DifFun2 - Uncertainty principles

Jonas Dahlbæk

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All functions are assumed to be measurable throughout. Furthermore, unless otherwise explicitly stated, all function spaces are spaces of functions $f : \mathbb{R} \to \mathbb{C}$, i.e. $L^1 = L^1(\mathbb{R})$, $C^\infty = C^\infty(\mathbb{R})$ etc..

1 Uncertainty principles

The following discussion is based on material by Tao [2009].

Definition 1.0.1 (Fourier transform)**.** *We define the Fourier transform of $f \in L^1$ by*

$$\forall p \in \mathbb{R} : \ \widehat{f}(p) := \int_{\mathbb{R}} f(x) e^{-2\pi i x p} dx,$$

and extend the Fourier transform so defined in the standard way to L^2.

Recall that we have the following relations between growth conditions on f and smoothness conditions on $\widehat{f}$:

1. $f \in L^2 \Rightarrow \widehat{f} \in L^2$,
2. $f \in L^1 \Rightarrow \widehat{f} \in C_0$,
3. $(\forall n \in \mathbb{N} : \ \sup_{x\in\mathbb{R}} |x^n f(x)| < \infty) \Rightarrow \widehat{f} \in C^\infty$,
4. f has exponential decay $\Rightarrow \widehat{f}$ extends to a holomorphic function in a strip,
5. f has super exponential decay $\Rightarrow \widehat{f}$ extends to an entire function.

Here L^2 is the space of square integrable functions, L^1 is the space of integrable functions, C_0 is the space of continuous functions converging to zero at infinity, and C^∞ is the space of smooth functions. We conclude that the quicker f decays, the smoother $\widehat{f}$ is.

4. and 5. above motivates another principle: If $f \neq 0$ decays very rapidly, $\widehat{f}$ cannot decay very rapidly. Note that by 4., if f has exponential decay, $\widehat{f}$ cannot have compact support. If f decays super exponentially, 5. implies that the set of zeroes of $\widehat{f}$ has no limit points at all. The principle that it is impossible for both f and $\widehat{f}$ to decay very rapidly (unless $f = 0$) is commonly known as the uncertainty principle. In the following, we review some manifestations of this principle.

2 Heisenberg's uncertainty principle [Heisenberg 1927]

Let $\mathscr{S}$ denote the space of Schwartz functions.

Definition 2.0.1. *Define $M : \mathscr{S} \to \mathscr{S}$ by*

$$\forall f \in \mathscr{S}, x \in \mathbb{R} : \ (Mf)(x) = xf(x).$$

2.1 The case of a Gaussian

If we consider the Gaussian function

$$\psi(x) = \frac{1}{\sqrt{\sigma}} \exp\left(-\frac{\pi x^2}{2\sigma^2}\right),$$

it is well known that

$$\widehat{\psi}(p) = \sqrt{2\sigma} e^{-\pi 2\sigma^2 p^2}.$$

A computation shows that

$$\|M\psi\|_{L^2} = \frac{\sigma}{\sqrt{2\pi}},$$
$$\|M\widehat{\psi}\|_{L^2} = \frac{1}{\sqrt{2\pi}2\sigma}.$$

We therefore arrive at the identity

$$\|M\psi\|_{L^2}\|M\widehat{\psi}\|_{L^2} = 1/4\pi.$$

In vague terms, the decay of ψ is inversely related to the decay of $\widehat{\psi}$.

