Complex Interpolation

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Motivation: Generalization of Riesz-Thorin to general Banach spaces. Recall that Riesz-Thorin said that given a linear map T which is bounded between two pairs of L^p-spaces, say from L^{p_0} to L^{q_0} and from L^{p_1} to L^{q_1}, then it will also be bounded between all the pairs in between, i.e. from L^{p_θ} to L^{q_θ}, $\theta \in [0,1]$. For general Banach spaces we will define interpolation spaces for a given $\theta \in (0,1)$. The main result is then that if T is bounded between two pairs of Banach spaces, then it will also be bounded between all of the interpolation spaces.

Let E, F be Banach spaces. We first consider the case where $F \subseteq E$ continuously injected. Put

$$\Omega := \{z \in \mathbb{C} \mid 0 < \text{Re} z < 1\}$$

Define

$$\mathcal{H}_{E,F}(\Omega) = \{u \in C_b(\bar{\Omega}, E) \mid u \text{ holomorphic on } \Omega,\ u(1+iy) \in F \ \text{ and } \exists C : \|u(1+iy)\|_F \leq C,\ \forall y \in \mathbb{R}\}$$

(For future notation, I will just write $\mathcal{H}_{E,F}$).

Definition 1. For $\theta \in [0,1]$ we define the interpolation space $[E,F]_\theta$ by

$$[E,F]_\theta = \{u(\theta) \mid u \in \mathcal{H}_{E,F}\}$$

Proposition 1. *$[E,F]_\theta$ is a Banach space.*

Proof. As vector spaces, we clearly have that $[E,F]_\theta \cong \mathcal{H}_{E,F}/\{u : u(\theta) = 0\}$ (by the isomorphism $u(\theta) \mapsto [u]$). We will show that $\mathcal{H}_{E,F}$ is a Banach space and that $\{u : u(\theta) = 0\}$ is a closed subspace. Then we know that the quotient is also a Banach space.

For $u \in \mathcal{H}_{E,F}$, define

$$\|u\|_{\mathcal{H}_{E,F}} = \sup_{y\in\mathbb{R}} \|u(iy)\|_E + \sup_{y\in\mathbb{R}} \|u(1+iy)\|_F$$

We need to check that this is a norm. It is easy to see that it satisfies the triangle inequality and behaves well under multiplication with scalars. It remains to check that $\|u\|_{\mathcal{H}_{E,F}} = 0$ implies $u = 0$. If $\|u\|_{\mathcal{H}} = 0$, then $u(iy) = 0$ and $u(1+iy) = 0$ for all $y \in \mathbb{R}$ by definition of the norm. Recall Lindelöf's Theorem from the lectures: If v is bounded by A on the left boundary of Ω and by B on the right boundary, and if v doesn't grow too fast, then $\|v(\theta+it)\| \leq A^{1-\theta}B^{\theta}$ for all $\theta \in [0,1]$. In this case, we have that u is bounded by 0 on the boundaries so it follows that $u=0$ everywhere.

We proceed to show that $\mathcal{H}_{E,F}$ is complete with this norm (hence a Banach space). Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{H}_{E,F}$. Since $\|\cdot\|_F \geq \|\cdot\|_E$, we get by Lindelöf's Theorem that

$$\|u\|_{\mathcal{H}_{E,F}} \geq \sup_{y\in\mathbb{R}} \|u(iy)\|_E + \sup_{y\in\mathbb{R}} \|u(1+iy)\|_E \geq \sup_{z\in\bar{\Omega}} \|u(z)\|_E$$

Hence (u_n) is also a Cauchy sequence in $C_b(\bar{\Omega}, E)$, and since we know that this is a Banach space, we get that u_n converges to some $u \in C_b(\bar{\Omega}, E)$. We will show that $u \in \mathcal{H}_{E,F}$ and that $u_n \to u$ in $\mathcal{H}_{E,F}$. For every $z \in \bar{\Omega}$ we have that $u_n(z) \to u(z)$ in E. Note that for every $y \in \mathbb{R}$ we have that $(u_n(1+iy))$ is a Cauchy sequence in F, hence converges to a limit $\tilde{u}(1+iy) \in F$ (hence also in E). By uniqueness of limits, we get that $u(1+iy) = \tilde{u}(1+iy) \in F$. Moreover, for $\varepsilon > 0$:

