E. Local Solvability: The Lewy Example

It was long believed that any "reasonable" partial differential equation (with no boundary conditions imposed) should have many solutions. In particular, suppose we have a linear equation $\sum_{|\alpha|\le k} a_\alpha \partial^\alpha u = f$ where f and the coefficients a_α are C^∞. Given $x_0 \in \mathbb{R}^n$, can we find a solution u (not necessarily C^∞) of this equation in some neighborhood of x_0? If f and the a_α's are analytic and $a_\alpha(x_0) \neq 0$ for some α with $|\alpha| = k$, the Cauchy-Kowalevski theorem shows that the answer is yes. Indeed, we can choose a vector ξ which is non-characteristic for $\sum a_\alpha \partial^\alpha$ at x_0 (and hence at all x in a neighborhood of x_0) and solve the Cauchy problem with zero Cauchy data on the hyperplane through x_0 orthogonal to ξ.

One might well expect that the assumption of analyticity can be omitted. But in 1957 Hans Lewy [33] destroyed all hopes for such a theorem with the following embarassingly simple counterexample. Consider the differential operator L defined on $\mathbb{R}^3$ with coordinates (x, y, t) by

$$L = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} - 2i(x+iy)\frac{\partial}{\partial t}. \tag{1.47}$$

(1.48) Theorem.
Let f be a continuous real-valued function depending only on t. If there is a C^1 function u of (x, y, t) satisfying $Lu = f$ in some neighborhood of the origin, then f is analytic at $t = 0$.

Proof: Suppose $Lu = f$ in the set where $x^2 + y^2 < R^2$ and $|t| < R$ $(R > 0)$. Set $z = x + iy$; write z in polar coordinates as $re^{i\theta}$ and set $s = r^2$. Consider the quantity V, a function of t and r (or equivalently of t and s) defined for $0 < r < R$ and $|t| < R$ by the contour integral

$$V = \int_{|z|=r} u(x,y,t)\, dz = ir\int_0^{2\pi} u(r\cos\theta,\, r\sin\theta,\, t)\, e^{i\theta}\, d\theta.$$

By Green's theorem,

$$\begin{aligned} V &= i\iint_{|z|\le r} \left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right](x,y,t)\, dx\, dy \\ &= i\int_0^r \int_0^{2\pi} \left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right](\rho\cos\theta,\, \rho\sin\theta,\, t)\, \rho\, d\rho\, d\theta. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial V}{\partial r} &= i\int_0^{2\pi} \left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right](r\cos\theta,\, r\sin\theta,\, t)\, r\, d\theta \\ &= \int_{|z|=r} \left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right](x,y,t)\, r\frac{dz}{z}. \end{aligned}$$

The equation $Lu = f$ then implies

$$\begin{aligned} \frac{\partial V}{\partial s} = \frac{1}{2r}\frac{\partial V}{\partial r} &= \int_{|z|=r} \left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right](x,y,t)\, \frac{dz}{2z} \\ &= i\int_{|z|=r} \frac{\partial u}{\partial t}(x,y,t)\, dz + \int_{|z|=r} f(t)\, \frac{dz}{2z} = i\frac{\partial V}{\partial t} + \pi i f(t). \end{aligned}$$

Thus if we set $F(t) = \int_0^t f(\tau)\, d\tau$, the quantity $U(t,s) = V(t,s) + \pi F(t)$ satisfies

$$\frac{\partial U}{\partial t} + i\frac{\partial U}{\partial s} = 0.$$

This is the Cauchy-Riemann equation, so U is a holomorphic function of $w = t + is$ in the region $0 < s < R^2$, $|t| < R$, and U is continuous up to the line $s = 0$. Moreover, $V = 0$ when $s = 0$, so $U(0,t) = \pi F(t)$ is real-valued. Therefore, by the Schwarz reflection principle, the formula $U(t,-s) = \overline{U(t,s)}$ gives a holomorphic continuation of U to a full neighborhood of the origin. In particular, $U(t,0) = \pi F(t)$ is analytic in t, hence so is $f = F'$. ∎

There is really nothing special about the origin in Theorem 1.48. In fact, a change-of-variable argument that we outline in Exercise 1 shows that for any $(x_0, y_0, t_0) \in \mathbb{R}^3$, the equation

$$Lu(x,y,t) = f(t + 2y_0 x - 2x_0 y)$$

has no C^1 solution in any neighborhood of (x_0, y_0, t_0) unless $f(\tau)$ is analytic at $\tau = t_0$.