2.2 The case of $\psi \in \mathscr{S}$

For $\psi \in \mathscr{S}$ (the space of Schwartz functions) we have a similar relation called Heisenberg's uncertainty principle. It states that if $\|\psi\|_2 = 1$, then $\|M\psi\|_{L^2}\|M\widehat{\psi}\|_{L^2} \geq 1/4\pi$. This follows quite easily by noting that

$$[\partial_1, M]\psi := \partial_1 M\psi - M\partial_1\psi = \psi,$$

so that a partial integration gives

$$1 = \langle \psi, [\partial_1, x]\psi \rangle = -\langle \partial_1\psi, x\psi \rangle - \langle x\psi, \partial_1\psi \rangle = -2\mathrm{Re}\,\langle x\psi, \partial_1\psi \rangle.$$

An application of the Cauchy-Schwarz inequality gives us

$$1 = 2|\mathrm{Re}\,\langle x\psi, \partial_1\psi \rangle| \leq 2|\langle x\psi, \partial_1\psi \rangle| \leq 2\|x\psi\|_2\|\partial_1\psi\|_2.$$

Now we can use Plancherel's Formula to obtain

$$\|x\psi\|_2\|p\widehat{\psi}\|_2 = \|x\psi\|_2\|\partial_1\psi\|_2/2\pi \geq 1/4\pi.$$

Again we see that, in a vague sense, if ψ (or $\widehat{\psi}$) decays quickly, $\widehat{\psi}$ (or ψ) cannot decay quickly.

3 Hardy's uncertainty principle

It is a result by Hardy [1933; see Theorem 2] that if $f : \mathbb{R} \to \mathbb{C}$ satisfies $|f(x)| \leq Ce^{-\pi a x^2}$ and $|\widehat{f}(y)| \leq Ce^{-\pi b y^2}$ for some positive constants a, b, C, then $f = 0$ if $ab > 1$ and $f = C' e^{-\pi a x^2}$ if $ab = 1$. We will give a proof of this, following material by Tao [2009]. We first establish some notation and a lemma which we shall need.

Definition 3.0.1. *For $0 < \alpha < \pi$ we denote the region in the complex plane bounded by the two rays $\{t \in \mathbb{C} \,|\, t \geq 0\}$ and $\{te^{i\alpha} \in \mathbb{C} \,|\, t \geq 0\}$ by*

$$\Gamma_\alpha = \left\{ te^{i\theta} \in \mathbb{C} \,\middle|\, 0 < t, 0 < \theta < \alpha \right\}. \tag{1}$$

We put

$$\mathcal{S} = \{\theta + it \in \mathbb{C} \,|\, 0 < \theta < 1, t \in \mathbb{R}\}.$$

We denote the space of entire functions by $\mathcal{O}$, and for an open region $\Omega \in \mathbb{C}$ we denote the space of holomorphic functions in Ω by $\mathcal{O}(\Omega)$.

Lemma 3.0.2 (Phragmén-Lindelöf)**.** *Suppose $0 < \alpha < \pi/2$ and $A, a > 0$. Suppose further that $f \in \mathcal{O}$ satisfies the conditions*

- $\forall r > 0 : \; |f(r)| \leq 1,$
- $\forall r > 0 : \; |f(re^{i\alpha})| \leq 1,$
- $\forall z \in \Gamma_\alpha : \; |f(z)| \leq A \exp(a|z|^2).$

Then $\forall z \in \overline{\Gamma}_\alpha : \; |f(z)| \leq 1.$

In words, we suppose that f is bounded on the two defining rays of Γ_α, and that f satisfies a growth condition in Γ_α. The conclusion is then that f is bounded on $\overline{\Gamma}_\alpha$. By a rescaling argument, it suffices to prove the lemma for $a = 1$.

Proof. The lemma may be proven directly, see for instance Lemma 4.2 on page 108 in the book by Stein and Weiss [1971]. Alternatively, the result can be realised as a consequence of Lindelöf's Theorem (sometimes called the Hadamard three lines theorem) as found in the lecture notes [Gimperlein] in the following way:

Consider $\phi \in \mathcal{O}$ defined by $\phi(z) = \exp(-i\alpha(z-1))$. We note that

- $\phi(i\mathbb{R}) = \{t \,|\, t > 0\}$,
- $\phi(1 + i\mathbb{R}) = \{te^{i\alpha} \,|\, t > 0\}$,
- ϕ maps a neighborhood U of $\overline{\mathcal{S}}$ bi-holomorphically to a neighborhood of $\overline{\Gamma}_\alpha$.

