Stationary Phase

[Based on notes by Kim Petersen, TEX'ed by Jakob Østergaard Pedersen]

Introduction The stationary phase approximation deals with bounding the oscillation of sine integrals. The methods were developed in the 19th century by George Gabriel Stokes and Lord Kelvin.
We shall study integrals of the form

$$I_{u,\varphi}(\lambda) = \int_{\mathbb{R}^n} u(x)e^{i\lambda\varphi(x)}dx = \int_{\mathbb{R}^n} ue^{-\lambda\varphi}dm \qquad , \ n \in \mathbb{N}$$

for $u \in C_c^\infty(\mathbb{R}^n)$, $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and $\lambda \in \mathbb{R}$. Here $u(x)$ is the amplitude and $\lambda\varphi(x)$ is the phase.

Example 1. When $n = 1$ and $\varphi = -\mathrm{id}$ we have

$$I_{u,-\mathrm{id}}(\lambda) = \int_{\mathbb{R}} u(x)e^{-i\lambda x}dx = \mathcal{F}u(\lambda) \to 0$$

as $\lambda \to \pm\infty$ by the Riemann-Lebesgue Lemma for $u \in L^1(\mathbb{R})$.

How does $I_{u,\varphi}$ behave for general φ when $\lambda \to \pm\infty$? Actually, we see that

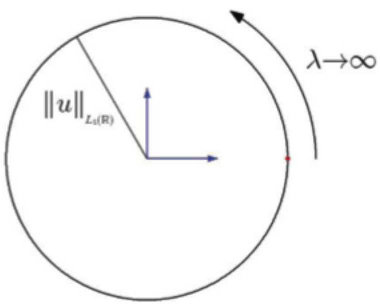
$$I_{u,\varphi}(-\lambda) = \int_{\mathbb{R}^n} u(x)e^{-i\lambda\varphi(x)}dx = \overline{\int_{\mathbb{R}^n} \overline{u(x)}e^{i\lambda\varphi(x)}dx} = \overline{I_{\overline{u},\varphi}(\lambda)}$$

so we only need to concentrate on $\lambda \to \infty$.

Example 2. Setting $n = 1$, $u > 0$, and $\varphi = 1$ we get

$$I_{u,1}(\lambda) = \int_{\mathbb{R}^n} u(x)e^{i\lambda}dx = e^{i\lambda}\|u\|_{L^1}$$

so in this case the values of $I_{u,1}$ moves around the circle with centre 0 and radius $\|u\|_{L^1}$ as depicted below.



Theorem 3 (Principle of non-stationary phase)**.** *Let $u \in C_c^\infty(\mathbb{R}^n)$ and let $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ such that $\nabla\varphi \neq 0$ on* supp u*. Then*

$$|I_{u,\varphi}(\lambda)| \leq C_{N,u,\varphi}\lambda^{-N}$$

for all $N \in \mathbb{N}_0$ and $\lambda > 0$.

Proof: First observe that on supp u we have

$$\frac{1}{i\lambda}\frac{\nabla\varphi}{|\nabla\varphi|^2} \cdot \nabla e^{i\lambda\varphi} = \frac{1}{i\lambda}\frac{\nabla\varphi}{|\nabla\varphi|^2} \cdot \left(e^{i\lambda\varphi} \cdot i\lambda\nabla\varphi\right) = e^{i\lambda\varphi}$$

so

$$\begin{aligned} I_{u,\varphi}(\lambda) &= \frac{1}{i\lambda}\int_{\mathbb{R}^n} u\frac{\nabla\varphi}{|\nabla\varphi|^2}\nabla e^{i\lambda\varphi}dm \\ &= -\frac{1}{i\lambda}\int_{\mathbb{R}^n} \nabla\left(u \cdot \frac{\nabla\varphi}{|\nabla\varphi|^2}\right)e^{i\lambda\varphi}dm \\ &= -\frac{1}{i\lambda}\int_{\mathbb{R}^n} u_1 e^{i\lambda\varphi}dm \\ &= -\frac{1}{i\lambda}I_{u_1,\varphi}(\lambda) \\ &= \left(-\frac{1}{i\lambda}\right)^2 I_{u_2,\varphi}(\lambda) \\ &\vdots \\ &= \left(-\frac{1}{i\lambda}\right)^N I_{u_N,\varphi}(\lambda) \end{aligned}$$