Once this is known, it is not hard to show that there are C^∞ functions g on $\mathbb{R}^3$ such that the equation $Lu = g$ has no solution $u \in C^{1+\alpha}$ $(\alpha > 0)$ in any neighborhood of any point. The idea is as follows. Pick a C^∞ periodic function f on $\mathbb{R}$ that is not analytic at any point, and pick a countable dense set $\{(x_j, y_j, t_j)\}_1^\infty$ in $\mathbb{R}^3$. Then there is a sequence of positive constants $\{c_j\}$ tending to zero rapidly enough so that the series

$$g_a(x,y,t) = \sum_{j=1}^\infty a_j c_j f(t + 2y_j x - 2x_j y)$$

defines a C^∞ function on $\mathbb{R}^3$ for any bounded sequence $a = \{a_j\}$. One can then show that for "most" sequences a, in the sense of Baire category in the space l^∞ of bounded sequences, the equation $Lu = g_a$ has no $C^{1+\alpha}$

solution near any point. For the details of this argument, see Lewy [33] or John [30].

This construction also leads immediately to an example of a *homogeneous* equation with no nontrivial solution. Namely, if f is a C^∞ function on $\mathbb{R}^3$ such that the equation $Lu = f$ has no solution near any point, then the equation $Lv - fv = 0$ has no solution except $v \equiv 0$. Indeed, suppose v is a solution that is nonzero on an open set Ω. By shrinking Ω we can assume that a single-valued branch of the logarithm can be defined on $v(\Omega)$, and then $u = \log v$ is a solution of $Lu = f$ on Ω.

A couple of years after Lewy proved Theorem (1.48), Hörmander embedded it into a more general result that initiated the theory of local solvability of differential operators. We make a formal definition: A linear differential operator L with C^∞ coefficients is said to be **locally solvable** at x_0 if there is a neighborhood Ω of x_0 such that for every $f \in C_c^\infty(\Omega)$ there exists $u \in \mathcal{D}'(\Omega)$ with $Lu = f$. Hörmander's theorem is then as follows; see [26] for the proof.

(1.49) Theorem.
Let L be a linear differential operator with C^∞ coefficients on Ω, let $P(x,\xi) = \chi_L(x,\xi)$ be the characteristic form of L, and let

$$Q(x,\xi) = \sum_1^n \left[\frac{\partial P}{\partial \xi_j}(x,\xi)\frac{\partial \overline{P}}{\partial x_j}(x,\xi) - \frac{\partial P}{\partial x_j}(x,\xi)\frac{\partial \overline{P}}{\partial \xi_j}(x,\xi)\right].$$

a. *If $x_0 \in \Omega$ and there is a $\xi \in \mathbb{R}^n$ such that $P(x_0,\xi) = 0$ but $Q(x_0,\xi) \neq 0$, then L is not locally solvable at x_0.*
b. *If for each $x \in \Omega$ there is a $\xi \in \mathbb{R}^n$ such that $P(x,\xi) = 0$ but $Q(x,\xi) \neq 0$, then there is an $f \in C^\infty(\Omega)$ such that the equation $Lu = f$ has no distribution solution on any open subset of Ω.*

It is easy to check that the Lewy operator (1.47) satisfies the hypothesis of (b) on $\Omega = \mathbb{R}^n$. Thus we obtain a strengthening of Theorem (1.48): there exist functions $f \in C^\infty$ for which the Lewy equation has no solution in $\mathcal{D}'$, not merely in C^1.

Theorem (1.49) was the starting point for a considerable body of research into necessary and sufficient conditions for local solvability. For an account of this work, we refer the reader to the expository articles of Treves [50], [51], and to Beals and Fefferman [6]. We mention also that Greiner, Kohn, and Stein have found a necessary and sufficient condition on f for the Lewy equation $Lu = f$ to be locally solvable; see Stein [46, §XIII.4].