$$\|u(1+iy)\|_F \leq \|u(1+iy) - u_n(1+iy)\|_F + \|u_n(1+iy)\|_F \leq C + \varepsilon$$

if n is big enough. Hence $\|u(1+iy)\|_F \leq C$. It remains to show that u is holomorphic. By the Cauchy Integral Theorem we have that

$$u_n(z_0) = \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{u_n(z)}{z_0 - z}\,\mathrm{d}z \to \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{u(z)}{z_0 - z}\,\mathrm{d}z$$

as $n \to \infty$ since u_n converges uniformly to u. Hence u satisfies the Cauchy Integral Theorem so u is holomorphic. Thus $u \in \mathcal{H}_{E,F}$ and it is clear that $u_n \to u$ in $\mathcal{H}_{E,F}$ (since we have pointwise convergence).

Finally, we prove that $U = \{u \in \mathcal{H}_{E,F} \mid u(\theta) = 0\}$ is a closed subspace of $\mathcal{H}_{E,F}$. Let (u_n) be a sequence in U that converges to $u \in \mathcal{H}_{E,F}$. We saw above that this implies that u_n converges uniformly to u, in particular $u_n(\theta) = 0$ converges to $u(\theta)$ so $u(\theta) = 0$ and hence $u \in U$. □

Note that for $\theta = 0$ we have that $[E,F]_0 = E$ and for $\theta = 1$ we have that $[E,F]_1 = F$ so we can interpret the spaces $[E,F]_\theta$ as being the spaces lieing in between E and F.

Proposition 2. *Let (E,F) and $(\tilde{E},\tilde{F})$ be as above. Let $T : E \to \tilde{E}$ be continuous and linear such that $T : F \to \tilde{F}$ continuously. Then for all $\theta \in [0,1]$: $T : [E,F]_\theta \to [\tilde{E},\tilde{F}]_\theta$ continuously.*

Proof. Let $x \in [E,F]_\theta$. There exists a $u \in \mathcal{H}_{E,F}(\Omega)$ such that $u(\theta) = x$. I want to show that $Tu \in \mathcal{H}_{\tilde{E},\tilde{F}}$.

- $Tu : \bar{\Omega} \to \tilde{E}$ continuous: clear
- Tu holomorphic: From KomAn: u holomorphic $\Leftrightarrow$ Cauchy integral formula holds. Hence we can write $u(z_0) = \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{u(z)}{z - z_0}\,\mathrm{d}z$ where $\overline{B_r(z_0)} \subseteq \Omega$. The integral is a limit of sums so since T is linear and continuous we get that

$$Tu(z_0) = T\left(\frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{u(z)}{z - z_0}\,\mathrm{d}z\right) = \frac{1}{2\pi i}\int_{\partial B_r(z_0)} \frac{Tu(z)}{z - z_0}\,\mathrm{d}z$$

  Hence Cauchys integral formula holds for Tu so Tu is holomorphic.

- Tu bounded on $\bar{\Omega}$: We know $\|u(z)\|_E \leq C$ for all $z \in \bar{\Omega}$ so by boundedness of T we get $\|Tu(z)\|_{\tilde{E}} \leq \|T\| \|u(z)\|_E \leq C\|T\|$.
- $Tu(1+iy) \in \tilde{F}$: We know $u(1+iy) \in F$ and $T : F \to \tilde{F}$.
- $\|Tu(1+iy)\|_{\tilde{F}}$ bounded: T bounded $F \to \tilde{F}$ (continuous and linear) so $\|Tu(1+iy)\|_{\tilde{F}} \leq \|T\| \|u(1+iy)\|_F \leq \|T\|\tilde{C}$.

