CHAPTER 1

The Basic Theorems of Fourier Analysis

The material contained in this chapter forms the core of our subject and is used throughout the later part of this book. Various approaches are possible; the same subject matter is treated, from different points of view, in Cartan and Godement [1], Loomis [1], and Weil [1].

Unless the contrary is explicitly stated, any group mentioned in this book will be abelian and locally compact, with addition as group operation and 0 as identity element (see Appendix B). The abbreviation LCA will be used for "locally compact abelian."

1.1. Haar Measure and Convolution

1.1.1. On every *LCA* group G there exists a non-negative regular measure m (see Appendix E), the so-called *Haar measure* of G, which is not identically 0 and which is *translation-invariant.* That is to say,

$$m(E + x) = m(E) \tag{1}$$

for every $x \in G$ and every Borel set E in G.

For the construction of such a measure, we refer to any of the following standard treatises: Halmos [1], Loomis [1], Montgomery and Zippin [1], and Weil [1]. The idea of the proof is to construct a positive translation-invariant linear functional T on $C_c(G)$, the space of all continuous complex functions on G with compact support. This means that $Tf \geqq 0$ if $f \geqq 0$ and that $T(f_x) = Tf$, where f_x is the translate of f defined by

$$f_x(y) = f(y - x) \qquad (y \in G). \tag{2}$$

As soon as this is done, the Riesz representation theorem shows that there is a measure m with the required properties, such that

(3) $$Tf = \int_G f\,dm \qquad (f \in C_c(G)).$$

1.1.2. *If V is a non-empty open subset of G, then $m(V) > 0$.* For if $m(V) = 0$ and K is compact, finitely many translates of V cover K, and hence $m(K) = 0$. The regularity of m then implies that $m(E) = 0$ for all Borel sets E in G, a contradiction.

1.1.3. We have spoken of *the* Haar measure of G. This is justified by the following uniqueness theorem:

If m and m' are two Haar measures on G, then $m' = \lambda m$, where λ is a positive constant.

Proof: Fix $g \in C_c(G)$ so that $\int_G g\,dm = 1$. Define λ by

$$\int_G g(-x)dm'(x) = \lambda.$$

For any $f \in C_c(G)$ we then have

$$\begin{aligned}\int_G f\,dm' &= \int_G g(y)dm(y) \int_G f(x)dm'(x)\\ &= \int_G g(y)dm(y) \int_G f(x+y)dm'(x)\\ &= \int_G dm'(x) \int_G g(y)f(x+y)dm(y)\\ &= \int_G dm'(x) \int_G g(y-x)f(y)dm(y)\\ &= \int_G f(y)dm(y) \int_G g(y-x)dm'(x) = \lambda \int_G f\,dm.\end{aligned}$$

Hence $m' = \lambda m$. Note that the use of Fubini's theorem was legitimate in the preceding calculation, since the integrands $g(y)f(x+y)$ and $g(y-x)f(y)$ are in $C_c(G \times G)$.

Thus Haar measure is unique, up to a multiplicative positive constant. If G is compact, it is customary to normalize m so that $m(G) = 1$. If G is discrete, any set consisting of a single point is assigned the measure 1. These requirements are of course contradictory if G is a finite group, but this will cause us no difficulty.

Having established the uniqueness of m, we shall now change our notation, and write $\int_G f(x)dx$ in place of $\int_G f\,dm$. Thus $dx, dy, \ldots$ will always denote integration with respect to Haar measure.

1.1.4. *For any Borel set E in G, $m(-E) = m(E)$.* For if we set $m'(E) = m(-E)$, m' is a Haar measure on G, and so there is a constant λ such that $m(-E) = \lambda m(E)$ for all Borel sets E. Taking E so that $-E = E$, we see that $\lambda = 1$.

1.1.5. Translation in $L^p(G)$. If G is a *LCA* group and $1 \leqq p \leqq \infty$, we shall write $L^p(G)$ in place of $L^p(m)$ (see Appendix E7). It is clear that the L^p-norms are translation invariant, i.e., that

(1) $$\|f_x\|_p = \|f\|_p \qquad (x \in G),$$

where, we recall, f_x is the translate of f defined by

(2) $$f_x(y) = f(y - x) \qquad (y \in G).$$

THEOREM. *Suppose $1 \leqq p < \infty$ and $f \in L^p(G)$. The map*

(3) $$x \to f_x$$

is a uniformly continuous map of G into $L^p(G)$.

Proof: Let $\varepsilon > 0$ be given. Since $C_c(G)$ is dense in $L^p(G)$ (Appendix E8) there exists $g \in C_c(G)$, with compact support K, such that $\|g - f\|_p < \varepsilon/3$, and the uniform continuity of g (Appendix B9) implies that there is a neighborhood V of 0 in G such that

(4) $$\|g - g_x\|_\infty < \frac{\varepsilon}{3}[m(K)]^{-1/p}$$

for all $x \in V$. Hence $\|g - g_x\|_p < \varepsilon/3$, and so

$$\|f - f_x\|_p \leqq \|f - g\|_p + \|g - g_x\|_p + \|g_x - f_x\|_p < \varepsilon$$

if $x \in V$. Finally, $f_x - f_y = (f - f_{y-x})_x$, so that $\|f_x - f_y\|_p < \varepsilon$ if $y - x \in V$, and the proof is complete.

Note that the same theorem (with the same proof) is true with $C_0(G)$ in place of $L^p(G)$, but that it is false for $L^\infty(G)$, unless G is discrete.

1.1.6. Convolutions. For any pair of Borel functions f and g on the *LCA* group G we define their convolution $f * g$ by the formula

(1) $$(f * g)(x) = \int_G f(x - y)g(y)dy$$

provided that

$$\int_G |f(x-y)g(y)|\,dy < \infty. \tag{2}$$

Note that the integral (1) can also be written in the form

$$\int_G f_y(x)g(y)\,dy \tag{3}$$

so that $f * g$ may be regarded as a limit of linear combinations of translates of f; this statement may be made precise, but we assign it only heuristic value at present. (See Theorem 7.1.2.)

THEOREM. (a) *If (2) holds for some $x \in G$, then $(f * g)(x) = (g * f)(x)$.*

(b) *If $f \in L^1(G)$ and $g \in L^\infty(G)$, then $f * g$ is bounded and uniformly continuous.*

(c) *If f and g are in $C_c(G)$, with compact supports A and B, then the support of $f * g$ lies in $A + B$, so that $f * g \in C_c(G)$.*

(d) *If $1 < p < \infty$, $1/p + 1/q = 1$, $f \in L^p(G)$, and $g \in L^q(G)$, then $f * g \in C_0(G)$.*

(e) *If f and g are in $L^1(G)$, then (2) holds for almost all $x \in G$, $f * g \in L^1(G)$, and the inequality*

$$\|f * g\|_1 \leqq \|f\|_1 \|g\|_1$$

holds.

(f) *If f, g, h are in $L^1(G)$, then $(f * g) * h = f * (g * h)$.*

Proof: Replacing y by $y + x$ in (1) and applying 1.1.4, we obtain

$$(f * g)(x) = \int_G f(-y)g(y+x)\,dy = \int_G f(y)g(-y+x)\,dy = (g * f)(x),$$

and (a) is proved.

Under the hypotheses of (b), it is clear that

$$|(f * g)(x)| \leqq \|f\|_1 \|g\|_\infty \qquad (x \in G)$$

so that $f * g$ is bounded. For $x \in G$, $z \in G$, we have

$$\begin{aligned}|(f * g)(x) - (f * g)(z)| &\leqq \int_G |f(x-y) - f(z-y)|\,|g(y)|\,dy \\ &\leqq \|f_{-x} - f_{-z}\|_1 \|g\|_\infty.\end{aligned}$$

Theorem 1.1.5 shows that the last expression can be made arbitrarily small by restricting $x - z$ to lie in a suitably chosen neighborhood of 0 and (b) follows.

If f vanishes outside A and g vanishes outside B, then $f(x-y)g(y) = 0$ unless $y \in B$ and $x - y \in A$, i.e., unless $x \in A + B$. Thus $f * g$ vanishes outside $A + B$, and (c) is proved.

To prove (d), choose sequences $\{f_n\}$ and $\{g_n\}$ in $C_c(G)$ such that $\|f_n - f\|_p \to 0$ and $\|g_n - g\|_q \to 0$ as $n \to \infty$. Hölder's inequality shows that $f_n * g_n \to f * g$ uniformly. By (c), $f_n * g_n \in C_c(G)$. Hence $f * g \in C_0(G)$, and (d) follows.

The proof of (e) will depend on Fubini's theorem, and we first have to show that the integrand in (1) is a Borel function on $G \times G$. Fix an open set V in the plane, put $E = f^{-1}(V)$, $E' = E \times G$, and let $E'' = \{(x, y) : x - y \in E\}$. Then E' is a Borel set in $G \times G$, and since the homeomorphism of $G \times G$ onto itself which carries (x, y) to $(x + y, y)$ maps E' onto E'', E'' is also a Borel set. Since $f(x - y) \in V$ if and only if $(x, y) \in E''$, we see that $f(x - y)$ is a Borel function on $G \times G$, and so is the product $f(x - y)g(y)$.