EXERCISES

1. Given $(x_0, y_0, t_0) \in \mathbb{R}^3$, define the transformation T of $\mathbb{R}^3$ by
$$T(x,y,t) = (x - x_0,\ y - y_0,\ t - t_0 - 2y_0x + 2x_0y).$$
Show that if L is the Lewy operator (1.47), then $L(u \circ T) = (Lu) \circ T$ for any function u, and conclude that solving $Lu = f(t + 2y_0x - 2x_0y)$ near (x_0, y_0, t_0) is equivalent to solving $L(u \circ T) = f(t - t_0)$ near the origin.

2. (Addendum to Exercise 1.) Show that the binary operation $*$ on $\mathbb{R}^3$ defined by
$$(a,b,c) * (x,y,t) = (a + x,\ b + y,\ c + t + 2bx - 2ay)$$
makes $\mathbb{R}^3$ into a group. (This group is known as the 3-dimensional *Heisenberg group*. Exercise 1 says that the Lewy operator is invariant under left translations on this group, as the transformation T is left translation by $(-x_0, -y_0, -t_0)$.)

3. The Lewy operator (1.47) arises in complex analysis because the equation $Lu = 0$ is, in a sense, the restriction of the Cauchy-Riemann equations on $\mathbb{C}^2$ to the hypersurface $\{(z_1, z_2) : \operatorname{Im} z_2 = |z_1|^2\}$. More precisely, define $\Phi : \mathbb{R}^3 \to \mathbb{C}^2$ by
$$\Phi(x,y,t) = \big(x + iy,\ t + i(x^2 + y^2)\big).$$
Suppose f is a holomorphic function on $\mathbb{C}^2$, so that it satisfies the Cauchy-Riemann equations $(\partial_{u_j} + i\partial_{v_j})f = 0$ for $j = 1, 2$, where $z_j = u_j + iv_j$. Show that $L(f \circ \Phi) = 0$.

4. Local non-solvability may occur for relatively trivial reasons when the characteristic form of an operator vanishes at a point. Show, for example, that the equation $x\partial_y u - y\partial_x u = x^2 + y^2$ has no continuous solutions in any neighborhood of $(0,0)$ in $\mathbb{R}^2$. (Hint: show that $x\partial_y - y\partial_x = \partial_\theta$ in polar coordinates.)

F. Constant-Coefficient Operators: Fundamental Solutions

A couple of years before Lewy discovered his example, Malgrange and Ehrenpreis independently proved that every linear differential operator with constant coefficients has a fundamental solution (a concept we shall define below). An immediate corollary is that every constant-coefficient operator is locally solvable, and one can deduce regularity properties of the solutions by examination of the fundamental solution. In this section we shall derive these results, following an argument of Nirenberg [38].

Let

$$L = \sum_{|\alpha|\le k} c_\alpha \partial^\alpha$$

be a differential operator with constant coefficients. The natural tool for studying such operators is the Fourier transform; if f is any tempered distribution, we have

$$(Lu)\widehat{\ }(\xi) = P(\xi)\widehat{u}(\xi), \tag{1.50}$$

where

$$P(\xi) = \sum_{|\alpha|\le k} c_\alpha (2\pi i \xi)^\alpha .$$

P is called the **symbol** of L; this notation relating L and P will be maintained throughout this section.

We begin by considering the question of local solvability of L. In view of (1.50), if $f \in C_c^\infty$, it would seem that we should be able to solve $Lu = f$ by taking $\widehat{u} = \widehat{f}/P$, that is,

$$u(x) = \int e^{2\pi i x\cdot\xi}\frac{\widehat{f}(\xi)}{P(\xi)}\,d\xi. \tag{1.51}$$

The trouble with this is that usually the polynomial P will have zeros, so that $\widehat{f}/P$ is not a locally integrable function and the integral (1.51) is not well-defined. However, since $f \in C_c^\infty$, $\widehat{f}$ extends to an entire holomorphic function on $\mathbb{C}^n$ by Proposition (0.30). The idea will therefore be to make sense of (1.51) by deforming the contour of integration so as to avoid the zeros of P.

