Exercise sheet 4 — Pointwise a.e. convergence

Exercise class week 20+21

Exercise 13:

Let $\psi : [0, \infty) \to [0, \infty]$ be non-increasing, measurable and $\Psi : \mathbb{R}^n \to [0, \infty)$, $\Psi(x) = \psi(|x|)$. Furthermore let $\Psi_t(x) = t^{-n}\Psi\left(\frac{x}{t}\right)$.

a) Show $\forall f \in L^p(\mathbb{R}^n)$ and for almost every $x \in \mathbb{R}^n$ that

- $|f * \Psi(x)| \leq \left(\int_{\mathbb{R}^n} \Psi\right) \sup_{r>0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} |f(y)| \,\mathrm{d}y$
- $|f * \Psi_t(x)| \leq \left(\int_{\mathbb{R}^n} \Psi\right) \sup_{r>0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} |f(y)| \,\mathrm{d}y \quad \forall t > 0$

Hint: You may assume that $\Psi \in L^1(\mathbb{R}^n)$. Also, first take $\psi = \sum_{j=1}^{N} a_j \mathbb{1}_{(0,r_j)},\ a_j, r_j \in (0, \infty)$. An arbitrary ψ can be approximated by such sums.

b) Let $\varphi \in L^1(\mathbb{R}^n)$, $\int \varphi = 1$, $|\varphi(x)| \leq \psi(|x|)$. Assume ψ ist bounded. Show $\forall f \in L^p(\mathbb{R}^n)$ and almost every $x \in \mathbb{R}^n$:

$$\sup_{t>0} |f * \varphi_t(x)| \leq C_\varphi \sup_{r>0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} |f(y)| \,\mathrm{d}y$$

c) Check that the proof of Lebesgue's differentiation theorem (*using b)!*) yields $\forall f \in L^p(\mathbb{R}^n)$

$$\lim_{t\to 0^+} f * \varphi_t = f(x) \quad \text{for almost every } x \in \mathbb{R}^n.$$

Exercise 14:

Let $f \in L^1_{loc}(\mathbb{R}^d)$. A point $x \in \mathbb{R}^d$ is called a Lebesgue point of f provided that

$$\exists c \in \mathbb{C} : \lim_{r\to 0^+} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} |f - c| = 0.$$

a) Show that almost every $x \in \mathbb{R}^d$ is a Lebesgue point of f and that $c = f(x)$ for almost every $x \in \mathbb{R}^d$.

b) Fundamental theorem of calculus: Show that $F(x) = \int\limits_0^x f(y)\,\mathrm{d}y$ is differentiable at every Lebesgue point of f and that $F' = f$ almost everywhere.

Exercise 15: For reading Tao's notes only - conditional expectations

Let $\left(X, \tilde{B}, \mu\right)$ be a measure space, B a σ-finite σ-subalgebra of $\tilde{B}$. Denote the orthogonal projection from $L^2\left(X, \tilde{B}, \mu\right)$ to its closed subspace $L^2(X, B, \mu)$ by $E(\cdot, B)$. Show that:

a) $$\int_X f\overline{E(g,B)}\,\mathrm{d}\mu = \int_X E(f,B)\overline{g}\,\mathrm{d}\mu = \int_X E(f,B)E(g,B)\,\mathrm{d}\mu \quad \forall f,g \in L^2\left(X, \tilde{B}, \mu\right)$$

b) $E(\cdot, B)$ is the unique map $L^2\left(X, \tilde{B}, \mu\right) \to L^2(X, B, \mu)$ sucht that

$$\int_X E(f,B)g\,\mathrm{d}\mu = \int_X fg\,\mathrm{d}\mu \quad \forall f \in L^2\left(X, \tilde{B}, \mu\right),\ g \in L^2(X, B, \mu).$$

c) Deduce that $E(\overline{f}, B) = \overline{E(f,B)}$, $f \leq g \Rightarrow E(f,B) \leq E(g,B)$,
$\forall h \in L^\infty(X, B, \mu):\ E(hf, B) = hE(f,B)$ and $|E(f,B)| \leq E(|f|, B)$.

d) $E(\cdot, B) : L^p\left(X, \tilde{B}, \mu\right) \to L^p(X, B, \mu)$ is continuous $\forall 1 \leq p \leq \infty$ and $||E(\cdot, B)||_{p \to p} \leq 1$.

Hint: Show this for $p = 1$ and $p = \infty$.

e) Let $B_1 \subset B_2 \subset \ldots$ be an increasing family of B's and let $\bigvee_{n+1}^{\infty} B_n$ the σ-algebra generated by $\bigcup_{n=1}^{\infty} B_n$. Show $\forall 1 \leq p \leq \infty$ and $\forall f \in L^p\left(X, \tilde{B}, \mu\right)$: $E(f, B_n) \overset{n\to\infty}{\longrightarrow} E\left(f, \bigvee_{n+1}^{\infty} B_n\right)$ in L^p.

Hint: See Tao, Chapter 2, Prop. 3.5.

f) Let $X = \mathbb{R}$, $\tilde{B} =$ Borel σ-algebra, $\mu =$ Lebesgue measure, $B_n = \sigma$-algebra generated by $\left\{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right\}_{k\in\mathbb{Z}}$. Show $\bigvee_{n+1}^{\infty} B_n = \tilde{B}$ and $E(f, B_n)(x) = 2^n \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} f(y)\,\mathrm{d}y$.