By Fubini's theorem,

$$\int_G \int_G |f(x-y)g(y)|\,dx\,dy = \|f\|_1 \|g\|_1.$$

Setting $\phi(x) = \int_G |f(x-y)g(y)|\,dy$, it follows that $\phi \in L^1(G)$. In particular, $\phi(x) < \infty$ for almost all x, and so $(f * g)(x)$ exists for almost all x. Finally, $|(f * g)(x)| \leqq \phi(x)$, and the proof of (e) is complete.

The proof of (f) is also an application of Fubini's theorem, justified by (e) for almost all x:

$$\begin{aligned}(f * (g * h))(x) &= \int_G f(x-z)(g * h)(z)\,dz \\ &= \int_G \int_G f(x-z)g(z-y)h(y)\,dy\,dz \\ &= \int_G \int_G f(x-z-y)g(z)h(y)\,dy\,dz \\ &= \int_G (f * g)(x-y)h(y)\,dy = ((f * g) * h)(x).\end{aligned}$$

1.1.7. THEOREM. *For any LCA group G, $L^1(G)$ is a commutative Banach algebra, if multiplication is defined by convolution. If G is discrete, $L^1(G)$ has a unit.*

Proof: The first statement follows from parts (e), (f), and (a) of Theorem 1.1.6, since the distributive law holds: $f * (g + h) = f * g + f * h$.

If G is discrete and the Haar measure is normalized as indicated in Section 1.1.3, then

$$(f * g)(x) = \sum_{y \in G} f(x - y)g(y),$$

and if $e(0) = 1$ but $e(x) = 0$ for all $x \neq 0$, then $e \in L^1(G)$ and $f * e = f$. Thus e is the unit of $L^1(G)$.

1.1.8. If G is not discrete, then $L^1(G)$ has no unit (see Section 1.7.3), but *approximate units* are always available.

THEOREM. *Given $f \in L^1(G)$ and $\varepsilon > 0$, there exists a neighborhood V of 0 in G with the following property: if u is a non-negative Borel function which vanishes outside V, and if $\int_G u(x)dx = 1$, then*

$$\|f - f * u\|_1 < \varepsilon.$$

Proof: By Theorem 1.1.5, we can choose V so that $\|f - f_y\|_1 < \varepsilon$ for all $y \in V$. If u satisfies the hypotheses, we have

$$(f * u)(x) - f(x) = \int_G [f(x - y) - f(x)]u(y)dy$$

so that

$$\|f * u - f\|_1 \leqq \int_G |u(y)|dy \int_G |f(x - y) - f(x)|dx$$
$$= \int_V \|f - f_y\|_1 u(y)dy < \varepsilon.$$

1.2. The Dual Group and the Fourier Transform

1.2.1. Characters. A complex function γ on a LCA group G is called a *character* of G if $|\gamma(x)| = 1$ for all $x \in G$ and if the functional equation

(1) $$\gamma(x + y) = \gamma(x)\gamma(y) \qquad (x, y \in G)$$

is satisfied. The set of all *continuous* characters of G forms a group Γ, the *dual group* of G, if *addition* is defined by

(2) $$(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x) \qquad (x \in G;\ \gamma_1, \gamma_2 \in \Gamma).$$

Throughout this book, the letter Γ will denote the dual group of the LCA group G.

In view of the duality between G and Γ which will be established in Section 1.7, it is customary to write

(3) $$(x, \gamma)$$

in place of $\gamma(x)$. With this notation, (1) and (2) become

(4) $$(x + y, \gamma) = (x, \gamma)(y, \gamma) \text{ and } (x, \gamma_1 + \gamma_2) = (x, \gamma_1)(x, \gamma_2).$$

It follows immediately that

(5) $$(0, \gamma) = (x, 0) = 1 \qquad (x \in G, \gamma \in \Gamma)$$

and

(6) $$(-x, \gamma) = (x, -\gamma) = (x, \gamma)^{-1} = \overline{(x, \gamma)}.$$

We shall presently endow Γ with a topology with respect to which Γ will itself be a LCA group. But first we identify Γ with the maximal ideal space of $L^1(G)$ (Appendix D).

1.2.2. THEOREM. *If $\gamma \in \Gamma$ and if*

(1) $$\hat{f}(\gamma) = \int_G f(x)(-x, \gamma)dx \qquad (f \in L^1(G)),$$

then the map $f \to \hat{f}(\gamma)$ is a complex homomorphism of $L^1(G)$, and is not identically 0. Conversely, every non-zero complex homomorphism of $L^1(G)$ is obtained in this way, and distinct characters induce distinct homomorphisms.

Proof: Suppose $f, g \in L^1(G)$, and $k = f * g$. Then

$$\hat{k}(\gamma) = \int_G (f * g)(x)(-x, \gamma)dx = \int_G (-x, \gamma)dx \int_G f(x - y)g(y)dy$$
$$= \int_G g(y)(-y, \gamma)dy \int_G f(x - y)(-x + y, \gamma)dx = \hat{g}(\gamma)\hat{f}(\gamma).$$

Thus the map $f \to \hat{f}(\gamma)$ is multiplicative on the Banach algebra

$L^1(G)$, and since it is clearly linear, it is a homomorphism. Since $|(-x, \gamma)| = 1$, $\hat{f}(\gamma) \neq 0$ for some $f \in L^1(G)$.

For the converse, suppose h is a complex homomorphism of $L^1(G)$, $h \neq 0$. Then h is a bounded linear functional of norm 1 (Appendix D4), so that

$$h(f) = \int_G f(x)\phi(x)dx \qquad (f \in L^1(G)) \tag{2}$$

for some $\phi \in L^\infty(G)$ with $||\phi||_\infty = 1$ (Appendix E10). If f and g are in $L^1(G)$, we have

$$\int_G h(f)g(y)\phi(y)dy = h(f)h(g) = h(f * g) = \int_G (f * g)(x)\phi(x)dx$$
$$= \int_G g(y)dy \int_G f(x - y)\phi(x)dx = \int_G g(y)h(f_y)dy,$$

so that

$$h(f)\phi(y) = h(f_y) \tag{3}$$

for almost all $y \in G$. By Theorem 1.1.5 and the continuity of h, the right side of (3) is a continuous function on G, for each $f \in L^1(G)$. Choosing f so that $h(f) \neq 0$, (3) shows that $\phi(y)$ coincides with a continuous function almost everywhere, and hence we may assume that ϕ is continuous, without affecting (2). Then (3) holds for all $y \in G$.

If we replace y by $x + y$ and then f by f_x in (3), we obtain $h(f)\phi(x + y) = h(f_{x+y}) = h((f_x)_y) = h(f_x)\phi(y) = h(f)\phi(x)\phi(y)$, so that

$$\phi(x + y) = \phi(x)\phi(y) \qquad (x, y \in G). \tag{4}$$

Since $|\phi(x)| \leqq 1$ for all x and since (4) implies that $\phi(-x) = \phi(x)^{-1}$, it follows that $|\phi(x)| = 1$. Hence $\phi \in \Gamma$.

Finally, if $\hat{f}(\gamma_1) = \hat{f}(\gamma_2)$ for all $f \in L^1(G)$, (1) implies that $(-x, \gamma_1) = (-x, \gamma_2)$ for almost all $x \in G$, and since γ_1 and γ_2 are continuous, 1.1.2 shows that the equality holds for all $x \in G$, so that $\gamma_1 = \gamma_2$.

1.2.3. The Fourier transform. For all $f \in L^1(G)$, the function $\hat{f}$ defined on Γ by

$$\hat{f}(\gamma) = \int_G f(x)(-x, \gamma)dx \qquad (\gamma \in \Gamma)$$

is called the *Fourier transform* of f. The set of all functions $\hat{f}$ so obtained will be denoted throughout by $A(\Gamma)$.

By Theorem 1.2.2, $\hat{f}$ is precisely the Gelfand transform of f. If we give Γ the weak topology induced by $A(\Gamma)$ (Appendix A10), the basic facts of the Gelfand theory (Appendix D4) show that $A(\Gamma)$ is a separating subalgebra of $C_0(\Gamma)$. We summarize some of the properties of $A(\Gamma)$.

1.2.4. THEOREM. (a) *$A(\Gamma)$ is a separating self-adjoint subalgebra of $C_0(\Gamma)$, so that $A(\Gamma)$ is dense in $C_0(\Gamma)$, by the Stone-Weierstrass theorem.*

(b) *The Fourier transform of $f * g$ is $\hat{f}\hat{g}$.*

(c) *$A(\Gamma)$ is invariant under translation and under multiplication by (x, γ), for any $x \in G$.*

(d) *The Fourier transform, considered as a map of $L^1(G)$ into $C_0(\Gamma)$, is norm-decreasing and therefore continuous: $||\hat{f}||_\infty \leqq ||f||_1$.*

(e) *For $f \in L^1(G)$ and $\gamma \in \Gamma$, $(f * \gamma)(x) = (x, \gamma)\hat{f}(\gamma)$.*

Proof: For $f \in L^1(G)$, define $\tilde{f}$ by

$$\tilde{f}(x) = \overline{f(-x)}.$$

The Fourier transform of $\tilde{f}$ is the complex conjugate of $\hat{f}$, and (a) follows; (b) is implicit in Theorem 1.2.2. If $\gamma_0 \in \Gamma$ and $g(x) = (x, \gamma_0)f(x)$, then $\hat{g}(\gamma) = \hat{f}(\gamma - \gamma_0)$, so that $A(\Gamma)$ is translation invariant. If $g = f_x$, then

$$\hat{g}(\gamma) = \int_G f(y - x)(-y, \gamma)dy$$
$$= (-x, \gamma) \int_G f(y - x)(x - y, \gamma)dy = (-x, \gamma)\hat{f}(\gamma).$$

This proves (c); (d) and (e) are trivial; (e) allows us to interpret the Fourier transform as a convolution:

$$\hat{f}(\gamma) = (f * \gamma)(0) \qquad (f \in L^1(G), \gamma \in \Gamma).$$

1.2.5. THEOREM. *If G is discrete, Γ is compact. If G is compact, Γ is discrete.*

Proof: If G is discrete, then $L^1(G)$ has a unit (Theorem 1.1.7) and its maximal ideal space Γ is therefore compact (Appendix D4).

