)\,ds}_{\substack{\text{integral operator} \\ f \mapsto u}} + u(0)$

$u''(x) = f(x) \rightsquigarrow u(x) = \overbrace{\int_0^x \left(\int_0^s f(t)\,dt\right) ds}$

$+ u'(0)x + u(0)$

For ODEs: (a,b) open Intervall

solution $\in C^m(a,b)$ = space of m-times continuously diff. functions on (a,b)

$C_0^m(a,b) = \{ f \in C^m(a,b) : f \equiv 0 \text{ near } a \text{ and } b \}$

For PDE solutions often <u>not</u> continuously diff.: shockwaves, wave crests, cracks

<u>Weak derivatives</u>: If $u \in C^1(a,b)$ and $\varphi \in C_0^1(a,b)$:

(Integration by parts) $-\int_a^b u\,\partial_x \varphi = \int_a^b (\partial_x u)\,\varphi.$

<u>Def</u>: ∂u is any function v s.t.

$$-\int_a^b u\,\partial_x\varphi = \int_a^b v\,\varphi \qquad \forall \varphi \in C_0^\infty(a,b)$$

Sometimes concept of weak derivative not sufficient: ∂u not a function.

$\longrightarrow$ distributions + measure theory.

Examples of distributions:
- Functions in L^1_{loc}
- δ_0 Dirac delta distributions
- Radon measures
- weak derivatives of the above

This course: Understand function spaces + distributions, differential and integral operators between them, apply to classical analysis and PDE.

Applications:
- convergence of Fourier series
- analysis of big data sets
- counting prime numbers
- solving PDE

Structure:
- Weekly lectures
- Edinburgh: student talks
- independent reading? (Tao's notes)

Weak derivative: $-\int_a^b u\, \partial_x \varphi = \int_a^b v \varphi \quad \forall \varphi \in C_0^\infty(a,b)$

test functions

Question: $\underset{\text{open}}{\Omega} \subseteq \mathbb{R}^n$. Do there exist test functions?

Def: $\operatorname{supp} u = \Omega \setminus \left(\bigcup \{ \omega \subseteq \Omega \text{ open} : u|_\omega = 0 \} \right)$

support of $u \in L^1_{loc}(\Omega)$.

Lemma: Let $R > r > 0$

a) $\exists\ \chi_{r,R} \in C_0^\infty(\mathbb{R}^n)$ s.t. $\chi_{r,R}(x) = \begin{cases} 1 & \forall |x| \leq r \\ 0 & \forall |x| \geq R \\ \in [0,1] & \forall r \leq |x| \leq R \end{cases}$

b) $C_0^\infty(\Omega) \neq \{0\}$.

c) $\exists\ h \in C_0^\infty(\mathbb{R}^n)$ s.t. $\operatorname{supp} h = \overline{B(0,1)}$, $h(x) > 0\ \forall |x| < 1$, $\int h(x)\, dx = 1$.

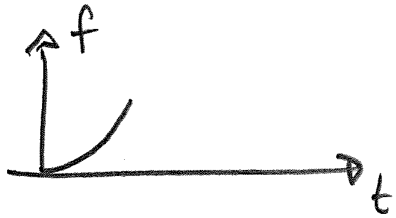
Proof: a) $f(t) = \begin{cases} e^{-1/t} & , t>0 \\ 0 & \text{else} \end{cases} \quad \in C^\infty(\mathbb{R})$

obvious for $t \neq 0$.

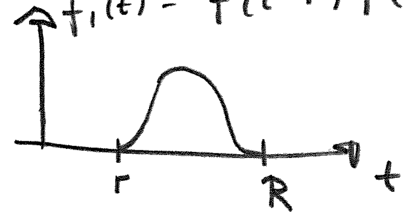
At $t=0$: $f(t) \to 0$ as $t \downarrow 0$

$$\partial_t^k f(t) = \begin{cases} \text{Polynomial}_k(1/t)\, e^{-1/t} & , t>0 \\ 0 & , t<0 \end{cases}$$