To this end, we make a simplification. By a rotation of coordinates we can assume that the vector $(0,\ldots,0,1)$ is non-characteristic for L, which means that the coefficient of ξ_n^k in $P(\xi)$ is nonzero. After dividing everything through by it, we may — and shall — assume that this coefficient is 1, so that

$$P(\xi) = \xi_n^k + \text{terms of lower order in } \xi_n .$$

For each fixed $\xi' = (\xi_1,\ldots,\xi_{n-1}) \in \mathbb{R}^{n-1}$, we consider $P(\xi) = P(\xi',\xi_n)$ as a polynomial in the single *complex* variable ξ_n. Let $\lambda_1(\xi'),\ldots,\lambda_k(\xi')$ be its zeros, counted according to multiplicity and arranged so that for $i \le j$, $\operatorname{Im}\lambda_i(\xi') \le \operatorname{Im}\lambda_j(\xi')$ and $\operatorname{Re}\lambda_i(\xi') \le \operatorname{Re}\lambda_j(\xi')$ if $\operatorname{Im}\lambda_i(\xi') = \operatorname{Im}\lambda_j(\xi')$. By Rouché's theorem, a small perturbation of ξ' produces a small perturbation of the zeros of $P(\xi',\xi_n)$, so the functions $\operatorname{Im}\lambda_j(\xi')$ are continuous functions of ξ'. Before proceeding to the main results, we need two lemmas.

(1.52) Lemma.
There is a measurable function $\phi : \mathbb{R}^{n-1} \to [-k,k]$ such that for all $\xi' \in \mathbb{R}^{n-1}$,

$$\min\{|\phi(\xi') - \operatorname{Im}\lambda_j(\xi')| : 1 \le j \le k\} \ge 1.$$

Proof: The idea is simple: There are at most k distinct points among $\operatorname{Im}\lambda_j(\xi')$ $(1 \le j \le k)$, so at least one of the $k+1$ intervals $[2m-k-1, 2m-k+1)$ $(0 \le m \le k)$ must contain none of them, and we can take $\phi(\xi')$ to be the midpoint of that interval. That is, for $0 \le m \le k$, let

$$V_m = \{\xi' : \operatorname{Im}\lambda_j(\xi') \notin [2m-k-1,\, 2m-k+1) \text{ for } j = 1,\ldots,k\}.$$

Then the sets V_m cover $\mathbb{R}^{n-1}$, and they are Borel sets since $\operatorname{Im}\lambda_j$ is continuous, so we can take

$$\phi(x) = 2m - k \text{ for } x \in V_m \setminus \bigcup_0^{m-1} V_l \qquad (0 \le m \le k).$$ ∎

(1.53) Lemma.
Let $g(z)$ be a monic polynomial of degree k in the complex variable z such that $g(0) \ne 0$, and let $\lambda_1,\ldots,\lambda_k$ be its zeros. Then $|g(0)| \ge (d/2)^k$ where $d = \min|\lambda_j|$.

Proof: We have $g(z) = (z-\lambda_1)\cdots(z-\lambda_k)$, so

$$\left|\frac{g(z)}{g(0)}\right| = \prod_1^k \left|1 - \frac{z}{\lambda_k}\right| \le 2^k \text{ for } |z| \le d.$$

Moreover, $g^{(k)}(z) \equiv k!$, so by the Cauchy integral formula,

$$k! = |g^{(k)}(0)| = \left|\frac{k!}{2\pi i}\int_{|z|=d}\frac{g(z)}{z^{k+1}}\,dz\right| \le \frac{k!\,2^k|g(0)|}{d^k},$$

which is the desired result. ∎

(1.54) Theorem.
If L is a differential operator with constant coefficients on $\mathbb{R}^n$ and $f \in C_c^\infty(\mathbb{R}^n)$, there exists $u \in C^\infty(\mathbb{R}^n)$ such that $Lu = f$.

Proof: We employ the notation introduced above. Let ϕ be as in Lemma (1.52), and set

$$u(x) = \int_{\mathbb{R}^{n-1}} \int_{\operatorname{Im}\xi_n = \phi(\xi')} e^{2\pi i x\cdot\xi} \frac{\widehat{f}(\xi)}{P(\xi)}\, d\xi_n\, d\xi'. \tag{1.55}$$

By Lemma (1.53) (applied to $g(z) = P(\xi', \xi_n + z)$) together with Lemma (1.52), we see that $|P(\xi)| \geq 2^{-k}$ when $\operatorname{Im}\xi_n = \phi(\xi')$. Moreover, by Proposition (0.30), $\widehat{f}(\xi)$ is rapidly decaying as $|\operatorname{Re}\xi| \to \infty$ when $|\operatorname{Im}\xi|$ remains bounded. Hence the integrand in (1.55) is bounded and rapidly decaying at infinity, so the integral is absolutely convergent. For the same reason, we can differentiate under the integral as often as we please and conclude that u is C^∞ and that

$$Lu(x) = \int_{\mathbb{R}^{n-1}} \int_{\operatorname{Im}\xi_n = \phi(\xi')} e^{2\pi i x\cdot\xi} \widehat{f}(\xi)\, d\xi_n\, d\xi'.$$