Hence $Tu \in \mathcal{H}_{\tilde{E},\tilde{F}}(\Omega)$ so $Tx = Tu(\theta) \in [\tilde{E}, \tilde{F}]_\theta$ as wanted. It remains to show continuity:

$$\begin{aligned}
\|Tu(\theta)\|_{[\tilde{E},\tilde{F}]_\theta} &= \inf_{v(\theta)=0} \|Tu + v\|_{\mathcal{H}_{\tilde{E},\tilde{F}}} \leq \inf_{v'=Tv(\theta), v(\theta)=0} \|Tu + v'\|_{\mathcal{H}_{\tilde{E},\tilde{F}}} \\
&= \inf_{v(\theta)=0} \|T(u+v)\|_{\mathcal{H}_{\tilde{E},\tilde{F}}} \\
&= \inf_{v(\theta)=0} \left(\sup_{y\in\mathbb{R}} \|T(u+v)(iy)\|_{\tilde{E}} + \sup_{y\in\mathbb{R}} \|T(u+v)(1+iy)\|_{\tilde{F}} \right) \\
&\leq C \inf_{v(\theta)=0} \left(\sup_{y\in\mathbb{R}} \|(u+v)(iy)\|_E + \sup_{y\in\mathbb{R}} \|(u+v)(1+iy)\|_F \right) \\
&= C \inf_{v(\theta)=0} \|u+v\|_{\mathcal{H}_{E,F}} = C\|u(\theta)\|_{[E,F]_\theta}
\end{aligned}$$

where $C = \max\{\|T\|_{E\to E}, \|T\|_{F\to F}\}$. □

Next: H Hilbert space, $\mathcal{D}(A)$ the domain of a closed operator A on H (Banach space with the graph norm). Want to identify $[H, \mathcal{D}(A)]_\theta$.

We will consider operators on the form $A = U^{-1}BU$ where $U : H \to L^2(X, \mu)$ is unitary and B is a multiplication operator on $L^2(X, \mu)$ for some measure space (X, μ), that is

$$Bu(x) = M_b u(x) = b(x)u(x)$$

for some function b. (For example, A could be a positive selfadjoint operator (by the Spectral Theorem).) Here $\mathcal{D}(B) = \{u \in L^2 \mid bu \in L^2\}$. Since $AU^{-1} = U^{-1}B$ we have that $\mathcal{D}(A) = U^{-1}\mathcal{D}(B)$. We will assume that $b(x) \geq 1$. Since $b(x) \in \mathbb{R}$ it can be shown that B is selfadjoint which implies that it is also closed (DifFun1). (That B is selfadjoint is clear if b is bounded, but not clear in general.)

For $n \in \mathbb{N}$ we clearly have that $A^n = U^{-1}B^nU$. For $z \in \mathbb{C}$ we define $A^z = U^{-1}B^zU$ where $B^z u(x) = b(x)^z u(x)$. Then $\mathcal{D}(A^z) = U^{-1}\mathcal{D}(B^z)$ where $\mathcal{D}(B^z) = \{u \in L^2 \mid b^z u \in L^2\}$.

Proposition 3. *Let A be as above. For $\theta \in [0, 1] : [H, \mathcal{D}(A)]_\theta = \mathcal{D}(A^\theta)$.*

Proof. $\supseteq$: Let $v \in \mathcal{D}(A^\theta)$. Want to find $u \in \mathcal{H}_{H,\mathcal{D}(A)}(\Omega)$ such that $u(\theta) = v$. Define $u(z) = A^{-z+\theta}v$. Then $u(\theta) = v$. Check that $u \in \mathcal{H}_{H,\mathcal{D}(A)}(\Omega)$:

- $u : \Omega \to H$ welldefined (i.e. $v \in \mathcal{D}(A^{-z+\theta})$): Note that

$$u(z) = A^{-z+\theta}v = U^{-1}B^{-z+\theta}Uv = U^{-1}B^{-z}B^\theta Uv$$

$v \in \mathcal{D}(A^\theta) = U^{-1}\mathcal{D}(B^\theta)$ by assumption. Hence $Uv \in \mathcal{D}(B^\theta)$ so $B^\theta Uv \in L^2$. It remains to show that $B^\theta Uv \in \mathcal{D}(B^{-z})$. We have that $|b(x)^{-z}| = |b(x)|^{-\mathrm{Re}(z)} \leq 1$ since $b(x) \geq 1$ and $\mathrm{Re} z \geq 0$ on Ω. Hence $b^{-z}w \in L^2$ for all $w \in L^2$ so $B^\theta Uv \in L^2 \subseteq \mathcal{D}(B^{-z})$. Thus u is welldefined and maps Ω to H.

- u continuous on $\bar{\Omega}$ and holomorphic on Ω: easy to see since A is essentially a multiplication operator.