$\to \partial_t^k f$ continuous at 0

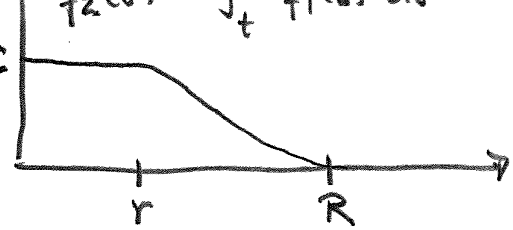


$\longrightarrow$ $f_1(t) = f(t-r)\, f(R-t)$



$f_2(t) = \int_t^\infty f_1(s)\, ds$

$\int_r^R f_1(s)\, ds = C$



Set $\chi_{r,R}(x) = \frac{1}{C} f_2(|x|)$,
for $x \in \mathbb{R}^n$.

b) $\Omega \subseteq \mathbb{R}^n$ open $\to$ contains open ball around $x_0 \in \Omega$

$\to \chi_{r,R}(x - x_0) \in C_0^\infty(\Omega)$ for small r, R.

c) Take $h(x) = \dfrac{\chi_{1/2,1}(x)}{\int \chi_{1/2,1}(s)\, ds}$ $\boxed{\xi}$

For later reference: χ denotes a function (cut-off function)

s.t. $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 2 \\ \in [0,1] & \text{for } |x| \in [1,2] \end{cases}$

The proof used (in b)): <u>Extension by 0</u>

$$\varphi \in C_0^\infty(\Omega) \longrightarrow \tilde{\varphi} \in C_0^\infty(\mathbb{R}^n)$$

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x) & , x \in \Omega \\ 0 & \text{else} \end{cases}$$

<u>Lemma:</u> $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ open $\implies \exists$ sequence $K_j \subset \Omega$ compact

- $K_1 \subset \mathring{K}_2 \subset K_2 \subset \mathring{K}_3 \subset K_3 \subset \dots$
- $\bigcup_j \mathring{K}_j = \Omega$

<u>Proof:</u> Set $K_j = \{ x \in \Omega : |x| \leq j \text{ and } \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega) \geq \frac{1}{j} \}$

$\boxed{\equiv}$

<u>Seminorms on $C_0^\infty(\mathbb{R}^n)$:</u> For $k, j \in \mathbb{N}$:

$$p_{k,j}(f) = \sup \{ \partial^\alpha f(x) : |\alpha| \leq k \text{ and } x \in K_j \}$$

Here we use $\partial^\alpha f(x) = \dfrac{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n} f(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$

$\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$.

<u>Definition / Theorem:</u> a) A sequence $\varphi_\ell \to \varphi_0$ in $C_0^\infty(\Omega)$

$\iff \exists j \in \mathbb{N} \, \forall \ell : \operatorname{supp} \varphi_\ell \subseteq K_j$

and $p_{k,j}(\varphi_\ell - \varphi_0) \xrightarrow{\ell \to \infty} 0 \quad \forall k$.

b) A linear map $\Lambda : C_0^\infty(\Omega) \to \mathbb{C}$ <u>continuous</u>

$\iff \forall j \, \exists N_j \, \exists c_j > 0 \, \forall \varphi \in C_0^\infty(\Omega)$ with $\operatorname{supp} \varphi \subseteq K_j$:

$|\Lambda(\varphi)| \leq c_j \, p_{N_j, j}(\varphi)$

c) A linear map $T: C_0^\infty(\Omega) \to C_0^\infty(\Omega)$ is <u>continuous</u> $\Leftrightarrow$ $\forall n,k \; \exists m \; \exists C$

$$p_{ke}(T\varphi) \le C \, p_{mn}(\varphi)$$

<u>Remark</u>: $C_0^\infty(\Omega) \subseteq C^\infty(\Omega)$

$C_0^\infty(\Omega) \subseteq L^p(\Omega) \quad \forall p \in [1,\infty]$

<u>Examples</u>: a) Dirac delta: $\delta_{x_0}(u) = u(x_0)$ point evaluation

b) $\partial^\alpha : C_0^\infty \to C_0^\infty$

c) $M_f(u) = fu$ multiplication by $f \in C^\infty(\Omega)$

d) $\sum_\alpha f_\alpha \partial^\alpha$ Differential operators

all continuous operations from $C_0^\infty(\Omega)$ to a) $\mathbb{C}$ b,c,d) C_0^∞.