where $u_i = \nabla\left(u_{i-1}\frac{\nabla\varphi}{|\nabla\varphi|^2}\right)$, $i = 1, 2, \ldots, N$ and $u_0 = u$. Now using the triangle inequality

$$|I_{u,\varphi}(\lambda)| = \left|\left(-\frac{1}{i\lambda}\right)^N I_{u_N,\varphi}(\lambda)\right| \leq \lambda^{-N}\left|\int_{\mathbb{R}^n} u_N e^{i\lambda\varphi}dm\right| = \lambda^{-N}\underbrace{\int_{\text{supp } u}|u_N|dm}_{:=C_{N,u,\varphi}}$$

□

A consequence of this is that the essential contributions to the asymptotic behavior of $I_{u,\varphi}$ come from the stationary points of φ. (Recall that a stationary point is a point $x_0 \in \mathbb{R}^n$ where $\nabla\varphi(x_0) = 0$.) We shall in general assume that the stationary points of φ satisfy $\det(\partial_i\partial_j\varphi(x_0))_{ij} \neq 0$, i.e. the stationary points are *non-degenerate.*

Lemma 4 (The Morse Lemma)**.** *Let $x_0 \in \mathbb{R}^n$ be a non-degenerate stationary point for $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$. Then there exists a neighbourhood V of x_0 and U of 0, numbers $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$ and a diffeomorphism $\mathcal{H} : V \to U$ with $\mathcal{H}(x_0) = 0$ such that*

$$\varphi \circ \mathcal{H}^{-1}(x) = \varphi(x_0) + \varepsilon_1 x_1^2 + \ldots + \varepsilon_n x_n^2 = \varphi(x_0) + \langle x, \mathcal{E}x\rangle$$

with $\mathcal{E} := \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$.

(Recall that a *diffeomorphism* is a bijective map with both it and its inverse being differentiable.)

Remark. It can be shown that the number of $+1$'s amongst the $\varepsilon_1, \ldots, \varepsilon_2$ is equal to the number of positive eigenvalues of $(\partial_i\partial_j\varphi(x_0))_{ij}$.

Remark. The lemma is named after Marston Morse for use in differential topology; he wished to study the topology of a manifold using differential functions on that manifold.

Proof of the Morse Lemma: We first prove the theorem for $x_0 = 0$ and $\varphi(0) = 0$ and then apply the result to the function $x \mapsto \varphi(x + x_0) - \varphi(x_0)$. We shall prove the following statement by induction:
For all $N \in \{1, 2, \ldots, n+1\}$ there exists neighbourhoods $V_n, U_n \subseteq \mathbb{R}^n$ of 0, a diffeomorphism $\mathcal{H}_N : V_n \to U_N$ with $\mathcal{H}_N(0) = 0$, numbers $\varepsilon_m \in \{\pm 1\}$, and a set of functions $\{q_{ij}^{(N)} \mid i, j \in \mathbb{N}, N \leq i, j \leq n\}$ with

$i_N)$ $q_{ij}^{(N)} \in C^\infty(V_n)$

$ii_N)$ $q_{ij}^{(N)} = q_{ji}^{(N)}$

$iii_N)$ $q_{lk}^{(N)}(0) \neq 0$ for some l, k (non-zero for some particular l, k)

such that

$$\varphi \circ \mathcal{H}^{-1}(x) = \sum_{m=1}^{N-1} \varepsilon_m x_m^2 + \sum_{N \leq i,j \leq n} q_{ij}^{(N)}(x) x_i x_j. \tag{1}$$

