Commutative Banach algebras

We recall some basic notions:

Def: A Banach algebra $:\Longleftrightarrow$ • A Banach space over $\mathbb{C}$

• $\cdot : A \times A \to A$ bilinear, associative, $\|a \cdot b\| \leq \|a\| \|b\|$

A commutative (or abelian) $:\Longleftrightarrow$ $a \cdot b = b \cdot a \;\forall a, b \in A$.

A unital $:\Longleftrightarrow$ $\exists e \in A \;\forall a \in A: e \cdot a = a \cdot e = a$, $\|e\| = 1$.

Examples: • Closed subalgebras of $B(H)$, H Hilbert space

commutative:

• $L^1(\mathbb{R})$ with convolution product

• $\mathcal{F}L^1(\mathbb{R})$ with pointwise multiplication

• $C_0(X)$ with pointwise multiplication, X locally compact Hausdorff space

Def: $h : A \to \mathbb{C}$ (complex) homomorphism $:\Longleftrightarrow$ • h linear

• $h(ab) = h(a)\,h(b) \quad \forall a, b \in A$.

Exercise 7.1.c: homomorphisms are continuous, norm ≤ 1. If A is unital $\Rightarrow$ norm $\in \{0, 1\}$.

$$\Delta := \{\text{homomorphisms} \not\equiv 0\} \subseteq \text{unit ball in } A^* \subseteq A^*$$

Every $a \in A$ induces a function $\hat{a} : \Delta \to \mathbb{C}$, $h \mapsto h(a) =: \hat{a}(h)$, s.t. $\widehat{ab}(h) = \hat{a}(h)\hat{b}(h)$

Gelfand topology on Δ := topology of pointwise convergence
$\equiv$ coarsest topology s.t. all $\hat{a} \in C_0(\Delta)$.
$\equiv$ weak*-topology on A^* restricted to Δ.

Thm: (Banach-Alaoglu) The unit ball in A^* is weak*-compact.

As $\Delta \cup \{0\} = \{\text{homomorphisms}\}$ is a closed subset of the unit ball of A^* $\Rightarrow$ Δ locally compact.
As for any $h_1 \neq h_2 \in \Delta$ there's an $a \in A$ s.t. $\hat{a}(h_1) = h_1(a) \neq h_2(a) = \hat{a}(h_2)$, Δ is also Hausdorff.

A unital $\Rightarrow$ $\hat{e} = 1 \in C_0(\Delta) \Leftrightarrow C_0(\Delta) = C(\Delta) \Leftrightarrow \Delta$ compact.

To sum up: $\hat{}: A \to C_0(\Delta)$ is a homomorphism of A onto
$a \mapsto \hat{a}$

a subalgebra of the Banach algebra $C_0(\Delta)$ and hence $\|\hat{a}\|_\infty \leq \|a\|$.

$\hat{}$ is called the Gelfand transform.

Thm: (Stone-Weierstrass)
Let Δ be a locally compact Hausdorff space, $\hat{A} \subset C_0(\Delta)$ a subalgebra s.t.

- $\forall h \in \Delta \; \exists \hat{a} \in \hat{A} : \hat{a}(h) \neq 0$
- $\forall h_1 \neq h_2 \in \Delta \; \exists \hat{a} \in \hat{A} : \hat{a}(h_1) \neq \hat{a}(h_2)$
- If $\hat{a} \in \hat{A}$, also the complex conjugate $\bar{\hat{a}} \in \hat{A}$.

$\Longrightarrow$ $\hat{A}$ dense in $C_0(\Delta)$.

Cor: The range of the Gelfand transform is dense ~~iff~~ if it is closed under complex conjugation.

Def: spectrum $\sigma(a) := \text{range}(\hat{a} : \Delta \cup \{0\} \to \mathbb{C})$

spectral radius $r(a) := \|\hat{a}\|_\infty$.

We have $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$, and the spectrum coincides with the usual definition.