<u>Def</u>: A <u>distribution on Ω</u> is a continuous linear map

$$C_0^\infty(\Omega) \to \mathbb{C},$$

The vector space of distributions on Ω is denoted by

$$\mathcal{D}'(\Omega).$$

<u>Notation</u>: $\Lambda(\varphi) = \langle \Lambda, \varphi \rangle$

<u>Examples</u>:

- $f \in L^1_{loc}(\Omega) \to \Lambda_f(\varphi) = \int_\Omega f(x)\varphi(x)\,dx$

  $\Lambda_f \in \mathcal{D}'(\Omega)$
- Radon measures: $\Lambda_\mu(\varphi) = \int_\Omega \varphi(x)\,d\mu(x)$
- δ_{x_0}, $\delta_{x_0} \circ M_f \circ \partial^\alpha \circ M_g$

Lecture 2:

Exercise: $\Lambda \in \mathcal{D}'(\Omega) \iff \forall \varphi_n \to \varphi_0$ in $C_0^\infty(\Omega)$:

$\Lambda(\varphi_n) \to \Lambda(\varphi_0)$

Examples of distributions included

$$L^1_{loc}(\Omega) \ni f \longmapsto \Lambda_f(\varphi) = \int_\Omega f(x)\,\varphi(x)\,dx$$

Lemma: If $\Lambda_f = \Lambda_g \Rightarrow f = g$ a.e.

In particular, $f \longmapsto \Lambda_f$ is injective from L^1_{loc} to $\mathcal{D}'$

Proof: It is enough to show this for $f=0$.

i.e. if $\forall \varphi \in C_0^\infty(\Omega)$: $\int_\Omega f(x)\,\varphi(x)\,dx = 0$, then $f=0$ a.e.

To show this, Let $\phi \in C^\infty(\mathbb{R})$, $\phi(t) = \begin{cases} e^{-1/t}, & t>0 \\ 0 & \text{else} \end{cases}$

$\Omega \subset \mathbb{R}$: $\phi_{\lambda,a,b}(x) = \phi(\lambda(b-x)(x-a)) \in C_0^\infty(\mathbb{R})$

$\phi \to \chi_{(a,b)}$ in $L^1(\mathbb{R})$.

If $0 = \int_{\mathbb{R}} f(x)\,\phi_{\lambda,a,b}(x)\,dx \xrightarrow[\lambda\to\infty]{} \int_a^b f(x)\,dx$

by dominated convergence

for all $(a,b) \subseteq \Omega$.

The theory of Lebesgue integration $\Rightarrow f=0$.

$\Omega \subset \mathbb{R}^n$: $\phi_\lambda(x_1, \ldots, x_n) = \prod_{i=1}^n \phi_{\lambda,a_i,b_i}(x_i) \longrightarrow \chi_{\prod_{i=1}^n (a_i,b_i)}$

$$0 = \int_{\mathbb{R}^n} f\,\phi_\lambda \xrightarrow[\lambda\to\infty]{} \int_{\prod(a_i,b_i)} f\,dx \quad \forall \prod_{i=1}^n (a_i,b_i)$$

$\Longrightarrow f=0$ $\square$

In the following we will therefore identify f with Λ_f and treat $L^1_{loc}(\Omega)$ as a subspace of $\mathcal{D}'(\Omega)$.

I.e. functions $\subseteq$ distributions.

<u>Operations</u>: • $u, v \in \mathcal{D}'(\Omega)$, $a, b \in \mathbb{C} \Rightarrow au + bv \in \mathcal{D}'(\Omega)$.

$\Rightarrow$ vector space

• $$\left\langle \frac{\partial u}{\partial x_i}, \varphi \right\rangle := -\left\langle u, \frac{\partial \varphi}{\partial x_i} \right\rangle \quad \text{(int by parts)}$$

$$\longrightarrow \quad \frac{\partial u}{\partial x_i} \in \mathcal{D}'(\Omega)$$

<u>Check</u>: If $f \in C^1(\Omega)$, then $\frac{\partial f}{\partial x_i}$ coincides with the usual derivative:

From integration by parts:

$$\left\langle \Lambda_{\frac{\partial f}{\partial x_i}}, \varphi \right\rangle = \left\langle \frac{\partial}{\partial x_i} \Lambda_f, \varphi \right\rangle, \text{ because}$$

$$\left\langle \Lambda_{\frac{\partial f}{\partial x_i}}, \varphi \right\rangle = \int_\Omega \frac{\partial f}{\partial x_i} \varphi \, dx \overset{\text{int. by parts}}{=} -\int_\Omega f \frac{\partial \varphi}{\partial x_i} \, dx$$

$$= -\left\langle \Lambda_f, \frac{\partial \varphi}{\partial x_i} \right\rangle \equiv \left\langle \frac{\partial}{\partial x_i} \Lambda_f, \varphi \right\rangle$$

Then one defines $\left\langle \frac{\partial^\alpha u}{\partial x^\alpha}, \varphi \right\rangle = (-)^{|\alpha|} \left\langle u, \frac{\partial^\alpha \varphi}{\partial x^\alpha} \right\rangle$.