The last statement is a consequence of the assumption $\alpha < \pi/2 < \pi$. From these properties, we may define a holomorphic function $g : U \to \mathbb{C}$ (in particular $g \in \mathcal{O}(\mathcal{S}) \cap C(\overline{\mathcal{S}})$) by $g(z) = f(\phi(z))$. g then satisfies $|g(it)| \leq 1$, $|g(1+it)| \leq 1$ and

$$|g(\theta + it)| \leq A \exp(\exp(2\alpha t)).$$

Since $2\alpha < \pi$, we can apply Lindelöf's Theorem, and the proof is done. □

We now come to the main theorem of this section.

Theorem 3.0.3 (Hardy's uncertainty principle)**.** *Suppose $C, a, b > 0$ and suppose $f : \mathbb{R} \to \mathbb{C}$ satisfies*

- $\forall x \in \mathbb{R} : \ |f(x)| \leq Ce^{-\pi a x^2}$,
- $\forall y \in \mathbb{R} : \ |\widehat{f}(y)| \leq Ce^{-\pi b y^2}$.

If $ab > 1$, then $f = 0$. If $ab = 1$, then there is a constant $C' \in \mathbb{C}$ such that $f(x) = C'e^{-\pi a x^2}$.

By considering $g(x) = f(x/\sqrt{a})/\max\{C, C\sqrt{a}\}$, it suffices to prove the theorem with $C = a = 1$.

Proof. We first note that we can extend $\widehat{f}$ to an entire function: define $f : \mathbb{C} \to \mathbb{C}$ by

$$\widehat{f}(z) = \int f(x)e^{-2\pi i x z}\, dx.$$

The integral is well-defined since the integrand is bounded by an L^1 function, for each $z \in \mathbb{C}$:

$$|f(x)e^{-2\pi i x z}| = |f(x)|e^{2\pi x z_2} \leq e^{-\pi x^2 + 2\pi x z_2} \leq e^{-\pi x^2 + 2\pi |x||z_2|},$$

where $z_2 = \operatorname{Im} z$. Suppose $z_n \to z$, then $r = \sup |\operatorname{Im} z_n| < \infty$, so that the above bound, with r in place of z_2, allows us to apply Lebesgue's Dominated Convergence Theorem to yield $\widehat{f}(z_n) \to \widehat{f}(z)$. $\widehat{f}$ is therefore continuous. We want to use Morera's Theorem in order to show that f is entire: Let Δ be a compact triangle in $\mathbb{C}$, then yet another application of the above bound lets us use Fubini's Theorem

$$\int_\Delta \widehat{f}(z)\, dz = \int_\Delta \int_\mathbb{R} f(x)e^{-2\pi i x z}\, dx\, dz = \int_\mathbb{R} f(x) \left[\int_\Delta e^{-2\pi i x z} dz \right] dx = 0$$

so that $\widehat{f}$ is an entire function. Noting that

$$|\widehat{f}(z)| \leq \int_\mathbb{R} |f(x)e^{-2\pi i x z}|\, dx \leq \int_\mathbb{R} e^{-\pi x^2 + 2\pi x z_2}\, dx = e^{\pi z_2^2} \int_\mathbb{R} e^{-\pi (x - z_2)^2}\, dx = e^{\pi z_2^2},$$

we see that if we define an entire function F by $F(z) = e^{\pi z^2} \widehat{f}(z)$, then we have, for any $z_1, z_2 \in \mathbb{R}$, the two relations $|F(z_1 + iz_2)| \leq e^{\pi z_1^2}$ (by the inequality on $\widehat{f}$ just above) and $|F(z_1)| \leq e^{\pi(1-b)x^2}$ (by the decay assumption on $\widehat{f}$). We claim that this shows that F is actually a constant, so that $\widehat{f}(y) = F(0)e^{-\pi y^2}$ which finishes the proof if $b = 1$. If $b > 1$, the relation $|F(x)| \le e^{\pi(1-a)x^2}$ shows that $|F(z)| = \inf_{x\in\mathbb{R}} |F(x)| = 0$, which finishes the proof for $b > 1$.