If G is compact and its Haar measure is normalized so that $m(G) = 1$, the orthogonality relations

$$(1)\qquad \int_G (x, \gamma)dx = \begin{cases} 1 & \text{if } \gamma = 0 \\ 0 & \text{if } \gamma \neq 0 \end{cases}$$

hold. The case $\gamma = 0$ is clear. If $\gamma \neq 0$, then $(x_0, \gamma) \neq 1$ for some $x_0 \in G$, and

$$\int_G (x, \gamma)dx = (x_0, \gamma) \int_G (x - x_0, \gamma)dx = (x_0, \gamma) \int_G (x, \gamma)dx,$$

so that (1) is proved. If $f(x) = 1$ for all $x \in G$, then $f \in L^1(G)$ since G is compact, and $\hat{f}(0) = 1$, $\hat{f}(\gamma) = 0$, if $\gamma \neq 0$, by (1). Since $\hat{f}$ is continuous, the set consisting of 0 alone is open in Γ, and so Γ is discrete.

1.2.6. The topology of Γ. So far, Γ is a group and a locally compact Hausdorff space. We shall now prove that these two structures fit together so as to make Γ a LCA group. Our proof depends on an alternative description of the topology of Γ:

THEOREM. (a) *(x, γ) is a continuous function on $G \times \Gamma$.*

(b) *Let K and C be compact subsets of G and Γ, respectively, let U_r be the set of all complex numbers z with $|1 - z| < r$, and put*

$$N(K, r) = \{\gamma\colon (x, \gamma) \in U_r \text{ for all } x \in K\},$$
$$N(C, r) = \{x\colon (x, \gamma) \in U_r \text{ for all } \gamma \in C\}.$$

Then $N(K, r)$ and $N(C, r)$ are open subsets of Γ and G, respectively.

(c) *The family of all sets $N(K, r)$ and their translates is a base for the topology of Γ.*

(d) *Γ is a LCA group.*

Proof: Equation (3) of Section 1.2.2, rewritten in the form

$$(1)\qquad \hat{f}(\gamma)(x, \gamma) = \hat{f}_x(\gamma) \qquad (x \in G,\ \gamma \in \Gamma)$$

implies (a), as soon as it is proved that $\hat{f}_x(\gamma)$ is a continuous function on $G \times \Gamma$, for every $f \in L^1(G)$.

Fix x_0, γ_0, and $\varepsilon > 0$. There are neighborhoods V of x_0 and W of γ_0 such that

$$(2)\qquad \|f_x - f_{x_0}\|_1 < \varepsilon \text{ and } |\hat{f}_{x_0}(\gamma) - \hat{f}_{x_0}(\gamma_0)| < \varepsilon$$

for all $x \in V$, $\gamma \in W$, by Theorem 1.1.5 and the continuity of $\hat{f}_{x_0}$. Since $|\hat{f}_x(\gamma) - \hat{f}_{x_0}(\gamma)| \leqq \|f_x - f_{x_0}\|_1$, it follows that $|\hat{f}_x(\gamma) - \hat{f}_{x_0}(\gamma_0)| < 2\varepsilon$ if $x \in V$ and $\gamma \in W$, and (a) is proved.

Choose a compact set K in G, choose $r > 0$, and fix $\gamma_0 \in N(K, r)$. To every $x_0 \in K$ there correspond neighborhoods V of x_0 and W of γ_0 such that $(x, \gamma) \in U_r$, if $x \in V$ and $\gamma \in W$; this follows from (a). Since K is compact, finitely many of these sets V cover K, and if W^* is the intersection of the corresponding sets W, then $W^* \subset N(K, r)$. Since W^* is a neighborhood of γ_0, $N(K, r)$ is open.

The same proof applies to $N(C, r)$.

To prove (c), assume that V is a neighborhood of γ_0. We have to show that $\gamma_0 + N(K, r) \subset V$ for some choice of K and r. Take $\gamma_0 = 0$, without loss of generality. The definition of the Gelfand topology on Γ shows that there exist functions $f_1, \ldots, f_n \in L^1(G)$ and $\varepsilon > 0$ so that

$$(3)\qquad \bigcap_{i=1}^{n} \{\gamma\colon |\hat{f}_i(\gamma) - \hat{f}_i(0)| < \varepsilon\} \subset V.$$

Since $C_c(G)$ is dense in $L^1(G)$, we may assume that $f_1, \ldots, f_n$ vanish outside a compact set K in G. If

$$(4)\qquad r < \varepsilon/\max_i \|f_i\|_1$$

and if $\gamma \in N(K, r)$, then

$$(5)\qquad |\hat{f}_i(\gamma) - \hat{f}_i(0)| \leqq \int_K |(-x, \gamma) - 1||f_i(x)|dx \leqq r\|f_i\|_1 < \varepsilon.$$

Hence $N(K, r) \subset V$, and (c) follows.

Given $\gamma', \gamma'' \in \Gamma$ and $N(K, r)$, the obvious relation

$$(6)\qquad [\gamma' + N(K, r/2)] - [\gamma'' + N(K, r/2)] \subset \gamma' - \gamma'' + N(K, r)$$

shows, by (b) and (c), that the map $(\gamma', \gamma'') \to \gamma' - \gamma''$ of $\Gamma \times \Gamma$ onto Γ is continuous. This completes the theorem.

1.2.7. Examples. The "classical groups" of Fourier analysis are:

(a) the additive group R of the real numbers, with the natural topology of the real line;

(b) the additive group of the reals modulo 2π, or, equivalently, the circle group T, the multiplicative group of all complex numbers of absolute value 1;

(c) the additive group Z of the integers.

The circle group is of particular importance to us, since characters are nothing but homomorphisms into T.

Suppose $G = R$ and fix $\gamma \in \Gamma$. Write $\gamma(x)$ instead of (x, γ), for the moment; there exists $\delta > 0$ such that

$$(1) \qquad \int_0^\delta \gamma(t)dt = \alpha \neq 0.$$

The functional equation

$$(2) \qquad \gamma(x + t) = \gamma(x)\gamma(t) \qquad (x, t \in R)$$

then implies that

$$(3) \qquad \alpha \cdot \gamma(x) = \gamma(x) \int_0^\delta \gamma(t)dt = \int_0^\delta \gamma(x + t)dt = \int_x^{x+\delta} \gamma(t)dt.$$

Since γ is continuous, the last expression is differentiable, and so γ has a continuous derivative γ'. Differentiate (2) with respect to t and then set $t = 0$. The result is the differential equation

$$(4) \qquad \gamma'(x) = A\gamma(x), \qquad A = \gamma'(0).$$

Since $\gamma(0) = 1$ and since γ is bounded, (4) implies that

$$(5) \qquad \gamma(x) = e^{iyx}$$

for some $y \in R$. The correspondence $\gamma \leftrightarrow y$ is an isomorphism between Γ and R. Thus: *The dual group of R is R.*

We still have to check that the natural topology of R is the same as the Gelfand topology of the dual group. For $r > 0$ and $n = 1, 2, 3, \ldots$, let $V(n, r)$ be the set of all y such that $|1 - e^{ixy}| < r$ if $|x| \leqq n$. By Theorem 1.2.6, the sets $V(n, r)$ form a neighborhood base at 0 with respect to the Gelfand topology. But $y \in V(n, r)$ if and only if $|y| < (2/n)$ arc sin $(r/2)$. Thus the two topologies coincide.

If $G = T$, the same computation as above shows that every character of T must be of the form (5), but now we also must have $\gamma(x + 2\pi) = \gamma(x)$. Hence y must be an integer, and Γ is identified as the discrete group Z (compare Theorem 1.2.5).

If $G = Z$ and $\gamma \in \Gamma$, then $(1, \gamma) = e^{i\alpha}$ for some real α, and it follows that $(n, \gamma) = e^{in\alpha}$. The correspondence $\gamma \leftrightarrow e^{i\alpha}$ is an isomorphism between Γ and T, and we conclude that T is the dual group of Z (the two topologies coincide, as in the case $G = R$).

