Diffun2, Fredholm Operators

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H_1 and H_2 denote Hilbert spaces in the following.

Definition 1. A Fredholm operator is an operator $T \in B(H_1, H_2)$ such that $\ker T$ and $\mathrm{coker}T := H_2/\mathrm{im}T$ are finite dimensional. The dimension of the cokernel is called the codimension, and it is denoted $\mathrm{codim}T$.

Fredholm operators can also be studied on Banach spaces as well as on more general spaces, but here we will concentrate on Hilbert spaces.

Lemma 2. *If $T \in B(H_1, H_2)$ is a Fredholm operator, then $\mathrm{im}T$ is closed.*

Proof. Let $\tilde{T}$ denote the restriction of T to $(\ker T)^\perp$. $\tilde{T}$ is clearly bounded, and it is not hard to see that it is a Fredholm operator. Since T is a Fredholm operator, we can assume that $\mathrm{codim}T = n$. Let $S : \mathbb{C}^n \to H_2$ be a linear mapping onto the complement of $\mathrm{im}T$ in H_2, and define $T_1 : (\ker T)^\perp \oplus \mathbb{C}^n \to H_2$ by $T_1(x, y) = \tilde{T}x + Sy$. T_1 is bijective and continuous. By the closed graph theorem, the inverse of T_1 is bounded and hence continuous. Hence $\mathrm{im}T = T_1\left((\ker T)^\perp \oplus \{0\}\right)$ is closed. □

Definition 3. Let $T \in B(H_1, H_2)$ be a Fredholm operator. Then we define its index by

$$\mathrm{index}T = \dim\ker T - \dim \mathrm{coker}T = \dim\ker T - \mathrm{codim}T$$

Theorem 4. *Let $T \in B(H_1, H_2)$ be bijective, and let $K \in B(H_1, H_2)$ be compact. Then $T + K$ is a Fredholm operator.*

Before proving this theorem, we recall that a compact operator maps any bounded sequence into a sequence which has a convergent subsequence.

Proof. $\ker(T + K)$ is a Hilbert space, and in particular it is a linear space, so for $x \in \ker(T + K)$ we have $Tx = -Kx$. Let $(x_n)_{n\in\mathbb{N}} \subseteq \ker(T + K)$ be a bounded sequence. Since K is a compact operator, the sequence $(Kx_n)_{n\in\mathbb{N}}$ has a convergent subsequence, $(Kx_{n_k})_{k\in\mathbb{N}}$. But $x_{n_k} \in \ker(T + K)$ for each $k \in \mathbb{N}$, and thus

$(Kx_{n_k})_{k\in\mathbb{N}} = (-Tx_{n_k})_{k\in\mathbb{N}}$, which tells us that $(x_{n_k})_{k\in\mathbb{N}}$ is convergent, since T^{-1} is bounded. Hence any bounded sequence in $\ker(T+K)$ has a convergent subsequence, which means that $\dim\ker(T+K) < \infty$, since an infinite dimensional Hilbert space has an infinite orthonormal sequence with no convergent subsequences.

We know that $H_2 = \overline{\mathrm{im}(T+K)} \oplus \ker(T^* + K^*)$, and since T^* is invertible and K^* is compact, we get by the above that $\dim\ker(T^* + K^*) < \infty$. This means that we only have to check that $\mathrm{im}(T+K)$ is closed in order to see that $\mathrm{codim}(T+K) < \infty$. To see this we split H_1 into the direct sum $H_1 = \tilde{H}_1 \oplus \ker(T+K)$, and we consider the restriction of $T+K$ to $\tilde{H}_1$. We want to show that for all $x \in \tilde{H}_1$ the inequality