- u bounded on $\bar{\Omega}$: Put $b = Uv$. Then $b \in \mathcal{D}(B^\theta)$. We have that

$$\begin{aligned}\|u(z)\|_H &= \|U^{-1}B^{-z+\theta}Uv\|_H = \|U^{-1}B^{-z}B^\theta b\|_H \\ &= \|B^{-z}B^\theta b\|_{L^2} \leq \|B^{-z}\|\|B^\theta b\|_{L^2} \leq \|B^\theta b\|_{L^2} = C < \infty\end{aligned}$$

  since $\|B^{-z}\| \leq 1$ as above.

- $u(1+iy) \in \mathcal{D}(A)$: We need to show that $Au(1+iy) \in H$.

$$Au(1+iy) = AA^{-1-iy+\theta}v = A^{-iy}A^\theta v$$

  We have that $A^\theta v \in H$ since $v \in \mathcal{D}(A^\theta)$. The result follows if we can show that $\mathcal{D}(A^{-iy}) = H$ or equivalently that $\mathcal{D}(B^{-iy}) = L^2$. We have that $|b^{-iy}| = |b|^{-\mathrm{Re}(iy)} = |b|^0 = 1$. Hence $b^{-iy}w \in L^2$ for all $w \in L^2$ as wanted.

- $\|u(1+iy)\|_{\mathcal{D}(A)}$ bounded:

$$\|u(1+iy)\|_{\mathcal{D}(A)} = \|u(1+iy)\|_H + \|Au(1+iy)\|_H < \infty$$

  since u bounded on $\bar{\Omega}$ in $\|\cdot\|_H$ and we just showed that $\|Au(1+iy)\|_H = \|A^{-iy}A^\theta v\|_H \leq \|A^\theta v\| < \infty$.

$\subseteq$: Suppose $v \in [H, \mathcal{D}(A)]_\theta$ i.e. $v = u(\theta)$ for some $u \in \mathcal{H}_{H,\mathcal{D}(A)}$. We need to show that $A^\theta u(\theta) \in H$.

We would like to use the maximum modulus principle on the function $A^z u(z)$. Problem: Ω is not bounded. Instead of $A^z u(z)$ we will look at $A^z(1+i\varepsilon A)^{-1}u(z)$, which turns out to be a bounded function, and then let $\varepsilon \to 0$.

We first show that $A^z(1+i\varepsilon A)^{-1}u(z)$ is bounded on Ω:

$$A^z(1+i\varepsilon A)^{-1}u(z) = U^{-1}B^zU(U^{-1}(1+i\varepsilon B)U)^{-1}u(z) = U^{-1}B^z(1+i\varepsilon B)^{-1}Uu(z)$$

We know that $Uu(z)$ is a bounded holomorphic function $\Omega \to L^2$ since u by assumption is a bounded holomorphic function on Ω. Moreover, $B^z(1+i\varepsilon B)^{-1}$ is bounded since

$$\left|\frac{b^z}{1+i\varepsilon b}\right| = \frac{|b|^{\mathrm{Re} z}}{\sqrt{1+\varepsilon^2 b^2}} \leq \frac{|b|}{\sqrt{1+\varepsilon^2 b^2}} \leq \frac{|b|}{\sqrt{\varepsilon^2 b^2}} = \frac{1}{\varepsilon}$$

Here, we used that $\mathrm{Re} z \in [0,1]$ since $z \in \bar{\Omega}$. Hence $U^{-1}B^z(1+i\varepsilon B)^{-1}Uu(z)$ is a bounded holomorphic function from Ω into H so we can use the maximum modulus principle:

$$\begin{aligned}\sup_{z\in\bar{\Omega}} \|A^z(1+i\varepsilon A)^{-1}u(z)\|_H = \sup_{y\in\mathbb{R}} \max\big\{&\|A^{iy}(1+i\varepsilon A)^{-1}u(iy)\|_H, \\ &\|A^{1+iy}(1+i\varepsilon A)^{-1}u(1+iy)\|_H\big\}\end{aligned}$$

We need this to be bounded independently of ε.