The Fourier transforms, in these three cases, have the following forms:

$$G = R: \qquad \hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-iyx}dx \qquad (y \in R),$$

$$G = T: \qquad \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta \qquad (n \in Z),$$

$$G = Z: \qquad \hat{f}(e^{i\alpha}) = \sum_{n=-\infty}^{\infty} f(n)e^{-in\alpha} \qquad (e^{i\alpha} \in T).$$

1.3. *Fourier-Stieltjes Transforms*

1.3.1. Convolutions of measures. Suppose G is a LCA group, and μ, λ are members of $M(G)$ (Appendix E1), i.e., bounded regular complex valued measures on G. Let $\mu \times \lambda$ be their product measure on the product space $G^2 = G \times G$, and associate with each Borel set E in G the set

$$(1) \qquad E_{(2)} = \{(x, y) \in G^2 \colon x + y \in E\}.$$

Then $E_{(2)}$ is a Borel set in G^2 (see the proof of Theorem 1.1.6(d)) and we define $\mu * \lambda$ by

$$(2) \qquad (\mu * \lambda)(E) = (\mu \times \lambda)(E_{(2)}).$$

The set function $\mu * \lambda$ so defined is called the *convolution* of μ and λ.

1.3.2. Theorem. (a) *If $\mu \in M(G)$ and $\lambda \in M(G)$, then $\mu * \lambda \in M(G)$.*

(b) *Convolution is commutative and associative.*
(c) $\|\mu * \lambda\| \leqq \|\mu\| \cdot \|\lambda\|$.

COROLLARY. *$M(G)$ is a commutative Banach algebra with unit, if multiplication is defined by convolution.*

Proof: The Jordan decomposition theorem shows that in the proof of (a) it is enough to consider non-negative measures only. Since $\mu \times \lambda$ is a measure on G^2, it is clear that $(\mu * \lambda)(E) = \sum (\mu * \lambda)(E_i)$ if E is the union of the disjoint Borel sets E_i $(i = 1, 2, 3, \ldots)$. If E is a Borel set in G and if $\varepsilon > 0$, the regularity of $\mu \times \lambda$ shows that there is a compact set $K \subset E_{(2)}$ such that

$$(\mu \times \lambda)(K) > (\mu \times \lambda)(E_{(2)}) - \varepsilon.$$

If C is the image of K under the map $(x, y) \to x + y$, then C is a compact subset of E, $K \subset C_{(2)}$, and hence

$$(\mu * \lambda)(C) = (\mu \times \lambda)(C_{(2)}) \geqq (\mu \times \lambda)(K) > (\mu * \lambda)(E) - \varepsilon.$$

This establishes one half of the requirement that $\mu * \lambda$ be regular. The other half follows by complementation, and (a) is proved. (This argument applies to more general situations; see Stromberg [1].)

Since G is commutative, the condition $x + y \in E$ is the same as the condition $y + x \in E$, and hence $\mu * \lambda = \lambda * \mu$.

The simplest way to prove associativity is to extend the definition of convolution to the case of n measures $\mu_1, \ldots, \mu_n \in M(G)$: with each Borel set E in G associate the set

(1) $$E_n = \{(x_1, \ldots, x_n) \in G^n : x_1 + \ldots + x_n \in E\},$$

and put

(2) $$(\mu_1 * \mu_2 * \ldots * \mu_n)(E) = (\mu_1 \times \mu_2 \times \ldots \times \mu_n)(E_{(n)}),$$

where the measure on the right is the ordinary product measure on the product space G^n. Associativity now follows from Fubini's theorem, and (b) is proved.

Let χ_E be the characteristic function of the Borel set E in G. The definition of $(\mu * \lambda)(E)$ is equivalent to the equation

(3) $$\int_G \chi_E \, d(\mu * \lambda) = \int_G \int_G \chi_E(x + y) d\mu(x) d\lambda(y).$$

Hence if f is a *simple function* (a finite linear combination of characteristic functions of Borel sets), we have

(4) $$\int_G f d(\mu * \lambda) = \int_G \int_G f(x + y) d\mu(x) d\lambda(y),$$

and since every bounded Borel function is the uniform limit of a sequence of simple functions, (4) holds for every bounded Borel function f. (One could use (4) as the definition of $\mu * \lambda$.) If $|f(x)| \leqq 1$ for all $x \in G$, then $|\int_G f(x + y) d\mu(x)| \leqq \|\mu\|$ for all $y \in G$, and hence the right side of (4) does not exceed $\|\mu\| \cdot \|\lambda\|$. This proves part (c) of the theorem.

As to the Corollary, it only remains to be shown that $M(G)$ has a unit. Let δ_0 be the unit mass concentrated at the point $x = 0$; i.e., $\delta_0(E) = 1$ if $0 \in E$ and $\delta_0(E) = 0$ otherwise. Then $\mu * \delta_0 = \mu$ for all $\mu \in M(G)$, and the proof is complete.

1.3.3. Fourier-Stieltjes transforms. If $\mu \in M(G)$, the function $\hat{\mu}$ defined on Γ by

(1) $$\hat{\mu}(\gamma) = \int_G (-x, \gamma) d\mu(x) \qquad (\gamma \in \Gamma)$$

is called the Fourier-Stieltjes transform of μ. The set of all such functions $\hat{\mu}$ will be denoted by $B(\Gamma)$.

THEOREM. (a) *Each $\hat{\mu} \in B(\Gamma)$ is bounded and uniformly continuous.*

(b) *If $\sigma = \mu * \lambda$, then $\hat{\sigma} = \hat{\mu} \cdot \hat{\lambda}$. Hence the map $\mu \to \hat{\mu}(\gamma)$ is, for each $\gamma \in \Gamma$, a complex homomorphism of $M(G)$.*

(c) *$B(\Gamma)$ is invariant under translation, under multiplication by (x, γ) for any $x \in G$, and under complex conjugation.*

Proof: The definition of $\hat{\mu}$ shows immediately that $|\hat{\mu}(\gamma)| \leqq \|\mu\|$ for all $\gamma \in \Gamma$. Given $\delta > 0$, the regularity of $|\mu|$ shows that there is a compact set K in G such that $|\mu|(K') < \delta$, where K' is the complement of K. For any $\gamma_1, \gamma_2 \in \Gamma$ we have

$$|\hat{\mu}(\gamma_1) - \hat{\mu}(\gamma_2)| \leqq \int_G |1 - (x, \gamma_1 - \gamma_2)| d|\mu|(x) = \int_K + \int_{K'}.$$

If $\gamma_1 - \gamma_2 \in N(K, \delta)$, as defined in Theorem 1.2.6, the above integrand is less than δ for $x \in K$, hence $\int_K$ does not exceed $\delta\|\mu\|$. The second integral is less than $2|\mu|(K') < 2\delta$. Hence $\hat{\mu}$ is uniformly continuous.

Suppose $\sigma = \mu * \lambda$. Formula (4) in the proof of Theorem 1.3.2 then implies that

$$\begin{aligned}\hat{\sigma}(\gamma) &= \int_G (-z, \gamma) d(\mu * \lambda)(z) = \int_G \int_G (-x - y, \gamma) d\mu(x) d\lambda(y) \\ &= \int_G (-x, \gamma) d\mu(x) \int_G (-y, \gamma) d\lambda(y) = \hat{\mu}(\gamma)\hat{\lambda}(\gamma),\end{aligned}$$

and (b) is proved.

The proof of (c) is quite similar to that of the analogous part of Theorem 1.2.4. If $d\lambda(x) = (x, \gamma_0) d\mu(x)$, then $\hat{\lambda}(\gamma) = \hat{\mu}(\gamma - \gamma_0)$. If $\lambda(E) = \mu(E - x)$, then $\hat{\lambda}(\gamma) = (x, \gamma)\hat{\mu}(\gamma)$. If $\tilde{\mu}(E) = \overline{\mu(-E)}$, then the Fourier-Stieltjes transform of $\tilde{\mu}$ is the complex conjugate of $\hat{\mu}$.

1.3.4. $L^1(G)$ as a subalgebra of $M(G)$. Every $f \in L^1(G)$ generates a measure $\mu_f \in M(G)$, defined by

$$\mu_f(E) = \int_E f(x) dx, \tag{1}$$

and which is absolutely continuous with respect to the Haar measure of G. Conversely, the Radon-Nikodym theorem (Appendix E9) shows that every absolutely continuous $\mu \in M(G)$ is μ_f for some $f \in L^1(G)$. Since we identify functions in $L^1(G)$ which differ only on a set of Haar measure 0, the correspondence between f and μ_f is one-to-one, and we may therefore regard $L^1(G)$ as a subset of $M(G)$. It is easily seen that $\hat{f}(\gamma) = \hat{\mu}_f(\gamma)$ for all $\gamma \in \Gamma$ and that $\|f\|_1 = \|\mu_f\|$. Hence we may use f in place of μ_f without causing confusion. For instance, we may write $f * \sigma$ if $f \in L^1(G)$ and $\sigma \in M(G)$, instead of $\mu_f * \sigma$.

1.3.5. Let $M_c(G)$ and $M_d(G)$ denote the sets of all continuous and discrete members of $M(G)$, respectively (Appendix E6).