It remains only to prove the claim. Suppose F is an entire function which satisfies the bounds

$$\begin{aligned} |F(z_1 + iz_2)| &\le e^{\pi z_1^2}, \\ |F(z_1)| &\le 1, \end{aligned}$$

for all $z_1, z_2 \in \mathbb{R}$. We want to show that this implies that F is constant. Since F is entire, it suffices to show that F is bounded. Also, since the two functions $z \mapsto F(-z)$ and $z \mapsto \overline{F(\overline{z})}$ are also entire and also satisfy the two above bounds, it suffices to show that F is bounded in the upper right quadrant. Define, for each $\delta > 0$, an entire function F_δ by $F_\delta(z) = e^{i\delta z^2}F(z)$. We have the pointwise relation $F_\delta(z) \to F(z)$ as $\delta \downarrow 0$, and it therefore suffices to show that F_δ is bounded by 1 in the upper right quadrant for each $\delta > 0$. To this end, it obviously suffices to bound F_δ by 1 on $\overline{\Gamma}_\alpha$ (defined as in (1)) for each $0 < \alpha < \pi/2$, since F_δ is already known to be bounded by 1 on the imaginary axis.

Fix $0 < \alpha < \pi/2$. First, for $r \ge 0$, $|F_\delta(r)| = |F(r)| \le 1$. Our goal is therefore to bound F_δ by 1 on some ray $R_\theta = \left\{re^{i\theta} \,\middle|\, r > 0\right\}$ with $\pi/2 > \theta > \alpha$, since then the lemma above gives us the conclusion. Note that for $0 < \phi < \pi/2$, we have $|F_\delta(re^{i\phi})| \le \exp(-2\delta r^2 \cos\phi \sin\phi + \pi r^2 \cos^2\phi)$, and that we therefore have $|F_\delta(re^{i\phi})| \le 1$ whenever ϕ satisfies $\pi/2 > \phi \ge \tan^{-1}(\pi/2\delta)$. Taking $\pi/2 > \theta > \max\{\alpha, \tan^{-1}(\pi/2\delta)\}$, the proof is done. $\square$

4 Beurling's uncertainty principle

A theorem due to Beurling (whose proof was published by Hörmander [1991]) states that if $f \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)\widehat{f}(y)|e^{2\pi|xy|}\,dxdy < \infty, \tag{2}$$

then $f = 0$ (in L^1). The theorem has been extended into a characterization of Hermite functions by Bonami et al. [2003]. Here, however, we will instead follow work by Hedenmalm [2012].

First of all, we note that (2) is a quite strong condition; since $1 \le e^{2\pi xy}$, (2) implies $\|f\|_1\|\widehat{f}\|_1 < \infty$. But then we also have $\widehat{f} \in L^1$ so that $f \in C_0$, and therefore $f \in L^p$ for all $p \ge 1$. We therefore start our quest by attempting to weaken the condition (2), i.e. we look for other conditions which are implied by (2).

Suppose f satisfies (2). If we put $\mathcal{S} = \{\lambda \in \mathbb{C} \,|\, \operatorname{Im}\lambda < 1\}$, we obviously have

$$|f(x)\widehat{f}(y)\exp(2\pi ixy\lambda)| \le |f(x)\widehat{f}(y)|\exp(2\pi|xy|) \tag{3}$$

for all $\lambda \in \overline{\mathcal{S}}$, so that we can define a function $F : \overline{\mathcal{S}} \to \mathbb{C}$ by

$$F(\lambda) = \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{f(x)}\widehat{f}(y)e^{2\pi ixy\lambda}\,dxdy.$$

Continuity of F follows easily by (3) and Lebesgue's Dominated Convergence Theorem. Likewise, a combination of Morera's Theorem and Fubini's Theorem (which (3) allows us to use) shows easily that F is holomorphic in $\mathcal{S}$. Furthermore, since $\widehat{f} \in L^1$, we may apply the Fourier Inversion Formula to obtain

$$F(t) = \int_{\mathbb{R}} \overline{f(x)} f(tx)\, dx, \tag{4}$$

for $t \in \mathbb{R}$.