The sums are 0 if respectively $N = 1$ or $N = n+1$ (because $i, j = 0$ and thus $x_i = x_j = 0$).
Induction start ($N = 1$): We use Taylor's formula, [GG, (A.8)] and obtain

$$\begin{aligned}\varphi(x) &= \varphi(0+x) = \sum_{|\alpha|<2} \frac{x^\alpha}{\alpha!}\partial^\alpha\varphi(0) + \sum_{|\alpha|=2} \frac{2}{\alpha!}x^\alpha \int_0^1 (1-\theta)\partial^\alpha\varphi(\theta x)d\theta \\ &= \sum_{1\le i,j\le n} q_{ij}^{(1)}(x)x_i x_j\end{aligned}$$

with

$$q_{ij}^{(1)}(x) = \frac{2}{i!j!}\int_0^1 (1-\theta)\partial_i\partial_j\varphi(\theta x)d\theta$$

This equality is not really obvious but by being careful with the indices it can be verified Note also that in $\mathbb{R}^n$, if $|\alpha| = 2$, we have the following cases: Case 1: $\alpha_i = 2$ and $\alpha_j = 0$ for all $j \neq i$; Case 2: $\alpha_i = 1$, $\alpha_j = 1$ for each j, and $\alpha_k = 0$ for $k \neq i, j$ (the αs are pairwise 1).
We just need to verify the three conditions on $q_{ij}^{(1)}$:

$i_1)$ $\varphi \in C^\infty(\mathbb{R}^n)$ so $q_{ij}^{(1)}$ is too.

$ii_1)$ This is obvious.

$iii_1)$ $q_{ij}^{(1)}(0) = \frac{2}{i!j!}\partial_i\partial_j\varphi(0)\left[\theta - \frac{1}{2}\theta^2\right]_0^1 = \frac{1}{i!j!}\partial_i\partial_j\varphi(0)$ so the claim follows as 0 is a non-degenerate stationary point for φ.

So the formula holds for $N = 1$.

Induction step: Assume (1) holds for some $N \in \{1, 2, \ldots, n\}$ and without loss of generality we may furthermore assume that $q_{NN}^{(N)}(0) \neq 0$. So by continuity of $q_{NN}^{(N)}$ there exists a neighbourhood $W \subset V_N$ of 0 on which $q_{NN}^{(N)} \neq 0$, so setting $\varepsilon_N = \mathrm{sign}(q_{NN}^{(N)}(0))$

we get

$$\begin{aligned}
\varphi\circ\mathcal{H}^{-1}(x) &= \sum_{m=1}^{N-1}\varepsilon_m x_m^2 + \sum_{N\le i,j\le n} q_{ij}^{(N)}(x)x_ix_j \\
&= \sum_{m=1}^{N-1}\varepsilon_m x_m^2 + \sum_{N\le i,j\le n} q_{ij}^{(N)}(x)x_ix_j \\
&\quad + \sum_{N+1\le i,j\le n}\frac{q_{Ni}^{(N)}(x)q_{Nj}^{(N)}(x)}{q_{NN}^{(N)}(x)}x_ix_j - \sum_{N+1\le i,j\le n}\frac{q_{Ni}^{(N)}(x)q_{Nj}^{(N)}(x)}{q_{NN}^{(N)}(x)}x_ix_j \\
&= \sum_{m=1}^{N-1}\varepsilon_m x_m^2 + \sum_{N+1\le i,j\le n} q_{ij}^{(N)}(x)x_ix_j + \sum_{\substack{N\le i,j\le n\\ i=N\vee j=N}} q_{ij}^{(N)}x_ix_j \\
&\quad + \sum_{N+1\le i,j\le n}\frac{q_{Ni}^{(N)}(x)q_{Nj}^{(N)}(x)}{q_{NN}^{(N)}(x)}x_ix_j - \sum_{N+1\le i,j\le n}\frac{q_{Ni}^{(N)}(x)q_{Nj}^{(N)}(x)}{q_{NN}^{(N)}(x)}x_ix_j \\
&= \sum_{m=1}^{N-1}\varepsilon_m x_m^2 + \varepsilon_N\left(\sqrt{|q_{NN}^{(N)}(x)|}x_N + \frac{\varepsilon_N}{\sqrt{|q_{NN}^{(N)}(x)|}}\sum_{j=N+1}^{n} q_{Nj}^{(N)}(x)x_j\right)^2 \\
&\quad + \sum_{N+1\le i,j\le n}\left(q_{ij}^{(N)}(x) - \frac{q_{Ni}^{(N)}(x)q_{Nj}^{(N)}(x)}{q_{NN}^{(N)}(x)}\right)x_ix_j
\end{aligned}$$