THEOREM. (a) *$L^1(G)$ and $M_c(G)$ are closed ideals in $M(G)$.*
(b) *$M_d(G)$ is a closed subalgebra of $M(G)$.*

Proof: If we apply the Fubini theorem to the definition of $\mu * \lambda$, we obtain, for any Borel set E in G,

$$(\mu * \lambda)(E) = \int_G \mu(E - y) d\lambda(y). \tag{1}$$

If μ is absolutely continuous and $m(E) = 0$, then $m(E - y) = 0$ for all y, hence $\mu(E - y) = 0$, and so $(\mu * \lambda)(E) = 0$ for every $\lambda \in M(G)$. This says that $\mu * \lambda$ is absolutely continuous, and hence $L^1(G)$ is an ideal in $M(G)$. Since $\|f\|_1 = \|\mu_f\|$ and since $L^1(G)$ is complete, $L^1(G)$ is closed in $M(G)$. If E is countable, $\mu_n \in M_c(G)$, and $\|\mu - \mu_n\| \to 0$, then

$$|\mu(E)| = |(\mu - \mu_n)(E)| \leqq |\mu - \mu_n|(E) \leqq \|\mu - \mu_n\|,$$

so that $\mu(E) = 0$ and $\mu \in M_c(G)$. Thus $M_c(G)$ is closed, and part (a) is proved. Part (b) follows from the observation that the convolution of two point-measures is a point-measure.

1.3.6. A uniqueness theorem. We shall see later that $\hat{\mu}$ determines μ, i.e. if $\mu \in M(G)$ and $\hat{\mu} = 0$, then $\mu = 0$. At present, we can prove this for the inverse transform:

THEOREM. *If $\mu \in M(\Gamma)$ and if*

$$\int_\Gamma (x, \gamma) d\mu(\gamma) = 0$$

for every $x \in G$, then $\mu = 0$.

Proof: For every $f \in L^1(G)$,

$$\begin{aligned}\int_\Gamma \hat{f}(\gamma) d\mu(\gamma) &= \int_\Gamma \int_G f(x)(-x, \gamma) dx d\mu(\gamma) \\ &= \int_G f(x) dx \int_\Gamma (-x, \gamma) d\mu(\gamma) = 0.\end{aligned}$$

Since $A(\Gamma)$ is dense in $C_0(\Gamma)$ (Theorem 1.2.4), it follows that $\int_\Gamma \phi d\mu = 0$ for every $\phi \in C_0(\Gamma)$, and hence $\mu = 0$.

1.4. Positive-Definite Functions

1.4.1. A function ϕ, defined on G, is said to be *positive-definite* if the inequality

$$\sum_{n,m=1}^{N} c_n \bar{c}_m \phi(x_n - x_m) \geqq 0 \tag{1}$$

holds for every choice of $x_1, \ldots, x_N$ in G and for every choice of complex numbers $c_1, \ldots, c_N$.

If ϕ is positive-definite, the following three relations hold:

$$\phi(-x) = \overline{\phi(x)}; \tag{2}$$

$$|\phi(x)| \leqq \phi(0); \tag{3}$$

$$|\phi(x) - \phi(y)|^2 \leqq 2\phi(0) \operatorname{Re} [\phi(0) - \phi(x-y)]. \tag{4}$$

We conclude from (3) that $\phi(0) \geqq 0$ and that ϕ is bounded; (4) implies that ϕ is uniformly continuous if ϕ is continuous at 0.

To prove these relations, take $N = 2$ in (1); $x_1 = 0$, $x_2 = x$; $c_1 = 1$, $c_2 = c$. This gives

$$\{1 + |c|^2\}\phi(0) + c\phi(x) + \bar{c}\phi(-x) \geqq 0. \tag{5}$$

Taking $c = 1$, we see that $\phi(x) + \phi(-x)$ is real; $c = i$ shows that $i(\phi(x) - \phi(-x))$ is real. Hence (2) holds.

If c is chosen so that $c\phi(x) = -|\phi(x)|$, (5) implies (3). To prove (4), take $N = 3$ in (1); $x_1 = 0$, $x_2 = x$, $x_3 = y$; $c_1 = 1$, λ real,

$$c_2 = \frac{\lambda|\phi(x) - \phi(y)|}{\phi(x) - \phi(y)},$$

and $c_3 = -c_2$. Then (1) simplifies to

$$\phi(0)(1 + 2\lambda^2) + 2\lambda|\phi(x) - \phi(y)| - 2\lambda^2 \operatorname{Re} \phi(x-y) \geqq 0. \tag{6}$$

The discriminant of the quadratic polynomial (6) in λ can therefore not be positive, and this gives (4).

1.4.2. Examples of positive-definite functions. (a) *Suppose $f \in L^2(G)$ and $\phi = f * \tilde{f}$. Then ϕ is positive-definite and continuous on G.*

The convolution of any two functions in $L^2(G)$ is continuous (Theorem 1.1.6(d)) and

$$\sum c_n \overline{c_m} \phi(x_n - x_m) = \sum c_n \overline{c_m} \int_G f(x_n - x_m - y) \overline{f(-y)}\, dy$$
$$= \sum c_n \overline{c_m} \int_G f(x_n - y)\overline{f(x_m - y)}dy = \int_G |\sum c_n f(x_n - y)|^2 dy \geqq 0.$$

(b) Every character is positive-definite, hence so is every finite linear combination of characters if the coefficients are positive. More generally, *if $\mu \in M(\Gamma)$, if $\mu \geqq 0$, and if*

$$\phi(x) = \int_\Gamma (x, \gamma) d\mu(\gamma) \qquad (x \in G), \tag{1}$$

then ϕ is continuous and positive definite.

Indeed, (1) shows that

$$\sum c_n \overline{c_m} \phi(x_n - x_m) = \int_\Gamma \sum_{n,m} c_n \overline{c_m} (x_n - x_m; \gamma) d\mu(\gamma)$$
$$= \int_\Gamma |\sum_n c_n (x_n, \gamma)|^2 d\mu(\gamma) \geqq 0,$$

so that ϕ is positive-definite. Since the sets $N(C, r)$ of Theorem 1.2.6 are open in G, our proof of the continuity of $\hat{\mu}$ (Theorem 1.3.3) shows equally well that ϕ is continuous if ϕ is defined by (1).

1.4.3. The previous example (1.4.2(b)) establishes the trivial half of the following important characterization of positive-definite functions:

BOCHNER'S THEOREM. *A continuous function ϕ on G is positive-definite if and only if there is a non-negative measure $\mu \in M(\Gamma)$ such that*

$$\phi(x) = \int_\Gamma (x, \gamma) d\mu(\gamma) \qquad (x \in G). \tag{1}$$

For $G = Z$, this is due to Herglotz [1]; for $G = R$, to Bochner [1]; for the general case, to Weil [1]. Bochner was the first to recognize the key role which this result plays in harmonic analysis. By 1.3.6, the above representation (1) is unique.

Proof: Suppose ϕ is continuous and positive-definite. By 1.4.1(3) we may assume, without loss of generality, that $\phi(0) = 1$.

If $f \in C_c(G)$ and has support K, then $f(x)\overline{f(y)}\phi(x-y)$ is uniformly continuous on $K \times K$, and K can be partitioned into disjoint sets $E_1, \ldots, E_n$ such that the sum

$$\sum_{i,j=1}^{n} f(x_i)\overline{f(x_j)}\phi(x_i - x_j) m(E_i) m(E_j) \qquad (x_i \in E_i) \tag{2}$$

differs from the integral

(3) $$\int_G \int_G f(x)\overline{f(y)}\phi(x-y)dx\,dy$$

by as little as we please. Since ϕ is positive-definite, (2) is always non-negative, and hence so is (3). Since $C_c(G)$ is dense in $L^1(G)$, it follows that (3) *is non-negative for every* $f \in L^1(G)$.

Define a functional T_ϕ by

(4) $$T_\phi(f) = \int_G f(x)\phi(x)dx \qquad (f \in L^1(G))$$

and put

(5) $$[f, g] = T_\phi(f * \tilde{g}) \qquad (f, g \in L^1(G)).$$

We recall that $\tilde{g}(x) = \overline{g(-x)}$, so that

(6) $$[f, g] = \int_G \int_G f(x)\overline{g(y)}\phi(x-y)dx\,dy.$$

Hence $[f, g]$ is linear in f, $[g, f]$ is the complex conjugate of $[f, g]$, and $[f, f] \geqq 0$. These are just the properties of the Hilbert space inner product which are needed for the standard proof of the Schwarz inequality. In our case, the inequality is

(7) $$|[f, g]|^2 \leqq [f, f][g, g].$$

Take for g the characteristic function of a symmetric neighborhood V of 0, divided by $m(V)$. By (6),

$$[f, g] - T_\phi(f) = \int_G f(x)\frac{1}{m(V)}\int_V [\phi(x-y) - \phi(x)]dy\,dx$$

and

$$[g, g] - 1 = \frac{1}{m(V)^2}\int_V \int_V [\phi(x-y) - 1]dx\,dy.$$

Since ϕ is uniformly continuous, these expressions can be made arbitrarily small by taking V small enough, and then (7) yields the inequality

(8) $$|T_\phi(f)|^2 \leqq [f, f] = T_\phi(f * \tilde{f}) \qquad (f \in L^1(G)).$$

Put $h = f * \tilde{f}$ and $h^n = h^{n-1} * h$ $(n = 2, 3, 4, \ldots)$. Since $||\phi||_\infty = 1$, we have $||T_\phi|| = 1$, and if we apply (8) with $h, h^2, h^4, \ldots$ in place of f, we obtain

$$|T_\phi(f)|^2 \leqq T_\phi(h) \leqq \{T_\phi(h^2)\}^{\frac{1}{2}} \leqq \ldots \leqq \{T_\phi(h^{2^n})\}^{2^{-n}} \leqq ||h^{2^n}||_1^{2^{-n}}.$$