Now, suppose only that $f \in L^2$ and define $F : \mathbb{R}^\times \to \mathbb{C}$ by (4). Here, $R^\times := \mathbb{R} \setminus \{0\}$. We want to show that F is continuous. Suppose $\epsilon > 0$ and $t \neq 0$. We have

$$\begin{aligned} |F(t+h) - F(t)| &\leq \int_{\mathbb{R}} |\overline{f(x)}||f(tx+hx) - f(tx)|\, dx \\ &\leq \|f\|_2 \left(\int_{\mathbb{R}} |f(tx+hx) - f(tx)|^2\, dx \right)^{1/2} \\ &= \|f\|_2 |t|^{-1} \left(\int_{\mathbb{R}} |f(x+(h/t)x) - f(x)|^2\, dx \right)^{1/2}. \end{aligned}$$

If $g \in C_c$ with support contained in a ball of radius M, then g is uniformly continuous, so there is a $\delta' > 0$ such that

$$\left(\int_{\mathbb{R}} |g(x+(h/t)x) - g(x)|^2\, dx \right)^{1/2} \leq \epsilon (2(M+1))^{1/2}$$

whenever $|h| < \delta := \min\{1, \delta'\}|t|/M$. Since C_c is dense in L^2, we can pick $g \in C_c$ such that $\|f - g\|_2 \leq \epsilon$, and therefore

$$|F(t+h) - F(t)| \leq \epsilon \|f\|_2 |t|^{-1} (2 + (2(M+1))^{1/2}),$$

whenever $|h| < \delta$. This is what we wanted. If we put $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and denote the area element on $\mathbb{C}$ by dA, then we will show the following result by Hedenmalm [2012], which shows a succesful weakening of the hypothesis (2):

Theorem 4.0.1. *Suppose $f \in L^2$ and define $F : \mathbb{R}^\times \to \mathbb{C}$ by (4). If F has a holomorphic extension to a neighborhood of $\overline{\mathbb{D}} \setminus \{i, -i\}$ (which we continue to denote by F), and this extension satisfies*

$$\int_{\mathbb{D}} |F(\lambda)|^2 |1 + \lambda^2|\, dA(\lambda) < \infty,$$

then there is a constant $c \geq 0$ such that $F(\lambda) = c(1+\lambda^2)^{-1/2}$. If $\inf_{\mathbb{D}} |F(\lambda)|^2 |1+\lambda^2| = 0$, then $\|f\|_2^2 = F(1) = 0$, and therefore $f(x) = 0$ almost everywhere.

We note that if f satisfies (2), then F is bounded in $\mathbb{D}$, by the above discussion. Therefore, the weighted square area-integral in the theorem is finite, and we also have $\inf_{\mathbb{D}} |F(\lambda)|^2 |1+\lambda^2| = 0$ (by letting λ tend to i). Beurling's uncertainty principle is therefore a special case of the above theorem. Before giving the proof of the theorem, we give a characterization of removable singularities in terms of a square area-integrability condition. For $z_0 \in \mathbb{C}$ and $\epsilon > 0$, Let $\mathbb{D}(z_0, \epsilon) = \{z \in \mathbb{C} \,|\, |z - z_0| < \epsilon\}$ denote the disc of radius ϵ at z_0 and let $\mathbb{D}'(z_0, \epsilon) = \{z \in \mathbb{C} \,|\, 0 < |z - z_0| < \epsilon\}$ denote the punctured disc of radius ϵ at z_0.