The left boundary: We saw earlier that A^{iy} is bounded on H. Since U is unitary we have that $\|(1+i\varepsilon A)^{-1}\| = \|(1+i\varepsilon B)^{-1}\|$. Since $\left|\frac{1}{1+i\varepsilon b}\right| = \frac{1}{\sqrt{1+\varepsilon^2 b^2}} \leq 1$, we thus get that $\|(1+i\varepsilon A)^{-1}\| \leq 1$ for all ε. Hence

$$\|A^{iy}(1+i\varepsilon A)^{-1}u(iy)\|_H \leq C\|u(iy)\|_H$$

for some C independent of ε.

The right boundary: An easy calculation shows that $A^{1+iy}(1+i\varepsilon A)^{-1}u(1+iy) = A^{iy}(1+i\varepsilon A)^{-1}Au(1+iy)$ (since multiplication operators commute). We saw before that $A^{iy}(1+i\varepsilon A)^{-1}$ is bounded by C. Since $u \in \mathcal{H}_{H,\mathcal{D}(A)}$ we have that $u(1+iy) \in \mathcal{D}(A)$ so

$$\|A^{iy}(1+i\varepsilon A)^{-1}Au(1+iy)\|_H \leq C\|Au(1+iy)\|_H \leq C\|u(1+iy)\|_{\mathcal{D}(A)}$$

We conclude that

$$\sup_{z\in\bar{\Omega}} \|A^z(1+i\varepsilon A)^{-1}u(z)\|_H \leq \sup_{y\in\mathbb{R}} \max\left\{C\|u(iy)\|_H, C\|u(1+iy)\|_{\mathcal{D}(A)}\right\} \leq \tilde{C}$$

for some $\tilde{C}$ independent of ε. By letting $\varepsilon \to 0$ we get that $A^z u(z) \in H$ with $\|A^z u(z)\| \leq \tilde{C}$ for all $z \in \bar{\Omega}$. In particular, if we let $z = \theta$ we get $A^\theta u(\theta) \in H$ as wanted. □

Application:

Recall from DifFun:

$$\begin{aligned} H^s(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \langle\xi\rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^n)\}, \quad s \in \mathbb{R} \\ &= \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}\langle\xi\rangle^s \hat{u}(\xi) \in L^2\} \end{aligned}$$

Put $\Lambda^s := \mathcal{F}^{-1}M_{\langle\xi\rangle^s}\mathcal{F}$ with $\mathcal{D}(\Lambda^s) = \{u \in \mathcal{S}' \mid \Lambda^s u \in L^2\}$. Then

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \Lambda^s u \in L^2(\mathbb{R}^n)\} = \mathcal{D}(\Lambda^s)$$

If $s \geq 0$ then we know from DifFun that $H^s \subseteq L^2$ so Λ^s is an operator on L^2. Hence it satisfies all of the assumptions of the above proposition so we get that

$$[L^2(\mathbb{R}^n), H^s(\mathbb{R}^n)]_\theta = H^{k\theta}(\mathbb{R}^n), \quad \theta \in [0,1]$$

for $s \geq 0$.

More generally, we can show that for any $\sigma, t \in \mathbb{R}$ $(t \geq \sigma)$:

$$[H^\sigma(\mathbb{R}^n), H^t(\mathbb{R}^n)]_\theta = H^{\theta t + (1-\theta)\sigma}(\mathbb{R}^n), \quad \theta \in [0,1]$$

Proof. We saw in DifFun1 that Λ^s is an isometry $H^t \to H^{t-s}$ for all $s, t \in \mathbb{R}$. We must then have that

$$\Lambda^s[E,F]_\theta = [\Lambda^s E, \Lambda^s F]_\theta$$

$(\Lambda^s[E,F]_\theta = \{\Lambda^s u(\theta)\}$ and $[\Lambda^s E, \Lambda^s F]_\theta = \{(\Lambda^s u)(\theta)\}$.)