As $n \to \infty$, the last expression converges to the spectral radius of h, i.e. to $||\hat{h}||_\infty$. (See Appendix D 6 and Theorem 1.2.2.) Hence

(9) $$|T_\phi(f)|^2 \leqq ||\hat{h}||_\infty = ||\hat{f}||_\infty^2, \text{ or } |T_\phi(f)| \leqq ||\hat{f}||_\infty \qquad (f \in L^1(G)).$$

This means that T_ϕ may be regarded as a bounded linear functional on $A(\Gamma)$, with respect to the supremum norm. (We have not yet proved that $\hat{f}_1 = \hat{f}_2$ implies $f_1 = f_2$, but (9) shows that $\hat{f}_1 = \hat{f}_2$ implies $T_\phi(f_1) = T_\phi(f_2)$, and this is sufficient.) We can extend T_ϕ to a bounded linear functional on $C_0(\Gamma)$, preserving its norm, and the Riesz representation theorem then implies that there is a $\mu \in M(\Gamma)$, with $||\mu|| \leqq 1$, such that

(10) $$T_\phi(f) = \int_\Gamma \hat{f}(-\gamma)d\mu(\gamma) = \int_G f(x)dx \int_\Gamma (x, \gamma)d\mu(\gamma).$$

Comparison of (10) and (4) shows that (1) holds for almost all $x \in G$, hence for all x, since both sides of (1) are continuous. Finally, taking $x = 0$ in (1), we have

$$1 = \phi(0) = \int_\Gamma d\mu(\gamma) = \mu(\Gamma) \leqq ||\mu|| = 1;$$

hence $\mu(\Gamma) = ||\mu||$, and this implies that $\mu \geqq 0$.

1.5. The Inversion Theorem

1.5.1. We let $B(G)$ be the set of all functions f on G which are representable in the form

(1) $$f(x) = \int_\Gamma (x, \gamma)d\mu(\gamma) \qquad (x \in G).$$

Bochner's theorem implies, in combination with the Jordan decomposition theorem, that $B(G)$ is exactly the set of all finite linear combinations of continuous positive-definite functions on G.

THEOREM. (a) *If* $f \in L^1(G) \cap B(G)$, *then* $\hat{f} \in L^1(\Gamma)$.

(b) *If the Haar measure of G is fixed, the Haar measure of* Γ *can be so normalized that the inversion formula*

$$f(x) = \int_\Gamma \hat{f}(\gamma)(x, \gamma)d\gamma \qquad (x \in G) \tag{2}$$

is valid for every $f \in L^1(G) \cap B(G)$.

Proof: Let us write B^1 in place of $L^1(G) \cap B(G)$, and if μ is associated with f as in (1) above, let us write $\mu = \mu_f$. (This notation has nothing to do with our earlier use of the symbol μ_f in Section 1.3.4.) If $f \in B^1$ and $h \in L^1(G)$, we then have

$$(h * f)(0) = \int_G h(-x)f(x)dx = \int_\Gamma \hat{h}(\gamma)d\mu_f(\gamma), \tag{3}$$

and if g is also in B^1, (3) implies that

$$\int_\Gamma \hat{h}\hat{g}d\mu_f = ((h * g) * f)(0) = ((h * f) * g)(0) = \int_\Gamma \hat{h}\hat{f}d\mu_g.$$

Since $A(\Gamma)$ is dense in $C_0(\Gamma)$, it follows that

$$\hat{g}d\mu_f = \hat{f}d\mu_g. \qquad (f, g \in B^1). \tag{4}$$

We shall now define a positive linear functional T on $C_c(\Gamma)$. Suppose K is the support of some $\psi \in C_c(\Gamma)$. To every $\gamma_0 \in K$ there corresponds a function $u \in C_c(G)$ with $\hat{u}(\gamma_0) \neq 0$, since $C_c(G)$ is dense in $L^1(G)$. The Fourier transform of $u * \tilde{u}$ is positive at γ_0, and is nowhere negative. Since K is compact, there is a finite number of such functions, say $u_1, \ldots, u_n$ such that the function $g = u_1 * \tilde{u}_1 + \ldots + u_n * \tilde{u}_n$ has $\hat{g} > 0$ on K. Since $g \in C_c(G)$, 1.4.2(a) shows that $g \in B^1$. Put

$$T\psi = \int_\Gamma (\psi/\hat{g})d\mu_g. \tag{5}$$

Note that $T\psi$ is well defined: if g were replaced by another function f in B^1 whose Fourier transform has no zero on K, the value of $T\psi$ would not be changed, since (4) implies that

$$\int \frac{\psi}{\hat{f}\hat{g}}\hat{f}d\mu_g = \int \frac{\psi}{\hat{f}\hat{g}}\hat{g}d\mu_f. \tag{6}$$

It is clear that T is linear. The function g in (5) is positive-definite, hence $\mu_g \geqq 0$, and it follows that $T\psi \geqq 0$ if $\psi \geqq 0$. There exists ψ and μ_f such that $\int \psi d\mu_f \neq 0$, and if g is as in (5), we have

$$T(\psi\hat{f}) = \int_\Gamma (\psi\hat{f}/\hat{g})d\mu_g = \int_\Gamma \psi d\mu_f \neq 0. \tag{7}$$

Thus $T \neq 0$.

Fix $\psi \in C_c(\Gamma)$ and $\gamma_0 \in \Gamma$. Construct g as above, so that $\hat{g} > 0$ on K and also on $K + \gamma_0$. Setting $f(x) = (-x, \gamma_0)g(x)$, we have $\hat{f}(\gamma) = \hat{g}(\gamma + \gamma_0)$ and $\mu_f(E) = \mu_g(E - \gamma_0)$. If $\psi_0(\gamma) = \psi(\gamma - \gamma_0)$, then

$$T\psi_0 = \int_\Gamma [\psi(\gamma - \gamma_0)/\hat{g}(\gamma)]d\mu_g(\gamma) = \int_\Gamma [\psi(\gamma)/\hat{f}(\gamma)]d\mu_f(\gamma) = T\psi.$$

Thus T is translation-invariant, and it follows that

$$T\psi = \int_\Gamma \psi(\gamma)d\gamma \qquad (\gamma \in C_c(\Gamma)), \tag{8}$$

where $d\gamma$ denotes a Haar measure on Γ.

If now $f \in B^1$ and $\psi \in C_c(\Gamma)$, (7) and (8) show that

$$\int_\Gamma \psi d\mu_f = T(\psi\hat{f}) = \int_\Gamma \psi\hat{f}d\gamma, \tag{9}$$

and since (9) holds for every $\psi \in C_c(\Gamma)$, we conclude that

$$\hat{f}d\gamma = d\mu_f \qquad (f \in B^1). \tag{10}$$

Since μ_f is a finite measure, it follows that $\hat{f} \in L^1(\Gamma)$, and substitution of (10) into (1) gives the inversion formula (2).

This completes the proof.

1.5.2. Consequences of the inversion theorem. Let V be a neighborhood of 0 in G, choose a compact neighborhood W of 0 such that $W - W \subset V$, let f be the characteristic function of W, divided by $m(W)^{\frac{1}{2}}$, and put $g = f * \tilde{f}$. Then g is continuous, positive-definite (by 1.4.2(a)), and 0 outside $W - W$. The inversion theorem therefore applies to g. Hence $\hat{g} = |\hat{f}|^2 \geqq 0$,

$$\int_\Gamma \hat{g}(\gamma)d\gamma = g(0) = 1, \tag{1}$$

and it follows that there is a compact set C in Γ such that

$$\int_C \hat{g}(\gamma)d\gamma > \frac{2}{3}. \tag{2}$$

If $x \in N(C, 1/3)$ (in the notation of Theorem 1.2.6), we write

$$g(x) = \left(\int_C + \int_{C'}\right) \hat{g}(\gamma)(x, \gamma)d\gamma; \tag{3}$$

for $\gamma \in C$, $|1 - (x, \gamma)| < 1/3$, hence Re $(x, \gamma) > 2/3$, and the integral over C is at least $2/3 \int_C \hat{g} > 4/9$. Since $|\int_{C'}| < 1/3$, we see that $g(x) > 1/9$ if $x \in N(C, 1/3)$, and our conclusion is: $N(C, 1/3) \subset V$.

Since the sets $N(C, r)$ are open in G (Theorem 1.2.6(b)), we now have the following analogue of 1.2.6(c):

The family of all sets $N(C; r)$ and their translates is a base for the topology of G.

If $x_0 \in G$, $x_0 \neq 0$, we can choose V in the preceding paragraph so that $x_0 \notin V$, and we conclude that $(x_0, \gamma) \neq 1$ for some $\gamma \in \Gamma$. Hence Γ *separates points* on G: If $x_1 \neq x_2$, then $(x_1 - x_2, \gamma) \neq 1$ for some γ, and so $(x_1, \gamma) \neq (x_2, \gamma)$.