Lemma 4.0.2. *If $\Phi \in \mathcal{O}(\mathbb{D}'(z_0, \epsilon))$ satisfies*

$$\int_{\mathbb{D}(z_0,\epsilon/2)} |\Phi|^2 \, dA < \infty,$$

then $\Phi \in \mathcal{O}(\mathbb{D}(z_0, \epsilon))$.

Thus, if Φ is square area-integrable in some neighborhood of z_0, then Φ extends holomorphically across z_0. Conversely, if Φ is holomorphic in some neighborhood of z_0, then Φ is bounded in some neighborhood of z_0, and is therefore square area-integrable in some neighborhood of z_0.

Proof of Lemma 4.0.2. Recall that if Φ is holomorphic in a punctured disc $\mathbb{D}'(z_0, \epsilon)$, then Φ has a Laurent expansion

$$\Phi(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

which converges in $\mathbb{D}'(z_0, \epsilon)$ and converges absolutely in the ring

$$\mathcal{R}(z_0, r, R) = \{z \in \mathbb{C} \,|\, 0 < r/2 \leq |z - z_0| \leq R\}$$

for any $0 < r < R < \epsilon$. From absolute convergence, Fubini's Theorem gives us

$$\begin{aligned} \int_0^1 |\Phi(z_0 + re^{2\pi i\theta})|^2 \, d\theta &= \int_0^1 \left(\sum_n \sum_m a_n \overline{a_m} r^{n+m} e^{2\pi i(n-m)\theta} \right) d\theta \\ &= \sum_n \sum_m a_n \overline{a_m} r^{n+m} \int_0^1 e^{2\pi i(n-m)\theta} \, d\theta = \sum_{n=-\infty}^{\infty} |a_n|^2 r^{2n}, \end{aligned}$$

which we identify as Parseval's Formula for a trigonometric system. Suppose that we know that Φ is square area-integrable in a disc of radius $\epsilon/2$ around z_0, i.e.

$$\int_0^{\epsilon/2} \int_0^1 |\Phi(z_0 + re^{2\pi i\theta})|^2 \, d\theta r dr < \infty.$$

Applying Parseval's Formula, we obtain

$$\int_0^{\epsilon/2} |a_n|^2 r^{2n+1} \, dr \leq \int_0^{\epsilon/2} \left(\sum_{n=-\infty}^{\infty} |a_n|^2 r^{2n} \right) r dr = \int_{\mathbb{D}(z_0,\epsilon/2)} |\Phi|^2 \, dA < \infty.$$

Consequently, we must have $a_n = 0$ for all $n \leq -1$, i.e. z_0 is a removable singularity and Φ is actually holomorphic in the entire disc $\mathbb{D}(z_0, \epsilon)$. $\square$

With this lemma at hand, we are ready for the proof of Theorem 4.0.1. Let us put

$$\mathcal{L} = \{it \in \mathbb{C} \,|\, t \in \mathbb{R} : \ |t| \geq 1\}$$

Proof of Theorem 4.0.1. We note that $\lambda \mapsto \sqrt{1+\lambda^2}$ is holomorphic in $\mathbb{C} \setminus \mathcal{L}$. We can therefore define a holomorphic function Φ in an open set containing $\overline{\mathbb{D}} \setminus \{i, -i\}$ by $\Phi(\lambda) = F(\lambda)\sqrt{1+\lambda^2}$. A change of variables gives, for all $t \in \mathbb{R}^\times$,

$$F(t) = \int_{\mathbb{R}} \overline{f(x)} f(tx)\, dx = |t|^{-1} \int_{\mathbb{R}} \overline{f(x/t)} f(x)\, dx = |t|^{-1} \overline{F(1/t)}.$$