But now the integrand is an entire function which is rapidly decaying as $|\operatorname{Re}\xi| \to \infty$, so by Cauchy's theorem we can deform the contour of integration in ξ_n back to the real axis. By the Fourier inversion theorem, then, $Lu = f$. ∎

The content of Theorem (1.54) can be usefully rephrased as follows. A **fundamental solution** for the constant-coefficient operator L is a distribution K on $\mathbb{R}^n$ such that $LK = \delta$, where δ is the point mass at the origin. On the one hand, Theorem (1.54) is an immediate corollary of the existence of a fundamental solution, for if $f \in C_c^\infty$ we can take $u = K * f$: then $Lu = LK * f = \delta * f = f$. On the other hand, the proof of Theorem (1.54) easily yields a fundamental solution.

(1.56) The Malgrange-Ehrenpreis Theorem.
Every differential operator L with constant coefficients has a fundamental solution.

Proof: With notation as above, define a linear functional K on C_c^∞ by

$$\langle K, f\rangle = \int_{\mathbb{R}^{n-1}} \int_{\operatorname{Im}\xi_n = \phi(\xi')} \frac{\widehat{f}(-\xi)}{P(\xi)}\, d\xi_n\, d\xi'.$$

As in the proof of Theorem (1.54), the integral is bounded by

$$C \sup_{|\operatorname{Im}\xi|\leq k} (1+|\xi|)^{-n-1}|\widehat{f}(\xi)| \leq C' \sum_{|\alpha|\leq n+1} \|\partial^\alpha f\|_\infty,$$

where C and C' depend only on the support of f, so K is a distribution. Moreover, $\langle LK, f\rangle = \langle K, L'f\rangle$ where L' is the operator with symbol $P(-\xi)$, so that $(L'f)\widehat{\ }(-\xi) = P(\xi)\widehat{f}(-\xi)$. Hence, as in the proof of Theorem (1.54),

$$\langle LK, f\rangle = \int_{\mathbb{R}^{n-1}} \int_{\operatorname{Im}\xi_n = \phi(\xi')} \widehat{f}(-\xi)\, d\xi = \int_{\mathbb{R}^n} \widehat{f}(\xi)\, d\xi = f(0) = \langle \delta, f\rangle.$$

Alternatively, one could observe that $K * f$ is the function u defined by (1.55), so that $LK * f = Lu = f$ for all f and hence $LK = \delta$. ∎

With a fundamental solution K in hand, we can solve the equation $Lu = f$ not only when $f \in C_c^\infty$ but when f is any distribution with compact support; of course, the solution u will then be a distribution. Indeed, if $f \in \mathcal{E}'$, we have

$$L(K * f) = K * Lf = f, \tag{1.57}$$

since $L(K * f)$ and $K * Lf$ are both equal to $LK * f = \delta * f = f$. These relations can often be extended to f's which are not compactly supported, but the class of f's for which they hold will depend on the nature of K. We shall see some specific examples in Chapters 2, 4, and 5.

Fundamental solutions are useful not only for producing solutions of differential equations but also for studying their regularity properties. In particular, we have the following important result.

A differential operator L with C^∞ coefficients is called **hypoelliptic** if any distribution u on an open set Ω such that Lu is C^∞ on Ω must itself be C^∞ on Ω, that is, if all solutions of the equation $Lu = f$ must be C^∞ wherever f is C^∞. (The origin of this term is the fact that all elliptic operators are hypoelliptic, a fact which we shall prove in Chapter 6.)

(1.58) Theorem.
If L is a differential operator with constant coefficients, the following are equivalent:

a. *Some fundamental solution for L is C^∞ on $\mathbb{R}^n \setminus \{0\}$.*
b. *Every fundamental solution for L is C^∞ on $\mathbb{R}^n \setminus \{0\}$.*
c. *L is hypoelliptic.*

Proof: If K is a fundamental solution for L, then $LK = \delta$ is C^∞ on $\mathbb{R}^n \setminus \{0\}$, so (c) implies (b). (b) trivially implies (a), so it remains to show that (a) implies (c). For this we need a lemma.