$$\|x\| \leq c\|(T+K)x\| \tag{$*$}$$

holds for some $c > 0$. In order to show this inequality, we assume that for all $c > 0$ there exists $x \in \tilde{H}_1$ such that $\|x\| \geq c\|(T+K)x\|$. Then there exist sequences $(c_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$, $(x_n)_{n\in\mathbb{N}} \subseteq \tilde{H}_1$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$, $c_n \to \infty$ for $n \to \infty$, and $1 = \|x_n\| \geq c_n\|(T+k)x_n\|$ for all $n \in \mathbb{N}$. Hence $\|(T+K)x_n\| \leq \frac{1}{c_n} \to 0$ for $n \to \infty$. K is compact, and x_n has norm 1 for each $n \in \mathbb{N}$, so there exists a subsequence $(Kx_{n_k})_{k\in\mathbb{N}}$ of $(Kx_n)_{n\in\mathbb{N}}$ which is convergent; assume $Kx_{n_k} \to v \in H_2$ for $k \to \infty$. This means $Tx_{n_k} \to -v \in H_2$ for $k \to \infty$. Thus

$$x_{n_k} = T^{-1}Tx_{n_k} \to w = -T^{-1}v$$

for $k \to \infty$, where $w \in \tilde{H}_1$ with $\|w\| = 1$, since $\|x_{n_k}\| = 1$ for each $k \in \mathbb{N}$. But

$$(T+K)w = \lim_{k\to\infty}(Tx_{n_k} + Kx_{n_k}) = \lim_{k\to\infty} 0 = 0$$

contradicting $\tilde{H}_1 \perp \ker(T+K)$, and the claim follows, which means that we can now conclude that $\mathrm{im}(T+K)$ is closed. □

Note that when $T \in B(H_1, H_2)$ is a Fredholm operator, then T^* will also be a Fredholm operator, for it can be shown that $\mathrm{im}T^*$ is closed if and only if $\mathrm{im}T$ is closed, which we know is the case, and so it follows that

$$\mathrm{index}T^* = \dim\ker T^* - \dim\ker T^{**} = \dim\ker T^* - \dim\ker T = -\mathrm{index}T$$

since $H_2 = \overline{\mathrm{im}T} \oplus \ker T^*$ implies that $\mathrm{codim}T = \dim\ker T^*$.

Theorem 5. *$T \in B(H_1, H_2)$ is Fredholm if and only if there exist $S_1, S_2 \in B(H_2, H_1)$ and operators K_1 and K_2 which are compact on H_1 and H_2 respectively such that $S_1T = I + K_1$ and $TS_2 = I + K_2$.*

Proof. First assume that $T \in B(H_1, H_2)$ is a Fredholm operator. T defines a bijective operator $\tilde{T} : \tilde{H}_1 \to \tilde{H}_2$, where $\tilde{H}_1 = (\ker T)^\perp$ and $\tilde{H}_2 = \mathrm{im} T = (\ker T^*)^\perp$. Define $S_2 \in B(H_2, H_1)$ by $S_2 = \iota_{\tilde{H}_1} (\tilde{T})^{-1} \mathrm{pr}_{\mathrm{im} T}$. Then

$$TS_2 = T\iota_{\tilde{H}_1}(\tilde{T})^{-1}\mathrm{pr}_{\mathrm{im}T} = \mathrm{pr}_{\mathrm{im}T} = \mathrm{pr}_{\tilde{H}_2} = I - \mathrm{pr}_{\ker T^*}$$

Put $K_2 = -\mathrm{pr}_{\ker T^*}$, and one of the equations follows, since K_2 is a finite rank operator and therefore compact.

Since T^* is a Fredholm operator it follows in the same way that there exist operators S_3, K_3 with the required properties such that $T^* S_3 = I + K_3$. Using S_3^* and K_3^* as S_1 and K_1 respectively yields the other equation:

$$S_1 T = S_3^* T = (T^* S_3)^* = (I + K_3)^* = I + K_3^* = I + K_1$$

Assume now that there exist operators $S_1, S_2 \in B(H_2, H_1)$ and operators K_1 and K_2 which are compact on H_1 and H_2 respectively such that $S_1 T = I + K_1$ and $TS_2 = I + K_2$. We have the inclusions

$$\begin{aligned} \ker T &\subseteq \ker S_1 T = \ker(I + K_1) \\ \mathrm{im} T &\supseteq \mathrm{im} T S_2 = \mathrm{im}(I + K_2) \end{aligned}$$