This stems from

$$\begin{aligned}
\ell_N &:= \sqrt{|q_{NN}^{(N)}(x)|}x_N + \frac{\varepsilon_N}{\sqrt{|q_{NN}^{(N)}(x)|}}\sum_{j=N+1}^{n} q_{Nj}^{(N)}(x)x_j \quad ; \\
\varepsilon_N(\ell_N)^2 &= \varepsilon_N\left(|q_{NN}^{(N)}|x_N^2 + \frac{\varepsilon_N^2}{|q_{NN}^{(N)}|}\left(\sum_{j=N+1}^{n} q_{Nj}^{(N)}x_j\right)\left(\sum_{i=N+1}^{n} q_{Ni}^{(N)}x_i\right) + 2\varepsilon_N\sum_{j=N+1}^{n} q_{Nj}^{(N)}x_Nx_j\right) \\
&= \sum_{N+1\le i,j\le n}\frac{q_{Ni}^{(N)}q_{Nj}^{(N)}}{q_{NN}^{(N)}}x_ix_j + \sum_{\substack{N\le i,j\le n\\ i=N\vee j=N}} q_{ij}^{(N)}x_ix_j
\end{aligned}$$

and

$$\sum_{N\le i,j\le n} q_{ij}^{(N)}(x)x_ix_j = \sum_{\substack{N\le i,j\le n\\ i=N\vee j=N}} q_{ij}^{(N)}x_ix_j + \sum_{N+1\le i,j\le n} q_{ij}^{(N)}(x)x_ix_j$$

Consider then ℓ_N. This is a C^∞-function on W with

$$\partial_k \ell_N(x) = \begin{cases} 0 & , \ 1 \leq k < N \\ \frac{\varepsilon_N}{\sqrt{|q_{NN}^{(N)}(x)|}} \left(\sum_{j=N}^{n} \left(\partial_k q_{Nj}^{(N)}(x) - \frac{\partial_k q_{NN}^{(N)}(x) q_{Nj}^{(N)}(x)}{2q_{NN}^{(N)}(x)} \right) x_j + q_{Nk}^{(N)}(x) \right) & , \ N \leq k \leq n \end{cases},$$

so defining $\mathcal{H} : W \to \mathbb{R}^n$ by

$$\mathcal{H}(x) = (x_1, x_2, \ldots, x_{N-1}, \ell_N(x), x_{N+1}, \ldots, x_n) - (y_1, \ldots, y_N, \ldots, y_n)$$

gives a C^∞-map with

$$\det J\mathcal{H}(0) = \begin{array}{r} \\ \\ \\ N\text{'th} \to \\ \\ \\ \\ \end{array} \begin{vmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & \frac{\varepsilon_N q_{NN}^{(N)}(0)}{\sqrt{|q_{NN}^{(N)}(0)|}} & \frac{\varepsilon_N q_{N(N+1)}^{(N)}(0)}{\sqrt{|q_{NN}^{(N)}(0)|}} & \cdots & \frac{\varepsilon_N q_{Nn}^{(N)}(0)}{\sqrt{|q_{NN}^{(N)}(0)|}} \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{vmatrix} = \sqrt{|q_{NN}^{(N)}(0)|} \neq 0$$

where $J\mathcal{H}$ is the Jacobian of $\mathcal{H}$ with 0s everywhere else. The Inverse Function Theorem gives then the existence of open sets $V_{N+1} \subset W, U_{N+1} \subset \mathbb{R}^n$ such that $0 \in V_{N+1}, 0 = \mathcal{H}(0) \in U_{N+1}$, and $\mathcal{H}|_{V_{N+1}}$ is a diffeomorphism $V_{N+1} \to U_{N+1}$.
Hence