(1.59) Lemma.
Suppose f and g are distributions on $\mathbb{R}^n$, f is C^∞ on $\mathbb{R}^n \setminus \{0\}$, and g has compact support. Then $f * g$ is C^∞ on $\mathbb{R}^n \setminus (\operatorname{supp} g)$.

Proof: Given $x \notin \operatorname{supp} g$, choose $\epsilon > 0$ small enough so that $B_\epsilon(x)$ and $\operatorname{supp} g$ are disjoint, and choose $\phi \in C_c^\infty(B_{\epsilon/2}(0))$ such that $\phi = 1$ on $B_{\epsilon/4}(0)$. Then we can write

$$f * g = (\phi f) * g + [(1-\phi)f] * g.$$

On the one hand, $(1-\phi)f$ is a C^∞ function, so $[(1-\phi)f] * g$ is C^∞ everywhere. On the other hand,

$$\operatorname{supp}[(\phi f) * g] \subset \{x + y : x \in \operatorname{supp} \phi,\ y \in \operatorname{supp} g\},$$

which is disjoint from $B_{\epsilon/2}(x)$. Hence, on $B_{\epsilon/2}(x)$, $f * g = [(1-\phi)f] * g$ is C^∞. ∎

Returning to the proof of Theorem (1.58), let K be a fundamental solution for L that is C^∞ on $\mathbb{R}^n \setminus \{0\}$. Suppose u is a distribution on an open set $\Omega \subset \mathbb{R}^n$ such that Lu is C^∞ on Ω. If $x \in \Omega$, we pick $\epsilon > 0$ so that $B_\epsilon(x) \subset \Omega$, and we shall show that u is C^∞ on $B_{\epsilon/2}(x)$.

Pick $\phi \in C_c^\infty(B_\epsilon(x))$ with $\phi = 1$ on $B_{\epsilon/2}(x)$. Then $L(\phi u) = \phi Lu + v$ where $v = 0$ on $B_{\epsilon/2}(x)$ and outside $B_\epsilon(x)$. $K * (\phi Lu)$ is C^∞ since $\phi Lu \in C_c^\infty$, and $K * v$ is C^∞ on $B_{\epsilon/2}(x)$ by Lemma (1.59). But by (1.57),

$$\phi u = K * L(\phi u) = K * \phi Lu + K * v,$$

so on $B_{\epsilon/2}(x)$, $u = \phi u$ is C^∞. ∎

EXERCISES

1. Let $L = \sum_0^k c_j (d/dx)^j$ be an ordinary differential operator with constant coefficients. Let v be the solution of $Lv = 0$ satisfying $v(0) = \cdots = v^{(k-2)}(0) = 0$, $v^{(k-1)}(0) = c_k^{-1}$. Define $K(x) = 0$ if $x \le 0$, $K(x) = v(x)$ if $x > 0$. Show that K is a fundamental solution for L.

2. Show that the characteristic function of $\{(x,y) : x > 0,\ y > 0\}$ is a fundamental solution for $\partial_x \partial_y$ in $\mathbb{R}^2$.

3. Show that $K(x,y) = [2\pi i(x+iy)]^{-1}$ is a fundamental solution for the Cauchy-Riemann operator $L = \partial_x + i\partial_y$ on $\mathbb{R}^2$. Hint: if $\phi \in C_c^\infty$,

$$\langle LK, \phi\rangle = -\langle K, L\phi\rangle = \frac{-1}{2\pi i} \lim_{\epsilon \to 0} \iint_{x^2+y^2>\epsilon^2} \frac{\partial_x \phi + i\partial_y \phi}{x+iy}\, dx\, dy.$$

Use Green's theorem to show that this equals

$$\lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_{x^2+y^2=\epsilon^2} \phi(x,y) \frac{dx + i\, dy}{x+iy} = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_0^{2\pi} \phi(\epsilon\cos\theta, \epsilon \sin\theta)\, d\theta.$$

4. Suppose L is a constant-coefficient differential operator. Modify the proof of Theorem (1.58) to show that if there is a distribution K that is C^∞ away from the origin and satisfies $LK = \delta + f$ where f is C^∞, then L is hypoelliptic.