Any function of the form

$$f(x) = \sum_{j=1}^{n} a_j(x, \gamma_j) \qquad (x \in G)$$

is called a *trigonometric polynomial* on G. The set of all trigonometric polynomials on G is an algebra over the complex field, with respect to pointwise multiplication, and is closed under complex conjugation. Since Γ separates points on G, the Stone-Weierstrass theorem yields the following result:

If G is compact, the trigonometric polynomials on G form a dense subalgebra of $C(G)$.

It follows that the trigonometric polynomials are also dense in $L^p(G)$, $1 \leq p < \infty$, if G is compact (see Appendix E8).

1.5.3. Normalization of Haar measure. If the Haar measure of G is given, the inversion theorem singles out a specific Haar measure of Γ, adjusted so that the inversion formula holds. In

Section 1.1.3 we introduced standard normalizations for the Haar measures of compact and discrete groups. Since Γ is compact [discrete] if G is discrete [compact] (Theorem 1.2.5) the question arises whether these normalizations are "correct," i.e., whether the inversion formula holds for them.

To prove that this is so, it suffices to consider just one function (not identically 0) and its Fourier transform.

If G is compact and $m(G) = 1$, take $f(x) = 1$. Then (see 1.2.5) $\hat{f}(0) = 1$ and $\hat{f}(\gamma) = 0$ if $\gamma \neq 0$. If m_Γ is the Haar measure of Γ, adjusted in accordance with the inversion theorem, then

$$1 = f(0) = \int_\Gamma \hat{f}(\gamma)d\gamma = m_\Gamma(\{0\}), \tag{1}$$

and so m_Γ assigns measure 1 to each point of Γ.

If G is discrete and each point has measure 1, take $f(0) = 1$, $f(x) = 0$ if $x \neq 0$. Then $\hat{f}(\gamma) = 1$, and

$$m(\Gamma) = \int_\Gamma \hat{f}(\gamma)d\gamma = f(0) = 1 \tag{2}$$

if the inversion theorem holds.

To consider a non-trivial case, take $G = R$ (see 1.2.7) so that $\Gamma = R$, and let $\alpha\, dx$, $\beta\, dt$ be Haar measures on G and Γ; here dx and dt denote ordinary Lebesgue measure on the real line. Since $e^{-|t|} > 0$, the easily verified formula

$$\frac{2\beta}{1 + x^2} = \int_{-\infty}^{\infty} e^{-|t|}\, e^{ixt} \beta\, dt \tag{3}$$

shows that $(1 + x^2)^{-1}$ is positive-definite, and the uniqueness of the inverse transform, combined with the inversion theorem, shows that

$$e^{-|t|} = 2\alpha\beta \int_{-\infty}^{\infty} \frac{e^{-ixt}}{1 + x^2}\, dx. \tag{4}$$

With $t = 0$, (4) becomes

$$1 = 2\alpha\beta \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = 2\pi\alpha\beta, \tag{5}$$

and this is the normalization condition which α and β must satisfy.

Two of the possible choices are frequently used: $\alpha = 1/2\pi$, $\beta = 1$ or $\alpha = \beta = (2\pi)^{-1/2}$.

From now on, it will allways be tacitly assumed that the Haar measures of G and Γ are so adjusted that the inversion theorem holds.

1.6. The Plancherel Theorem

1.6.1. THEOREM. *The Fourier transform, restricted to* $(L^1 \cap L^2)(G)$, *is an isometry (with respect to the* L^2*-norms) onto a dense linear subspace of* $L^2(\Gamma)$. *Hence it may be extended, in a unique manner, to an isometry of* $L^2(G)$ *onto* $L^2(\Gamma)$.

Proof: If $f \in (L^1 \cap L^2)(G)$ and $g = f * \tilde{f}$, then $g \in L^1(G)$, g is continuous and positive definite, $|\hat{g}| = |\hat{f}|^2$, and the inversion theorem gives

$$\int_G |f(x)|^2 dx = \int_G f(x)\bar{f}(-x)dx = g(0) = \int_\Gamma \hat{g}(\gamma)d\gamma = \int_\Gamma |\hat{f}(\gamma)|^2 d\gamma,$$

or $\|\hat{f}\|_2 = \|f\|_2$.

Let Φ be the set of all $\hat{f} \in A(\Gamma)$ with $f \in (L^1 \cap L^2)(G)$. Since $(L^1 \cap L^2)(G)$ is translation invariant, Φ is invariant under multiplication by (x, γ), for any $x \in G$. Thus if $\psi \in L^2(\Gamma)$ and $\int_\Gamma \phi\bar{\psi}d\gamma = 0$ for all $\phi \in \Phi$, then also

$$\int_\Gamma \phi(\gamma)\overline{\psi(\gamma)}(x, \gamma)d\gamma = 0 \qquad (\phi \in \Phi,\ x \in G).$$

Since $\phi\bar{\psi} \in L^1(\Gamma)$, the uniqueness theorem 1.3.6 implies that $\phi\bar{\psi} = 0$ almost everywhere, for every $\phi \in \Phi$. But $(L^1 \cap L^2)(G)$ is invariant under multiplication by (x, γ), for any $\gamma \in \Gamma$, and so Φ is translation invariant. Hence to every γ_0 there corresponds a $\phi \in \Phi$ which is different from 0 in a neighborhood of γ_0. It follows that $\psi = 0$ almost everywhere. Thus 0 is the only element of $L^2(\Gamma)$ which is orthogonal to Φ, and hence Φ is dense in $L^2(\Gamma)$ (see Appendix C12).

1.6.2. The above extension of the Fourier transform to $L^2(G)$ is sometimes referred to as the *Plancherel transform*; the symbol $\hat{f}$ will be used in this context as well. An important part of the theorem is the assertion that each function in $L^2(\Gamma)$ is the Plancherel transform of some $f \in L^2(G)$. For compact G this is a special case of the Riesz-Fischer theorem about orthogonal systems of functions (Zygmund [1], vol. I, p. 127).

If f and g are in $L^2(G)$, the identity

$$4f\bar{g} = |f + g|^2 - |f - g|^2 + i|f + ig|^2 - i|f - ig|^2,$$

combined with the isometric character of the Plancherel transform, yields the *Parseval formula*

$$\int_G f(x)\overline{g(x)}dx = \int_\Gamma \hat{f}(\gamma)\overline{\hat{g}(\gamma)}d\gamma.$$

1.6.3. THEOREM. $A(\Gamma)$ *consists precisely of the convolutions* $F_1 * F_2$, *with* F_1 *and* F_2 *in* $L^2(\Gamma)$.

Proof: Suppose $f, g \in L^2(G)$. Replacing g by $\bar{g}$, the Parseval formula assumes the form

(1) $$\int_G f(x)g(x)dx = \int_\Gamma \hat{f}(\gamma)\hat{g}(-\gamma)d\gamma,$$

and if we replace $g(x)$ by $(-x, \gamma_0)g(x)$ in (1), we obtain

(2) $$\int_G f(x)g(x)(-x, \gamma_0)dx = \int_\Gamma \hat{f}(\gamma)\hat{g}(\gamma_0 - \gamma)d\gamma = (\hat{f} * \hat{g})(\gamma_0).$$

On the one hand, every $h \in L^1(G)$ is a product $h = fg$, with $f, g \in L^2(G)$, and (2) shows that $\hat{h} = \hat{f} * \hat{g}$, with $\hat{f}, \hat{g} \in L^2(\Gamma)$, by the Plancherel theorem. On the other hand, we can start with $\hat{f}, \hat{g} \in L^2(\Gamma)$, and see from (2) that $\hat{f} * \hat{g} \in A(\Gamma)$.

1.6.4. THEOREM. *If E is a non-empty open set in Γ, there exists* $\hat{f} \in A(\Gamma)$, $\hat{f} \neq 0$, *such that* $\hat{f}(\gamma) = 0$ *outside E.*

Proof: Let K be a compact subset of E, with $m(K) > 0$, let V be a compact neighborhood of 0 such that $K + V \subset E$, and set $\hat{f} = \hat{g} * \hat{h}$, where $\hat{g}$ and $\hat{h}$ are the characteristic functions of K and V, respectively. Then $\hat{f}(\gamma) = 0$ outside $K + V$, $\hat{f} \in A(\Gamma)$ by Theorem 1.6.3, and $\int_\Gamma \hat{f}(\gamma)d\gamma = m(K)m(V) > 0$, so that $\hat{f}$ is not identically 0.

1.7. The Pontryagin Duality Theorem

1.7.1. If G is a LCA group, we have seen (Theorem 1.2.6) that its dual Γ is also a LCA group. Hence Γ has a dual group, say $\hat{\Gamma}$, and everything we have proved so far for the ordered pair (G, Γ) holds equally well for the pair $(\Gamma, \hat{\Gamma})$. The value of a character $\hat{\gamma} \in \hat{\Gamma}$ at the point $\gamma \in \Gamma$ will be written $(\gamma, \hat{\gamma})$. (This notation is temporary, and will be abandoned as soon as we prove that $\hat{\Gamma} = G$.)

By Theorem 1.2.6(a) every $x \in G$ may be regarded as a continuous character on Γ, and thus there is a natural map α of G into $\hat{\Gamma}$, defined by

$$(1) \qquad (x, \gamma) = (\gamma, \alpha(x)) \qquad (x \in G,\ \gamma \in \Gamma).$$

1.7.2. THEOREM. *The above map α is an isomorphism and a homeomorphism of G onto $\hat{\Gamma}$.*

Thus $\hat{\Gamma}$ may be identified with G, and a more informal statement of the result would be:

Every LCA *group is the dual group of its dual group.*

This is the Pontryagin duality theorem.