<u>Exercise</u>: $\partial_i \partial_j u = \partial_j \partial_i u$.

Examples: • $\left(\frac{\partial}{\partial x_i}\delta_{x_0}\right)(\varphi) = -\left\langle \delta_{x_0}, \frac{\partial \varphi}{\partial x_i}\right\rangle = -\frac{\partial \varphi}{\partial x_i}(x_0)$

$$\left(\frac{\partial^\alpha}{\partial x^\alpha}\delta_{x_0}\right)(\varphi) = (-)^{|\alpha|}\frac{\partial^\alpha \varphi}{\partial x^\alpha}(x_0)$$

• $\frac{\partial}{\partial x}\chi_{[0,\infty)} = \delta_0$

• $\frac{\partial}{\partial x}|x| = \chi_{[0,\infty)} - \chi_{(-\infty,0]}$

$$\frac{\partial^2}{\partial x^2}|x| = 2\delta_0$$

Convergence of distributions:

$u_n \to u_0$ in $\mathcal{D}'(\Omega)$ $\iff$ $\lim_{n\to\infty}\langle u_n,\varphi\rangle = \langle u_0,\varphi\rangle$ $\forall \varphi \in C_0^\infty(\Omega)$.

Seminorms p_φ : $p_\varphi(u) = |\langle u,\varphi\rangle|$

If $u_n \to u_0$, $\partial^\alpha u_n \to \partial^\alpha u$.

Example: $\Omega = \mathbb{R}$, $f_n(x) = \frac{1}{n}\sin(nx)$

$f_n \to 0$ uniformly and hence $\Lambda_{f_n} \xrightarrow{\mathcal{D}'(\Omega)} 0$

Therefore $\partial_x \Lambda_{f_n} \to 0$ (Riemann-Lebesgue Lemma)

$\partial_x \Lambda_{f_n} = \Lambda_{\frac{\partial f_n}{\partial x}}$ $\quad \frac{\partial f_n}{\partial x} = \cos(nx)$

Fun exercise: Show that $n^{1999}\sin(nx) \xrightarrow{\mathcal{D}'(\mathbb{R})} 0$.

More operations

- $u \in \mathcal{D}'(\Omega)$, $f \in C^\infty(\Omega)$: $\quad \langle fu, \varphi\rangle := \langle u, f\varphi\rangle$

  $\rightarrow$ product fu of fct with distribution

Here it is crucial that with φ also $f\varphi \in C_0^\infty(\Omega)$.

Exercise: • Product rule: $\partial(fu) = (\partial f)u + f\,\partial u$

• $x\,\delta_0 = 0$, $\quad f(x)\,\delta_0 = f(0)\,\delta_0$

$x\,\delta_0' = -\delta_0$, $\quad f(x)\,\delta_0' = -f'(0)\,\delta_0 + f(0)\,\delta_0'$

- $T: C_0^\infty(\Omega) \to C_0^\infty(\Omega)$ continuous s.t.

  $\rightsquigarrow$ operator T^*: $\mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$:

$$\langle T^* u, \varphi\rangle = \langle u, T\varphi\rangle$$

Special classes of distributions:

$u = 0$ on $\omega \subseteq \Omega$ if $\langle u, \varphi\rangle = 0 \quad \forall \varphi \in C_0^\infty(\omega)$

$$\operatorname{supp} u = \Omega \setminus \left(\bigcup \{\omega \subseteq \Omega \text{ open} : u = 0 \text{ on } \omega\}\right)$$

$$\mathcal{E}'(\Omega) = \{u \in \mathcal{D}'(\Omega) : \operatorname{supp} u \text{ is compact}\}$$
$$= \{u \in \mathcal{D}'(\Omega) : u \text{ extends continuously to } C^\infty(\mathbb{R}^n)\}$$

on $\mathbb{R}^n$:

$$S(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \underbrace{\sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta f|}_{=: p_{\alpha\beta}(f)} < \infty \ \forall \alpha, \beta\}$$