We are looking at $E = L^2 = H^0$, $F = H^k$. Then $\Lambda^s E = H^{-s}$ and $\Lambda^s F = H^{k-s}$. We get

$$[H^{-s}, H^{k-s}]_\theta = \Lambda^s[L^2, H^k]_\theta = \Lambda^s H^{k\theta} = H^{k\theta - s}$$

Given σ and t from above, we choose $s = -\sigma$ and $k = t - \sigma$. Then the above result yields

$$[H^\sigma, H^t]_\theta = H^{(t-\sigma)\theta+\sigma} = H^{\theta t+(1-\theta)\sigma}$$

as wanted. □

As an application of this, we can look at the multiplication operator M_φ given by $(M_\varphi u)(x) = \varphi(x)u(x)$. If $\varphi \in C^\infty_{L^\infty}$ (i.e. $D^\alpha\varphi \in L^\infty$ for all α) then it is clear that M_φ maps H^k to H^k if $k \in \mathbb{Z}$ (follows by Leibniz' rule for positive integers and by duality for negative integers). If $k \in \mathbb{R}$ is not an integer this is not clear. However, it easily follows from complex interpolation:

Given $s \in \mathbb{R}$ we know that $s \in [n, n+1]$ for some $n \in \mathbb{Z}$, so we can write $s = \theta n + (1-\theta)(n+1)$ for some $\theta \in [0,1]$. We know that M_φ maps H^n to H^n and H^{n+1} to H^{n+1} so by Proposition 2.3 it follows that M_φ maps $[H^n, H^{n+1}]_\theta$ to itself. But we just saw that $[H^n, H^{n+1}]_\theta = H^{\theta n+(1-\theta)(n+1)} = H^s$ so the result follows.

We can also use interpolation to generalize the definition of Sobolev spaces to arbitrary domains (so far only defined for $s \in \mathbb{N}_0$): Let $s \geq 0$ and let $\Omega \subseteq \mathbb{R}^n$ be open. Define

$$H^s(\Omega) = [L^2(\Omega), H^k(\Omega)]_\theta, \quad s = \theta k$$

Note: One has to prove that this does not depend on the choise of k, θ. It can be shown that

$$H^s(\Omega) \simeq H^s(\mathbb{R}^n)/\{u \in H^s(\mathbb{R}^n) : u|_\Omega = 0\}$$

This characterization can be used to define $H^s(\Omega)$ when $s \leq 0$.

We now want to define interpolation spaces for Banach spaces which are not contained in eachother. Let E, F be Banach spaces and suppose that they are both continuously injected into a locally convex topological vector space V. Put $G = E + F = \{e + f \mid e \in E, f \in F\}$. This is a Banach space with the norm

$$\|a\|_G = \inf\{\|e\|_E + \|f\|_F \mid a = e + f, e \in E, f \in F\}$$

As before we define

$$\mathcal{H}_{E,F}(\Omega) = \{u \in C_b(\bar{\Omega}, G) \mid u \text{ holomorphic on } \Omega,\ \|u(iy)\|_E \text{ and } \|u(1+iy)\|_F \text{ bounded for } y \in \mathbb{R}\}$$

Note that if $F \subseteq E$, this is the same definition as before since then $G = E$ (and we can choose $V = E$). The definition of the interpolation spaces is exactly the same as before:

$$[E, F]_\theta = \{u(\theta) \mid u \in \mathcal{H}_{E,F}(\Omega)\}, \quad \theta \in [0,1]$$

We can use this on L^p spaces, which are typically not contained in eachother (we can use $V = G = L^p + L^q$ when interpolating between L^p and L^q). We have the following result which I will not prove:

Proposition 4. *Let (X, μ) be a measure space. For $0 < \theta < 1$:*

$$[L^{p_0}(X, \mu), L^{p_1}(X, \mu)]_\theta = L^{p_\theta}(X, \mu)$$

where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

We have a similar proposition as before:

Proposition 5. *Let $(E, F), (\tilde{E}, \tilde{F})$ be as above. Suppose $T : G \to \tilde{G}$ is linear such that $T : E \to \tilde{E}$ is bounded and $T : F \to \tilde{F}$ is bounded. Then for all $\theta \in [0, 1]$: $T : [E, F]_\theta \to [\tilde{E}, \tilde{F}]_\theta$ is bounded.*

Remark 1. Proposition 4 and Proposition 5 give us Riesz-Thorin as stated in the lecture:

If T is bounded $L^{p_0} \to L^{q_0}$ and $L^{p_1} \to L^{q_1}$, then Proposition 5 says that $T : [L^{p_0}, L^{p_1}]_\theta \to [L^{q_0}, L^{q_1}]_\theta$ for all $\theta \in [0, 1]$. By Proposition 4: $T : L^{p_\theta} \to L^{q_\theta}$.