In terms of Φ, this translates to

$$\Phi(t) = F(t)\sqrt{1+t^2} = \overline{F(1/t)}\sqrt{1+1/t^2} = \overline{\Phi(1/t)} = \overline{\Phi(1/\overline{t})}, \tag{5}$$

for all $t \in \mathbb{R}^\times$. This motivates the following: Let $V \supset \overline{\mathbb{D}} \setminus \{i, -i\}$ be the open set which F extends holomorphically to. We can assume that V is connected (by considering only the connected component containing $\overline{\mathbb{D}} \setminus \{i, -i\}$). Put $V_0 = V \setminus \{0\}$. Then the inversion mapping $\iota \in \mathcal{O}(\mathbb{C} \setminus \{0\}$ defined by $\iota : \lambda \mapsto \lambda^{-1}$ maps V_0 biholomophically to a connected neighborhood $\iota(V_0) \supset \overline{\mathbb{D}}_e \setminus \{i, -i\}$, where $\mathbb{D}_e = \mathbb{C} \setminus \overline{\mathbb{D}}$ is the open exterrior disc. Define $\Psi \in \mathcal{O}(\iota(V_0))$ by

$$\Psi(\lambda) = \overline{\Phi(\iota(\overline{\lambda}))}.$$

Consider the open set $V \cap \iota(V_0)$, and let V_1 be the connected component of $V \cap \iota(V_0)$ containing 1, which is open since $\mathbb{C}$ is locally connected. Also, V_1 contains the 'right half' of the unit circle,

$$V_1 \supset \left\{x + iy \,\middle|\, x, y \in \mathbb{R}, x^2 + y^2 = 1, x > 0\right\},$$

since this is a connected subset which is in $V \cap \iota(V_0)$. Lastly, V_1 contains an interval from the real line, and by (5) we have $\Phi(t) = \overline{\Phi(\overline{t})} = \Psi(t)$ there. But then $\Phi(\lambda) = \Psi(\lambda)$ in all of V_1, since V_1 is connected. The same argument works for -1. Extend Φ to $\mathbb{C} \setminus \{i, -i\}$ by putting

$$\Phi(\lambda) = \begin{cases} \Phi(\lambda) & \text{if } \lambda \in \mathbb{D}, \\ \Phi(\lambda) = \Psi(\lambda) & \text{if } \lambda \in V_1 \cup V_{-1}, \\ \Psi(\lambda) = \overline{\Phi(\iota(\overline{\lambda}))} & \text{if } \lambda \in \mathbb{D}_e. \end{cases}$$

Then we have $\Phi \in \mathcal{O}(\mathbb{C} \setminus \{i, -i\}$. We wish to apply the lemma above to show that Φ extends to an entire function. The integrability assumption of the theorem gives us

$$\int_{\mathbb{D}} |\Phi|^2\, dA < \infty.$$

However, we can use (5) and a change of variables to obtain an integrability condition in $\mathbb{D}_e$ as well:

$$\begin{aligned} \int_{\mathbb{D}_e} |\Phi(\lambda)|^2 |\lambda|^{-4}\, dA(\lambda) &= \int_0^1 \int_1^\infty \left| \Phi\left(\frac{1}{re^{-2\pi i\theta}}\right)\right|^2 r^{-3} dr d\theta \\ &= \int_0^1 \int_0^1 |\Phi(re^{2\pi i\theta})|^2\, r dr d\theta = \int_{\mathbb{D}} |\Phi|^2\, dA. \end{aligned}$$

If we let $2\mathbb{D}$ denote that disc of radius 2 around 0, we thus have

$$\int_{(2\mathbb{D})\setminus\mathbb{D}} |\Phi(\lambda)|^2 \, dA(\lambda) \le 16 \int_{\mathbb{D}_e} |\Phi(\lambda)|^2 |\lambda|^{-4} \, dA(\lambda) < \infty,$$

so that Φ square area-integrable in some open set around both i and $-i$. Therefore Φ extends to an entire function. But Φ is bounded in the compact set $\overline{2\mathbb{D}}$, and by definition therefore also bounded in the open exterrior disc (this is obvious from how we extended Φ; see above). Therefore Φ is bounded, and thus constant. Thus, we have

$$F(\lambda) = c(1+\lambda^2)^{-1/2},$$

with $c = \Phi(1) = \sqrt{2}\|f\|_2^2 \ge 0$. □

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