Schwartz space

$$S'(\mathbb{R}^n) = \{u \in \mathcal{D}'(\mathbb{R}^n) : u \text{ extends continuously to } S(\mathbb{R}^n)\}$$

Convolution: $\quad f, g \in C_0^\infty(\mathbb{R}^n)$

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy \overset{?}{\in} C_0^\infty(\mathbb{R}^n)$$

(Indeed $\operatorname{supp} f*g \subseteq \operatorname{supp} f + \operatorname{supp} g$)

Note $(f*g)(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy = g*f(x) \quad (*)$

Theorem: $\quad \partial^\alpha (f*g) = (\partial^\alpha f)*g \underset{(*)}{=} f*(\partial^\alpha g)$

In particular, $f*g \in C_0^\infty$.

Proof: This follows from differentiating under the integral sign:

$$\partial^\alpha (f*g) \overset{(*)}{=} \partial^\alpha \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$
$$= \int_{\mathbb{R}^n} \partial^\alpha f(x-y)\, g(y)\, dy$$
$$= \int_{\mathbb{R}^n} (\partial^\alpha f)(x-y)\, g(y)\, dy.$$

When can we interchange $\int$ and ∂^α?

Theorem (measure theory):

- $G \subseteq \mathbb{R}^n$ measurable, $X \subseteq \mathbb{R}^k$ open
- $f : G \times X \to \mathbb{C}$ s.t. • $\forall\, x \in X : f(\cdot, x) \in L^1(G)$
  • $\forall\, x \in X\ \forall y \in G: \partial_x f(y,x)$ exists and $\exists\, g \in L^1(G) : |\partial_x f| \leq g$

Then the function

$$F(x) = \int_G f(y,x)\,dy$$

is differentiable and $\partial_x F(x) = \int_G \partial_x f(y,x)\,dy$. $\boxed{\varepsilon}$

Same reasoning shows: $g \in L^1(\mathbb{R}^n)$, $f \in C_0^\infty(\mathbb{R}^n) \Rightarrow f*g \in C^\infty(\mathbb{R}^n)$
$\partial^\alpha(f*g) = (\partial^\alpha f)*g$

More operations:

- $R\varphi(x) = \varphi(-x)$ extends to distributions $\rightsquigarrow$ even/odd distributions.

- Note
$$\langle f*g, \varphi\rangle = \int_{\mathbb{R}^n} (f*g)(y)\,\varphi(y)\,dy = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\,g(y-x)\,\varphi(y)\,dx\,dy = \int_{\mathbb{R}^n} g(x)\big((Rf)*\varphi\big)(x)\,dx = \langle g, (Rf)*\varphi\rangle$$

Therefore we define for $u \in D'(\mathbb{R}^n)$ and $f \in C_0^\infty(\mathbb{R}^n)$

$$\langle f*u, \varphi\rangle := \langle u, (Rf)*\varphi\rangle$$

The same formula defines $f*u$ for $u \in \mathcal{E}'(\mathbb{R}^n)$ and $f \in C^\infty(\mathbb{R}^n)$

Exercise: $\partial^\alpha(f*u) = (\partial^\alpha f)*u = f*(\partial^\alpha u)$

follows by definition and from the corresponding Theorem for C_0^∞:

$$\langle \partial^\alpha(f*u), \varphi\rangle = \langle f*u, (-\partial)^\alpha\varphi\rangle = \langle u, (Rf)*(-\partial)^\alpha\varphi\rangle$$
$$\overset{(\text{Thm})}{=} \langle u, (-\partial)^\alpha(Rf*\varphi)\rangle = \langle \partial^\alpha u, Rf*\varphi\rangle$$
$$= \langle f*\partial^\alpha u, \varphi\rangle \quad \text{etc.}$$

Fourier transform and $S(\mathbb{R}^n)$

Recall

$$C_0^\infty(\mathbb{R}^n) \subseteq \overset{\subseteq L^1(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n)}{S(\mathbb{R}^n)} = \{ f \in C^\infty(\mathbb{R}^n) : \underbrace{\sup_x |x^\alpha \partial_x^\beta f(x)|}_{=: p_{\alpha\beta}(f)} < \infty \}$$

$$\mathcal{E}'(\mathbb{R}^n) \subseteq S'(\mathbb{R}^n) = \{ \text{continuous linear functionals on } S(\mathbb{R}^n) \}$$

$$\| \quad \{ u : S(\mathbb{R}^n) \to \mathbb{C} \, :$$

$$\exists \alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k : \forall f \in S(\mathbb{R}^n) :$$

$$|u(f)| \leq C \max_{j,k} p_{\alpha_j \beta_k}(f) \}$$

$$\subseteq \mathcal{D}'(\mathbb{R}^n)$$

Fourier transform: $\quad đx = \frac{dx}{(2\pi)^{n/2}}$

$$\hat{f}(\xi) \equiv (\mathcal{F}f)(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\, đx \quad , \xi \in \mathbb{R}^n$$