$$\begin{aligned} (\varphi \circ \underbrace{\mathcal{H}_N^{-1} \circ \mathcal{H}|_{V_{N+1}}^{-1}}_{\mathcal{H}_{N+1}^{-1} \text{ diffeomorphism}})(y) &= \sum_{m=1}^{N-1} \varepsilon_m y_m^2 + \varepsilon_N y_N^2 + \sum_{N+1 \leq i,j \leq n} \underbrace{\left(q_{ij}^{(N)} - \frac{q_{Ni}^{(N)} q_{Nj}^{(N)}}{q_{NN}^{(N)}} \right) \circ \mathcal{H}|_{V_{N+1}}^{-1}(y)}_{q_{ij}^{(N+1)}(y)} y_i y_j \\ &= \sum_{m=1}^{N} \varepsilon_m y_m^2 + \sum_{N+1 \leq i,j \leq n} q_{ij}^{(N+1)}(y) y_i y_j \end{aligned}$$

where $q_{ij}^{(N+1)}$ satisfies i_{N+1})-iii_{N+1}):

i_{N+1}) $q_{ij}^{(N+1)} \in C^\infty(V_{N+1})$: $q_{ij}^{(N)}$ is C^∞ so this is trivial

ii_{N+1}) $q_{ij}^{(N+1)} = q_{ji}^{(N+1)}$: Obvious

iii_{N+1}) $q_{lk}^{(N+1)}(0) \neq 0$: By the chain rule

$$(\partial_l \partial_k (\varphi \circ \mathcal{H}_{N+1}^{-1})(0)) = [D\mathcal{H}_{N+1}^{-1}(0)]^T (\partial_i \partial_j \varphi(0)) D\mathcal{H}_{N+1}^{-1}(0)$$

so $[iii_{N+1})]^r$ would imply that $\det(\partial_i \partial_j \varphi(0)) = 0$

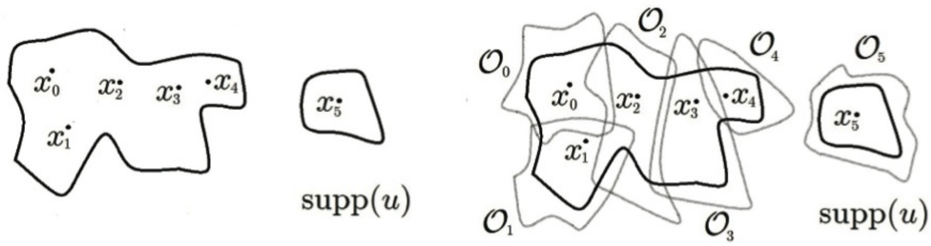
It is left as an exercise to apply the result to the function $x \mapsto \varphi(x + x_0) - \varphi(x_0)$. Thus The Morse Lemma is proved.

$\square$

A consequence of The Morse Lemma is the following corollary, which will not be proved:

Corollary 5. *A non-degenerate stationary point x_0 of $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ is an isolated stationary point.*

Furthermore, the compact set supp u can only contain finitely many non-degenerate stationary points of φ. Let $\{\mathcal{O}_j\}_{j=0}^N$ be a finite open cover of supp u such that $\mathcal{O}_j$ contains precisely one stationary point of φ as shown on the picture below.



Then Partition of Unity [GG, Theorem 2.17] gives that for $\psi_j \in C_c^\infty(\mathcal{O}_j, [0,1])$ with $\sum_{j=0}^N \psi_j = 1$ on supp u, we have

$$I_{u,\varphi}(\lambda) = \sum_{j=0}^N \int_{\mathbb{R}^n} \psi_j(x)u(x)e^{i\lambda\varphi(x)}dx = \sum_{j=0}^N I_{u\psi_j,\varphi}(\lambda) \qquad , \ \psi_j u \in C_c^\infty(\mathcal{O}_j)$$

so we can assume that φ has one and only one stationary point in supp u. (In the above, we just multiplied with 1 under the integral.)