Proof: For $x, y \in G$ and $\gamma \in \Gamma$, we have

$$(\gamma, \alpha(x + y)) = (x + y, \gamma) = (x, \gamma)(y, \gamma)$$
$$= (\gamma, \alpha(x))(\gamma, \alpha(y)) = (\gamma, \alpha(x) + \alpha(y)).$$

Hence $\alpha(x + y) = \alpha(x) + \alpha(y)$, and α is a homomorphism. Since Γ separates points on G (Section 1.5.2), α is one-to-one, and so α is an ismorphism of G into $\hat{\Gamma}$.

The rest of the proof may be broken into three steps:

(a) α is a homeomorphism of G into $\hat{\Gamma}$.

(b) $\alpha(G)$ is closed in $\hat{\Gamma}$.

(c) $\alpha(G)$ is dense in $\hat{\Gamma}$.

Choose a compact set C in Γ, choose $r > 0$, and put

$$(1) \qquad \begin{aligned} V &= \{x \in G\colon |1 - (x, \gamma)| < r \text{ for all } \gamma \in C\},\\ W &= \{\hat{\gamma} \in \hat{\Gamma}\colon |1 - (\gamma, \hat{\gamma})| < r \text{ for all } \gamma \in C\}. \end{aligned}$$

By 1.5.2 and 1.2.6(c), these sets V form a neighborhood base at 0 in G, and the sets W form a neighborhood base at 0 in $\hat{\Gamma}$. The definition of α shows that

$$(2) \qquad \alpha(V) = W \cap \alpha(G).$$

It follows that both α and its inverse are continuous at 0, and since α is an isomorphism, the same result holds, by translaticn, at any other point of G or of $\alpha(G)$.

This proves step (a), and so $\alpha(G)$ is locally compact, in the relative topology which $\alpha(G)$ has as a subset of $\hat{\Gamma}$. Suppose $\hat{\gamma}_0$ is in the closure of $\alpha(G)$, and let U be a neighborhood of $\hat{\gamma}_0$ whose closure $\bar{U}$ is compact. Since $\alpha(G)$ is locally compact, $\alpha(G) \cap \bar{U}$ is compact, and hence closed in $\hat{\Gamma}$. But $\hat{\gamma}_0$ is in the closure of $\alpha(G) \cap \bar{U}$, and it follows that $\hat{\gamma}_0 \in \alpha(G)$. Thus $\alpha(G)$ is closed, and step (b) is proved.

If $\alpha(G)$ is not dense in $\hat{\Gamma}$, there is a function $F \in A(\hat{\Gamma})$ which is 0 at every point of $\alpha(G)$ but is not identically 0 (see Theorem 1.6.4). For some $\phi \in L^1(\Gamma)$, we have

$$(3) \qquad F(\hat{\gamma}) = \int_\Gamma \phi(\gamma)(-\gamma, \hat{\gamma})d\gamma \qquad (\hat{\gamma} \in \hat{\Gamma}).$$

Since $F(\alpha(x)) = 0$ for all $x \in G$, it follows that

$$(4) \qquad \int_\Gamma \phi(\gamma)(-x, \gamma)d\gamma = \int_\Gamma \phi(\gamma)(-\gamma, \alpha(x))d\gamma = 0 \qquad (x \in G,)$$

and so $\phi = 0$, by the uniqueness theorem 1.3.6. Hence $F = 0$, by (3), and this contradiction proves step (c) and completes the proof.

1.7.3. Some consequences of the duality theorem. The symmetry between G and Γ which is now established shows that every theorem proved for the ordered pair (G, Γ) also holds for (Γ, G), and this enables us to complete some of the results which were previously established in provisional form only.

(a) *Every compact abelian group is the dual of a discrete abelian group, and every discrete abelian group is the dual of a compact abelian group.* This follows from Theorem 1.2.5.

(b) *If $\mu \in M(G)$ and $\hat{\mu}(\gamma) = 0$ for all $\gamma \in \Gamma$, then $\mu = 0$.* This is the dual of Theorem 1.3.6.

(c) *$M(G)$ and $L^1(G)$ are semi-simple Banach algebras.* (See Appendix D5). Since the map $\mu \to \hat{\mu}(\gamma)$ is a complex homomorphism of $M(G)$, for each $\gamma \in \Gamma$, the semi-simplicity of $M(G)$ follows from the uniqueness theorem (b). The same uniqueness theorem evidently holds for $L^1(G)$, and so $L^1(G)$ is semi-simple.

(d) *If G is not discrete, then $L^1(G)$ has no unit. Hence $L^1(G) = M(G)$ if and only if G is discrete.*

For if G is not discrete, then Γ is not compact, by (a), and since $A(\Gamma) \subset C_0(\Gamma)$, $A(\Gamma)$ contains no non-zero constants, hence has no unit. Since $A(\Gamma)$ is isomorphic, as an algebra, to $L^1(G)$, the proof is complete.

(e) *If $\mu \in M(G)$ and $\hat{\mu} \in L^1(\Gamma)$, there exists $f \in L^1(G)$ such that $d\mu(x) = f(x)dx$, and*

(1) $$f(x) = \int_\Gamma \hat{\mu}(\gamma)(x, \gamma)d\gamma \qquad (x \in G).$$

By hypothesis, $\hat{\mu} \in L^1(\Gamma) \cap B(\Gamma)$; hence if f is defined by (1), the inversion theorem (applied to the pair (Γ, G) instead of (G, Γ)), shows that $f \in L^1(G)$ and

(2) $$\hat{\mu}(\gamma) = \int_G f(x)(-x, \gamma)dx \qquad (\gamma \in \Gamma).$$

Since $\hat{\mu}(\gamma) = \int_G (-x, \gamma)d\mu(x)$, the uniqueness theorem now implies that $d\mu = f dx$, and the proof is complete.

1.8. *The Bohr Compactification*

1.8.1. Suppose Γ is the dual of the LCA group G, Γ_d is the group Γ with the discrete topology, and $\bar{G}$ is the dual of Γ_d. Then $\bar{G}$ is a compact abelian group which we call the *Bohr compactification* of G (Anzai and Kakutani [1]). Let β be the map of G into $\bar{G}$ defined by

(1) $$(x, \gamma) = (\gamma, \beta(x)) \qquad (x \in G, \gamma \in \Gamma).$$

1.8.2. Theorem. *β is a continuous isomorphism of G onto a dense subgroup $\beta(G)$ of $\bar{G}$.*

This theorem allows us to regard G as a dense subgroup of $\bar{G}$, so that $\bar{G}$ is indeed a compactification of G. Note, however, that $\beta(G)$ is not a locally compact subset of $\bar{G}$ and that β is not a

homeomorphism, unless G is compact, in which case $G = \bar{G}$ and $\Gamma = \Gamma_d$.

Proof: Since Γ separates points on G, β is one-to-one, and it is easy to verify, as in the beginning of the proof of the Pontryagin duality theorem, that β is an isomorphism.

Let W be a neighborhood of 0 in $\bar{G}$. Since a subset of Γ_d is compact if and only if it is finite, Theorem 1.2.6 shows that there exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $r > 0$, such that W contains the set

$$\{\bar{x} \in \bar{G}: |1 - (\gamma_i, \bar{x})| < r;\ i = 1, \ldots, n\}$$

which is a neighborhood of 0 in $\bar{G}$. Let

$$V = \{x \in G: |1 - (x, \gamma_i)| < r;\ i = 1, \ldots, n\}.$$

Then V is a neighborhood of 0 in G, and $x \in V$ implies $\beta(x) \in W$. Thus β is continuous at 0, and hence at all points of G, by translation.

Finally, let H be the closure in $\bar{G}$ of $\beta(G)$. If $H \neq \bar{G}$, then $\bar{G}/H$ is a non-trivial compact group, and hence there is a character ϕ on $\bar{G}/H$ which is not identically 1. The map $\bar{x} \to \phi(\bar{x} + H)$ is then a continuous character on $\bar{G}$, not identically 1, which is 1 if $\bar{x} \in H$. Consequently there exists $\gamma_0 \in \Gamma$, $\gamma_0 \neq 0$, such that $(x, \gamma_0) = (\gamma_0, \beta(x)) = 1$ for all $x \in G$. This last equation implies that $\gamma_0 = 0$, and this contradiction completes the proof.

1.8.3. We may interpret the theorem in the following way: G and Γ are given, G is the group of all continuous characters on Γ, $\bar{G}$ is the group of *all* characters on Γ, and the fact that G (or $\beta(G)$) is dense in $\bar{G}$ leads to an approximation theorem (Hewitt and Zuckerman [1]):

Theorem. *Given $\gamma_1, \ldots, \gamma_n \in \Gamma$, given $\varepsilon > 0$, and given any character ϕ on Γ, there is a continuous character ψ on Γ such that*

(1) $$|\psi(\gamma_i) - \phi(\gamma_i)| < \varepsilon \qquad (i = 1, \ldots, n).$$

Proof: $\phi \in \bar{G}$, and the set of all $\psi \in \bar{G}$ satisfying (1) is open in $\bar{G}$, hence intersects $\beta(G)$.