Properties: a) $\forall x_0 \in \mathbb{R}^n$: $\left(e^{ix_0 x} f \right)^\wedge (\xi) = \hat{f}(\xi - x_0)$

If $h(x) = f(x - x_0)$: $\hat{h}(\xi) = e^{-ix_0\xi} \hat{f}(\xi)$

b) $|\hat{f}(\xi)| \leq \|f\|_{L^1(\mathbb{R}^n)} \cdot (2\pi)^{-n/2}$

c) $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ continuous, $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

d) $g(x) = f(x/\lambda)$ satisfies $\hat{g}(\xi) = |\lambda|^n \hat{f}(\lambda\xi)$, $\lambda \neq 0$.

e) $$((-i\partial_x)^\alpha f)^\wedge(\xi) = \xi^\alpha \hat{f}(\xi)$$

$$(x^\alpha f)^\wedge(\xi) = (i\partial_\xi)^\alpha \hat{f}(\xi)$$

f) $$(f * g)^\wedge(\xi) = (2\pi)^{n/2} \hat{f}(\xi)\hat{g}(\xi)$$

g) $$\int \hat{f} g = \int f \hat{g}$$

These are all easy to check.

<u>Corollary</u>: $\mathcal{F}: S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is continuous.

⌈Proof: ... ⌋

<u>Key example</u>: $f(x) = e^{-|x|^2/2}$ satisfies $\hat{f} = f$.

<u>Proof</u>: As $\int_{\mathbb{R}^n} e^{-ix\xi} f(x)\,dx = \prod_{j=1}^{n} \int_{\mathbb{R}} e^{-ix_j\xi_j} f(x_j)\,dx_j$,

we only need to show this for $n=1$.

In this case f satisfies the ODE $(*)\ f'(x) = -xf(x)$.

For $\hat{f}$ we compute

$$\frac{d}{d\xi}\hat{f}(\xi) = (2\pi)^{-1/2}\int_{\mathbb{R}} \partial_\xi(e^{-ix\xi})\, f(x)\,dx$$

$$= (2\pi)^{-1/2}\int_{\mathbb{R}} (-ix)\, e^{-ix\xi} f(x)\,dx$$

$$\underset{\text{int by parts}}{=} (2\pi)^{-1/2}\int_{\mathbb{R}} i\, e^{-ix\xi} f'(x)\,dx \underset{e)}{=} -\xi \hat{f}(\xi)$$

$\Rightarrow \hat{f}$ satisfies $(*)$.

Also $u(0) = (2\pi)^{-1/2} = \hat{u}(0)$. As the $(*)$ has a unique solution

$\Rightarrow u = \hat{u}$ ☒

Theorem: For $f \in S(\mathbb{R}^n)$: $\mathcal{F}(\mathcal{F}f) = Rf$, where $Rf(x) = f(-x)$.
In particular, $\mathcal{F}^4 = \mathrm{Id}$, and therefore $\mathcal{F}: S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is an isomorphism.

Proof: Set $\Psi(x) = (2\pi)^{-n/2} e^{-|x|^2/2} \in S(\mathbb{R}^n)$. Then $\|\Psi\|_{L^1} = 1$.

Define $\Psi_\varepsilon(x) = \varepsilon^{-n} \Psi(x/\varepsilon)$.