The Morse Lemma inspires us to consider the case $\varphi(x) = \langle x, Ax\rangle$, where A is a real, symmetric, and invertible $n \times n$-matrix. We need the following lemma:

Lemma 6. *Let A be a real, symmetric, and invertible $n \times n$-matrix.*

$$\mathcal{F}(e^{i\lambda\langle x,Ax\rangle})(\xi) = \left(\det\left(\frac{A}{\pi i}\right)\right)^{-\frac{1}{2}} \lambda^{-\frac{n}{2}} e^{-i\frac{\langle \xi, A\xi\rangle}{4\lambda}}.$$

Proof: The Lemma will not be proved but it follows from Exercise 1a) with $B = -i\lambda A$ and then some convergence analysis of $B + \varepsilon I$ as $\varepsilon \to 0^+$.

Proposition 7. *Let A be a real, symmetric, and invertible $n \times n$-matrix. Then for all $u \in C_c^\infty(\mathbb{R}^n)$, $\lambda > 0$, and all integers $k > 0$ and $s > \frac{\pi}{2}$ we have*

$$\left| I_{u,\langle x,Ax\rangle}(\lambda) - \left(\det\left(\frac{A}{\pi i}\right)\right)^{-\frac{1}{2}} \cdot \sum_{j=0}^{k-1} \frac{\langle D, A^{-1}D\rangle^j u(0)}{(4i)^j j!} \lambda^{-\frac{n}{2}-j} \right| \leq C_k \left(\frac{\left\|A^{-1}\right\|}{\lambda}\right)^{\frac{n}{2}+k} \sum_{|\alpha|\leq s+2k} \|D^\alpha u\|_{L^2}$$

where $D = \frac{1}{i}(\partial_1, \ldots, \partial_n)$ *as usual.*

Proof: First note that

$$\begin{array}{rcl} I_{u,\langle x,Ax\rangle}(\lambda) & = & \displaystyle\int_{\mathbb{R}^n} u(x) e^{i\lambda\langle x,Ax\rangle} dx \\ \mathcal{F}^{-1} = (2\pi)^{-n}\overline{\mathcal{F}} \rightarrow & = & \langle e^{i\lambda\langle x,Ax\rangle}, \mathcal{F}(2\pi)^{-n}\overline{\mathcal{F}}u\rangle \\ & = & \left(\det\left(\dfrac{A}{\pi i}\right)\right)^{-\frac{1}{2}} \lambda^{-\frac{n}{2}} (2\pi)^{-n} \displaystyle\int_{\mathbb{R}^n} e^0 \left(e^{-i\frac{\langle \xi, A\xi\rangle}{4\lambda}} \mathcal{F}u(\xi)\right) d\xi \\ & = & \left(\det\left(\dfrac{A}{\pi i}\right)\right)^{-\frac{1}{2}} \lambda^{-\frac{n}{2}} \mathcal{F}^{-1}\left(e^{-i\frac{\langle \xi, A\xi\rangle}{4\lambda}} \mathcal{F}u\right)(0) \end{array}$$

so

$$\left| I_{u,\langle x,Ax\rangle}(\lambda) - \left(\det\left(\frac{A}{\pi i}\right)\right)^{-\frac{1}{2}} \cdot \sum_{j=0}^{k-1} \lambda^{-\frac{n}{2}-j} \frac{\langle D, A^{-1}D\rangle^j u}{(4i)^j j!}(0)\right|^2$$