By Theorem 4, $I + K_1$ and $I + K_2$ are Fredholm operators, and by the first inclusion above we conclude that $\dim \ker T \leq \dim \ker(I + K_1) < \infty$. By the second inclusion we conclude that $\mathrm{codim} T \leq \mathrm{codim}(I + K_2) < \infty$. Hence T is a Fredholm operator. □

Next we will look at some properties of Fredholm operators, but first we need a definition and a lemma:

Definition 6. Let $V_0, ..., V_n$ be vector spaces, and let $T_j : V_j \to V_{j+1}$, $0 \leq j \leq n-1$, be linear mappings. Then the sequence

$$V_0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-2}} V_{n-1} \xrightarrow{T_{n-1}} V_n$$

is called exact if $\mathrm{im} T_j = \ker T_{j+1}$, $j = 0, ..., n-2$.

Lemma 7. *Let*

$$V_0 = 0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-1}} V_{n-1} \xrightarrow{T_{n-1}} 0 = V_n$$

be an exact sequence with $\dim V_j < \infty$ *for all* $j = 0, ..., n$. *Then*

$$\sum_{j=0}^{n-1} (-1)^j \dim V_j = 0$$

Proof. For each j, we decompose $V_j = N_j \oplus Y_j$, where $N_j = \ker T_j$, and Y_j is some complement of N_j. The exactness of the sequence implies that $T_j : Y_j \to N_{j+1}$ is an isomorphism for each j. Hence $\dim V_j = \dim N_{j+1}$, which means that for $j \in \{0, ..., n-1\}$,

$$\dim V_j = \dim N_j + \dim Y_j = \dim N_j + \dim N_{j+1}$$

We also have that $\dim N_0 = 0$, and $\dim V_{n-1} = \dim N_{n-1}$. An easy calculation yields

$$\sum_{j=0}^{n-1} (-1)^j \dim V_j = 0$$

□

Theorem 8 (Multiplicative property of the index)**.** *If we are given two Fredholm operators* $T_1 \in B(H_1, H_2)$ *and* $T_2 \in B(H_2, H_3)$*, then* $T_2T_1 \in B(H_1, H_3)$ *is also a Fredholm operator, and it satisfies* $\text{index}T_2T_1 = \text{index}T_1 + \text{index}T_2$.

Proof. To see that T_2T_1 is a Fredholm operator, one can show that $\dim \ker T_2T_1 \leq \dim \ker T_1 + \dim \ker T_2 < \infty$ as well as $\text{codim}T_2T_1 \leq \text{codim}T_1 + \text{codim}T_2 < \infty$. Hence T_2T_1 is a Fredholm operator. To obtain the formula for the index, consider the exact sequence

$$0 \to \ker T_1 \xrightarrow{\iota} \ker T_2T_1 \xrightarrow{T_1} \ker T_2 \xrightarrow{q} H_2/\text{im}T_1 \xrightarrow{T_2} H_3/\text{im}T_2T_1 \xrightarrow{E} H_3/\text{im}T_2 \to 0$$

where $\iota : \ker T_1 \hookrightarrow \ker T_2T_1$ denotes the inclusion, $q : H_2 \supseteq \ker T_2 \to H_2/\text{im}T_1$ is the quotient map, and E maps equivalence classes modulo $\text{im}T_2T_1$ into equivalence classes modulo $\text{im}T_2$. Lemma 7 yields

$$\begin{aligned} 0 &= -\dim \ker T_1 + \dim \ker T_2T_1 - \dim \ker T_2 + \dim H_2/\text{im}T_1 - \dim H_3(\text{im}T_2T_1) + \dim H_3/\text{im}T_2 \\ &= -\text{index}T_1 - \text{index}T_2 + \text{index}T_2T_3 \end{aligned}$$

□

Theorem 9 (Invariance of Fredholm property and index under small pertubations)**.** *Let* $T \in B(H_1, H_2)$ *be a Fredholm operator. Then there exists a constant* $c > 0$ *such that for all operators* $S \in B(H_1, H_2)$ *with norm* $< c$*,* $T + S$ *is a Fredholm operator which satisfies* $\text{index}(T + S) = \text{index}T$.