Then
$$\widehat{\Psi(\varepsilon\cdot)}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} e^{-|\varepsilon x|^2/2} dx$$
$$= \varepsilon^{-n} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-iy\,\xi/\varepsilon} e^{-|y|^2/2} dy$$
$$= \varepsilon^{-n} e^{-|\xi/\varepsilon|^2/2} = \Psi_\varepsilon(\xi).$$

Therefore

$$\mathcal{F}(\mathcal{F}f)(x) = \int e^{-ix\xi} \hat{f}(\xi)\, d\xi = \lim_{\varepsilon \to 0^+} \int e^{-ix\xi} \Psi(\varepsilon\xi) \hat{f}(\xi) d\xi$$
$$= \lim_{\varepsilon \to 0^+} \int \Psi(\varepsilon\xi)\, \widehat{f(\cdot - x)}(\xi)\, d\xi$$
$$\overset{(g)}{=} \lim_{\varepsilon \to 0^+} \int \Psi_\varepsilon(y) f(y-x)\, dy$$
$$= \lim_{\varepsilon \to 0^+} (\Psi_\varepsilon * (Rf))(x)$$

One easily shows $\Psi_\varepsilon * g \to g$ for any $g \in L^1$, so that indeed $\mathcal{F}(\mathcal{F}f)(x) = Rf(x)$ a.e.
As both sides are continuous, the equality holds for every x. $\boxtimes$

Theorem: Parseval's identity: $\forall u, v \in S(\mathbb{R}^n)$: $\langle u, v\rangle_{L^2} = \langle \hat{u}, \hat{v}\rangle_{L^2}$

In particular, $\|u\|_{L^2} = \|\hat{u}\|_{L^2}$ $\forall u \in S(\mathbb{R}^n)$.

Proof: By g) $\int \hat{u}\, v\, dx = \int u \hat{v}\, dx$. Also, $\overline{\hat{u}}(x) = \hat{\bar{u}}(-x)$.
Therefore $\langle \hat{u}, \hat{v}\rangle_{L^2} = \int \hat{u}\, \overline{\hat{v}}\, dx = \int u\, \hat{\overline{\hat{v}}}\, dx = \int u \bar{v}\, dx = \langle u, v\rangle_{L^2}$ $\boxtimes$

Definition: Let $u \in S'(\mathbb{R}^n)$. Then we define the Fouriertransform $\mathcal{F}u \equiv \hat{u}$ of u as the distribution

$$\hat{u}(\varphi) := u(\hat{\varphi}) \qquad \forall \varphi \in S(\mathbb{R}^n).$$

By definition, $\mathcal{F}: S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$. It is an isomorphism which restricts to an operator $\mathcal{F}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

Theorem (Plancherel): $\mathcal{F}: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an isometric isomorphism.

This is an immediate consequence of the Parseval identity and the fact that $S(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$.

In the following, we will try to describe the range of $\mathcal{F}$ on interesting subspaces of $S'(\mathbb{R}^n)$:

Paley-Wiener theorems: Characterize image of $C_0^\infty(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$ under $\mathcal{F}$.

Hausdorff-Young inequality: For which p, q is $\mathcal{F}: L^p \to L^q$ continuous?

One of the first striking applications of distribution theory concerned the solution of (constant coefficient) differential equations:

Let $P(\partial_x) = \sum_{|\alpha|=0}^{m} a_\alpha \partial_x^\alpha$ a partial differential operator on $\mathbb{R}^n$, with constant coefficients $a_\alpha \in \mathbb{C}$, and not all $a_\alpha = 0$.

We say that $G \in \mathcal{D}'$ is a <u>fundamental solution</u> of $P(\partial_x)$, if $P(\partial_x) G = \delta_0$. (compare Applied Analysis + PDE 2 stream)

<u>Theorem</u>: (Malgrange-Ehrenpreis)

There exists a fundamental solution $G \in S'(\mathbb{R}^n)$.

<u>Idea of Proof</u>: $\mathcal{F}^{-1}$ $\mathcal{F} P(\partial_x) G = \mathcal{F}\delta_0 = (2\pi)^{-n/2}$

$$P(-i\xi)\, \mathcal{F}G =$$

Candidate: $G = \mathcal{F}^{-1}\left(\frac{(2\pi)^{-n/2}}{P(-i\xi)}\right)$

The proof needs to make sense of the right hand side as a distribution $\in S'(\mathbb{R}^n)$. $\square$

<u>Corollary</u>: For every $f \in S(\mathbb{R}^n)$ there exists a solution $u \in C^\infty(\mathbb{R}^n)$ to the differential equation

$$P(\partial_x) u = f.$$

<u>Proof</u>: $u = G * f$ satisfies $P(\partial_x) u = P(\partial_x)(G * f)$

$$= (P(\partial_x) G) * f$$
$$= \delta_0 * f = f. \quad \square$$