$$\begin{array}{rl} = & \underbrace{\left|\det\left(\dfrac{A}{\pi i}\right)\right|^{-1}}_{\propto|\det(A^{-1})|\leq\|A^{-1}\|^n} \lambda^{-n} \left| \mathcal{F}^{-1}\left(e^{-i\frac{\langle \xi, A\xi\rangle}{4\lambda}} \mathcal{F}u\right)(0) - \displaystyle\sum_{j=0}^{k-1} \lambda^{-j} \frac{\langle D, A^{-1}D\rangle^j u}{(4i)^j j!}(0)\right|^2 \\ \lesssim & \left(\dfrac{\|A^{-1}\|}{2}\right)^n \left\| \mathcal{F}^{-1}\left(e^{-i\frac{\langle \xi, A\xi\rangle}{4\lambda}} \mathcal{F}u\right) - \displaystyle\sum_{j=0}^{k-1} \lambda^{-j} \frac{\langle D, A^{-1}D\rangle^j u}{(4i)^j j!}\right\|_\infty^2 \\ \text{(Sobolev)} \quad \lesssim & \left(\dfrac{\|A^{-1}\|}{2}\right)^n \displaystyle\sum_{|\alpha|\leq s} \left\| D^\alpha \mathcal{F}^{-1}\left(e^{-i\frac{\langle \xi, A\xi\rangle}{4\lambda}} \mathcal{F}u\right) - D^\alpha \sum_{j=0}^{k-1} \lambda^{-j} \frac{\langle D, A^{-1}D\rangle^j}{(4i)^j j!} u\right\|_{L^2}^2 \\ \text{(Parseval)} \quad \backsimeq & \left(\dfrac{\|A^{-1}\|}{2}\right)^n \displaystyle\sum_{|\alpha|\leq s} \left\| \underbrace{\left| e^{-i\frac{\langle \xi, A\xi\rangle}{4\lambda}} - \sum_{j=0}^{k-1} \lambda^{-j} \frac{\langle \xi, A^{-1}\xi\rangle^j}{(4i)^j j!}\right|}_{(*)} \mathcal{F}D^\alpha u\right\|_{L^2}^2 \\ \lesssim & \left(\dfrac{\|A^{-1}\|}{2}\right)^n \displaystyle\sum_{|\alpha|\leq s} \left\| \underbrace{\left|\frac{\langle \xi, A^{-1}\xi\rangle}{\lambda}\right|^k}_{\leq(\|A^{-1}\|/\lambda)^k|\xi|^{2k}} \mathcal{F}D^\alpha u\right\|_{L^2}^2 \\ \lesssim & \left(\dfrac{\|A^{-1}\|}{2}\right)^{n+2k} \displaystyle\sum_{|\alpha|\leq s} \|D^\alpha u\|_{L^2}^2 \end{array}$$

Note that the D^α act on $\mathcal{F}u$ and u respectively in the above 4th line, (Sobolev) refers to Sobolev's Embedding Theorem [GG, Theorem 6.11], (Parseval) refers to the Parseval-Plancherel Theorem [GG, Theorem 5.5], and we remark that putting $w = -i\frac{\langle \xi, A\xi\rangle}{4\lambda}$, we get that

$$(*) = \left| e^w - \sum_{j=0}^{k-1} \frac{w^j}{j} \right| \leq \left| \frac{k}{k!} w^k \int_0^1 (1-\theta)^{k-1} e^{\theta w} dw \right| \leq \frac{|w|^k}{k!}$$

where the first inequality follows from Taylor's formula, [GG, (A.8)], and the second inequality stems form w being imaginary.
The desired result follows now by taking square roots on both sides and using the fact that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$.

□

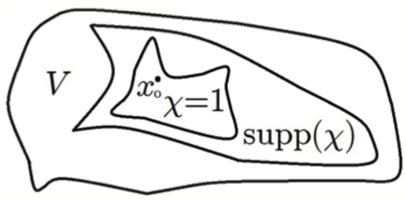
Now for the main theorem:

Theorem 8 (Principle of stationary phase)**.** *Let* $u \in C_c^\infty(\mathbb{R}^n)$ *and consider a* $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ *with one and only one stationary point* x_0 *in* supp u*; this is assumed to be non-degenerate. Then for all integers* $k > 0$ *we have*

$$\left| I_{u,\varphi}(\lambda) - e^{i\lambda\varphi(x_0)} \sum_{j=0}^{k-1} T_j u(x_0) \lambda^{-\frac{n}{2}-j} \right| \leq C_{k,n,u,\varphi} \lambda^{-\frac{n}{2}-k},$$

where T_j *is a differential operator of order* $2j$ *with* C^∞*-coefficients.*

Proof: Let $\mathcal{H} : V \to U$ be as in Morse's Lemma and choose $\chi \in C_c^\infty(V)$ with $\chi = 1$ near x_0.