Proof. Let R be such that $RT = I - \text{pr}_{\ker T}$. Then

$$R(T + S) = RT + RS = I - \text{pr}_{\ker T} + RS$$

For $\|S\| < \|R\|^{-1}$ we have that $\|RS\| < 1$. Hence $I + RS$ is invertible. In the same way, $T + S$ has a right Fredholm inverse, so by Theorem 5 we conclude that $T + S$ is a Fredholm operator.

When F is a finite rank operator on a Hilbert space H, $\text{index}(I + F) = 0$, for define $L := \text{im}F + (\ker F)^\perp$ with $\dim L < \infty$. Then $L \oplus L^\perp = H$, and we see that $(I + F)L \subseteq L + FL \subseteq L$ and $(I + F)|_{L^\perp} = I_{L^\perp}$, so L and $L^\perp$ are invariant under $I + F$, and we have that

$$\text{index}(I + F) = \text{index}((I + F)|_L) + \underbrace{\text{index}((I + F)|_{L^\perp})}_{=0}$$

Since $\dim L < \infty$, linear algebra yields $\text{index}((I + F)|_L) = 0$, since for any matrix A, $\dim L = \dim\ker A + \dim \text{im}A$.

Theorem 8 tells us that $\text{index}(I - \text{pr}_{\ker T}) = \text{index}RT = \text{index}R + \text{index}T$ from which we obtain the formula for the index:

$$\begin{aligned} \text{index}T &= -\text{index}R + \underbrace{\text{index}(I - \text{pr}_{\ker T})}_{=0} \\ &= -\text{index}((I + RS)^{-1}R) + \underbrace{\text{index}(I - (I - RS)^{-1}\text{pr}_{\ker T})}_{=0} \\ &= \text{index}(T + S) \end{aligned}$$

Where we used that $-\text{pr}_{\ker T}$ as well as $-(I - RS)^{-1}\text{pr}_{\ker T}$ are a finite rank operators.

□

Theorem 10 (Invariance of Fredholm property and index under compact pertubations)**.** *Let $T \in B(H_1, H_2)$ be a Fredholm operator. Then for any compact operator $S \in B(H_1, H_2)$, $T + S$ is a Fredholm operator, and* $\text{index}(T + S) = \text{index}T$ *holds.*

Proof. Let $T \in B(H_1, H_2)$ be a Fredholm operator, and let $S \in B(H_1, H_2)$ be a compact operator. Then by Theorem 5 there exist $S_1, S_2 \in B(H_2, H_1)$ and operators K_1 and K_2 which are compact on H_1 and H_2 respectively such that $S_1T = I + K_1$ and $TS_2 = I + K_2$. We see that

$$\begin{aligned} S_1(T + S) &= S_1T + S_1S = I + K_1 + S_1S = I + K_1' \\ (T + S)S_2 &= TS_2 + SS_2 = I + K_2 + SS_2 = I + K_2' \end{aligned}$$

where K_1' and K_2' are compact operators. By Theorem 5 we conclude that $T + S$ is a Fredholm operator.

$I + K_1$ has index 0 according to the proof of Theorem 9, so by Theorem 8,

$$0 = \text{index}(I + K_1) = \text{index}(S_1T) = \text{index}S_1 + \text{index}T$$

This tells us that $\text{index}S_1 = -\text{index}T$. Since K_1' is also a finite rank operator, $\text{index}(I + K_1') = 0$ so that

$$0 = \text{index}(S_1(T+S)) = \text{index}S_1 + \text{index}(T+S) = -\text{index}T + \text{index}(T+S)$$

Hence $\text{index}(T+S) = \text{index}T$. $\square$

The Fredholm property can also be attached to unbounded operators. Let $T : D(T) \to H_2$ be a closed operator with domain $D(T) \subseteq H_1$. Then T will be bounded as an operator on $D(T)$, which is a Hilbert space when equipped with the graph norm $\|u\|_{\text{graph}} = (\|u\|_{H_1}^2 + \|Tu\|_{H_2}^2)^{\frac{1}{2}}$. In this case, T is said to be a Fredholm operator when its kernel and cokernel are finite dimensional, and one defines the index in the exact same way as before. It can be shown that the image of T is still closed in H_2, and that Theorems 8-10 still hold when $D(T)$ is equipped with the graph norm.