Exercise sheet 3

Exercise class week 20

Integral operators and inequalities

Exercise 9:

For $\varphi \in L^1(-1,1)$ let $(F\varphi)(\lambda) := \int_{-1}^{1} e^{i\lambda x}\varphi(x)\,\mathrm{d}x$.

a) If $\varphi \in C_c^\infty(-1,1)$, show that $|(F\varphi)(\lambda)| \le C_k\,(1+|\lambda|)^{-k} \;\; \forall k \in \mathbb{N}$

b) If φ is the restriction of a function in $C^\infty(\mathbb{R})$, show that

$$\left|(F\varphi)(\lambda) - \frac{e^{i\lambda}}{i\lambda}\varphi(1) + \frac{e^{-i\lambda}}{i\lambda}\varphi(-1)\right| \le C\lambda^{-2}$$

Hint: Integration by parts!

Remark: Related expansions of integrals with oscillating integrands will be discussed later in this course.

Exercise 10:

Let $\Omega = \{(x,y) \in \mathbb{R}^2 : \mathfrak{Re}\, e^{-ixy} > 0\}$. Show that the integral operator associated to the kernel $k(x,y) = e^{-ixy}\mathbb{1}_\Omega(x,y)$ is a truncation of the Fourier transform, which is not bounded on $L^2(\mathbb{R})$.

Hint: Show that

$$\frac{\mathfrak{Re}\,\langle T_k \mathbb{1}_I, \mathbb{1}_I\rangle_{L^2(\mathbb{R})}}{\langle \mathbb{1}_I, \mathbb{1}_I\rangle_{L^2(\mathbb{R})}}$$

is large for large intervals $I \subset \mathbb{R}$.

Exercise 11: Hardy's inequalities

a) For $w : (0,\infty) \to (0,\infty)$ measurable, we denote by L^p_w the L^p-space associated to the weighted Lebesgue measure λ_w, $\lambda_w(A) = \int_A w(x)\,\mathrm{d}x$ for any measurable $A \subset (0,\infty)$. I.e.

$$L^p_w = \left\{ f : (0,\infty) \to \mathbb{C} \text{ measurable: } ||f||_{p,w} := \left(\int_0^\infty |f(x)|^p\, w(x)\,\mathrm{d}x \right)^{\frac{1}{p}} < \infty \right\}.$$

The triangle inequality in this case says

$$\left|\left| \int_0^\infty f(y,\cdot)\,\mathrm{d}y \right|\right|_{p,w} \le \int_0^\infty ||f(y,\cdot)||_{p,w}\,\mathrm{d}y.$$

Let $p \geq 1,\ \alpha \in \mathbb{R}$ and $K : (0,\infty)\times(0,\infty) \to \mathbb{C}$ a measurable function satisfying $K(\lambda x, \lambda y) = \lambda^{-1}K(x,y)$ for $\lambda > 0$. Assume

$$C := \int_0^\infty |K(x,1)|\, x^{-\frac{1+\alpha}{p}}\, \mathrm{d}x < \infty.$$

Show that the operator $T_K f(y) = \int_0^\infty K(x,y)f(x)\,\mathrm{d}x$ is a bounded operator on L^p_w for $w(x) = x^\alpha$ and that $||T_K||_{L^p_w \to L^p_w} \leq C.$

Hint: Write $T_K f(y) = \int_0^\infty K(x,1)f(xy)\,\mathrm{d}x$ *and use the triangle inequality.*

b) Use a) to show that the "Hilbert integral" given by $K(x,y) = \frac{1}{x+y}$ defines a continuous operator on $L^p(0,\infty)$ for $p > 1$.

c) Use a) to deduce Hardy's inequalities

$$\left(\int_0^\infty \left(\int_0^x f(y)\,\mathrm{d}y\right)^p x^{-r-1}\,\mathrm{d}x\right)^{\frac{1}{p}} \leq \frac{p}{r}\left(\int_0^\infty (yf(y))^p\, y^{-r-1}\,\mathrm{d}y\right)^{\frac{1}{p}},$$

$$\left(\int_0^\infty \left(\int_x^\infty f(y)\,\mathrm{d}y\right)^p x^{r-1}\,\mathrm{d}x\right)^{\frac{1}{p}} \leq \frac{p}{r}\left(\int_x^\infty (yf(y))^p\, y^{r-1}\,\mathrm{d}y\right)^{\frac{1}{p}}$$

where $f \geq 0,\ p \geq 1,\ r > 0.$

Exercise 12: "weak-type Schur test"
Slightly more sophisticated than Exercise 11 and Exercise 6 ist the weak-type Schur test (use Marcinkiewicz intead of Riesz-Thorin): If $k : X\times Y \to \mathbb{C}$ is measurable and $||k(x,y)||_{L^{q_0,\infty}(Y)} \leq B_0$, for almost every $x \in X$, $||k(x,y)||_{L^{p_1',\infty}(X)} \leq B_1$, for almost every $y \in Y$, $p_1,\ q_0 \in (1,\infty)$

$$\Rightarrow \quad \forall\theta \in (0,1):\ T_k : L^{p_\theta}(X) \to L^{q_\theta}(Y) \text{ bounded for}$$
$$\frac{1}{p_\theta} = 1-\theta+\frac{\theta}{p_1},\ \frac{1}{q_\theta} = \frac{1-\theta}{q_0} \text{ and } ||T_k||_{L^{p_\theta}\to L^{q_\theta}} \leq C_{p_1,q_0,\theta} B_0^{(1-\theta)} B_1^{\theta}.$$

Use this to show that $f \mapsto |x|^{-\alpha} * f$ defines a bounded operator $L^p(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ for $1 < p,r < \infty,\ 0 < \alpha < n$ and $\frac{1}{p} + \frac{\alpha}{n} = \frac{1}{r} + 1$.