Then

$$\begin{aligned}
I_{u,\varphi}(\lambda) &= \int_V e^{i\lambda\varphi(x)} (\chi u)(x) dx + \int_{\mathbb{R}^n} e^{i\lambda\varphi(x)} [(1-\chi)u](x) dx \\
&= \int_U e^{i\lambda\varphi\circ\mathcal{H}^{-1}(x)} (\chi u) \circ \mathcal{H}^{-1}(x) \left|\det(J\mathcal{H}^{-1}(x))\right| dx + I_{(1-\chi)u,\varphi}(\lambda) \\
&= \int_U e^{i\lambda\varphi(x_0) + \langle x, \mathcal{E}x\rangle} f_u(x) dx + I_{(1-\chi)u,\varphi}(\lambda) \\
&= e^{i\lambda\varphi(x_0)} I_{f_u, \langle x, \mathcal{E}x\rangle}(\lambda) + I_{(1-\chi)u,\varphi}(\lambda)
\end{aligned}$$

where $f_u \in C_c^\infty(\mathbb{R}^n)$. Then by setting

$$T_j u = \left(\det\left(\frac{\mathcal{E}}{\pi i}\right)\right)^{-\frac{1}{2}} \frac{\langle D, \mathcal{E}^{-1} D\rangle^j f_u}{(4i)^j j!}$$

and letting s be the smallest integer $> \frac{n}{2}$ we get

$$\begin{aligned}
&\left| I_{u,\varphi}(\lambda) - e^{i\lambda\varphi(x_0)} \sum_{j=0}^{k-1} T_j u(x_0) \lambda^{-\frac{n}{2}-j} \right| \\
&\quad \leq \left| I_{f_u,\langle x, \mathcal{E}x\rangle}(\lambda) - \left(\det\left(\frac{\mathcal{E}}{\pi i}\right)\right)^{-\frac{1}{2}} \sum_{j=0}^{k-1} \frac{\langle D, \mathcal{E}^{-1} D\rangle^j f_u}{(4i)^j j!} \right| + \left| I_{(1-\chi)u,\varphi}(\lambda) \right| \\
&\quad \leq C_k \left\| \mathcal{E}^{-1} \right\|^{\frac{n}{2}+k} \sum_{|\alpha| \leq 2k+s} \left\| D^\alpha f_u \right\|_{L^2} \lambda^{-\frac{n}{2}-k} + C'_{k,n,u,\varphi} \lambda^{-\frac{n}{2}-k} \\
&\quad \leq C_{k,n,u,\varphi} \lambda^{-\frac{n}{2}-k}
\end{aligned}$$

as we wanted.

□

Remark. Observe that by definition of T_j and f_u we have

$$T_0 u(0) = \left(\det\left(\frac{\mathcal{E}}{\pi i}\right)\right)^{-\frac{1}{2}} f_u(0) = \left(\det\left(\frac{\mathcal{E}}{\pi i}\right)\right)^{-\frac{1}{2}} \left|\det J\mathcal{H}^{-1}(0)\right| u(x_0) = C_{\partial_i \partial_j \varphi(x_0))ij} u(x_0)$$

so

$$\left| I_{u,\varphi}(\lambda) - C_{(\partial_i \partial_j \varphi(x_0))ij} e^{i\lambda\varphi(x_0)} u(x_0) \lambda^{-\frac{n}{2}} \right| \leq C_{k,n,u,\varphi} \lambda^{-\frac{n}{2}-1}.$$

For further studies of methods of stationary phase, one could consider $I_{u,\varphi}(\lambda)$ with complex λ and/or φ, or one could remove the smoothness assumption on u and φ, or one could allow degenerate stationary points of φ on supp u.

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