Locally compact abelian groups

$(G,+)$ abelian group, inverse of $x\in G$ is $-x$

$\mathbb{Z}$-module: $n\in\mathbb{Z} \Rightarrow nx := \begin{cases} x+\dots+x, & n>0 \\ 0, & n=0 \\ -(x+\dots+x), & n<0 \end{cases}$

homomorphisms: $\phi(x+y) = \phi(x)+\phi(y)$.

Def: G topological ~~abelian~~ group $:\Longleftrightarrow$
- G Hausdorff top. space which is also a group
- $G\times G \to G$ continuous, $(x,y)\mapsto x-y$

G locally compact abelian group (LCA) $:\Longleftrightarrow$ G top. abelian group and topology is loc. compact.

Remarks:
- The topology is uniquely determined by a basis of 0-neighborhoods: U neighborhood of $0 \Longleftrightarrow x+U$ neighborhood of $x\in G$. May assume $U=-U$ (otherwise replace U by $U\cap -U$)
- A open, $B\subset G \Rightarrow A+B = \bigcup_{b\in B} A+b$ open

  A,B compact $\Longrightarrow$ $A+B = \text{image}\begin{pmatrix} A\times B \to G \\ (x,y)\mapsto x+y \end{pmatrix}$ compact
- G abelian group w/ discrete topology

Examples:
- $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$, w/ the usual topology, classical examples
- $\mathbb{Z}$, $\mathbb{Z}/N\mathbb{Z}$ with the discrete topology, important e.g. in additive combinatorics
- $\mathbb{Q}_p$ = completion of $\mathbb{Q}$ with respect to the distance coming from the absolute value $|p^n \frac{a}{b}|_p := p^{-n}$, $a,b\in\mathbb{Z}$, $p\nmid a,b$, $|0|_p := 0$.

  (p prime) = "p-adic numbers"
- finite products of the above examples
- adele group $\mathbb{A} \subset \mathbb{R}\times \prod_{p \text{ prime}} \mathbb{Q}_p$, important in number theory ("Tate's thesis")

Non-example: A topological $\mathbb{R}$-vector space is locally compact

As on $\mathbb{R}^n$, continuous functions vanishing at ∞ are actually uniformly continuous, i.e. $\forall \varepsilon>0\ \exists$ neighborhood V of $0\in G$ s.t. $y-x\in V$ $\Rightarrow |f(x)-f(y)|<\varepsilon$:

<u>Theorem:</u> $f\in C_0(G) \Rightarrow f$ uniformly continuous.

<u>Proof:</u> Let $\varepsilon>0$, $K\subset G$ compact such that $\sup\limits_{G\setminus K}|f|<\frac{\varepsilon}{4}$.

By continuity of $+: G\times G\to G$, given a 0-neighborhood W $\Rightarrow \exists$ 0-neighborhood $V: V+V\subset W$. Replacing V by $-V\cap V$, we may assume $-V=V$.

Let $x\in K$. By continuity $\exists W_x$ 0-neighborhood: $\forall y\in K\cap(x+W_x)$: $|f(x)-f(y)|<\varepsilon/2$. Choose V_x as above w/ $V_x+V_x\subseteq W_x$.

Compactness: $\exists x_1,\dots,x_n\in K: K=\bigcup\limits_{i=1}^{n}(x_i+V_{x_i})$, $V=\bigcap\limits_{i=1}^{n}V_{x_i}$.

Now if $x,y\in K$, $x-y\in V \Rightarrow \exists i: x\in x_i+V_{x_i}$ and $y\in x+V\subset x_i+V_{x_i}+V$
$\subset x_i+W_{x_i}$

$\Rightarrow |f(x)-f(y)|\le|f(x)-f(x_i)|+|f(x_i)-f(y)|<2\frac{\varepsilon}{2}=\varepsilon$.

If $x\in K$, $y\in G\setminus K$, $x-y\in V \Rightarrow |f(x)-f(y)|\le \underbrace{|f(x)|}_{<\frac{\varepsilon}{4}+\frac{\varepsilon}{2}}+\underbrace{|f(y)|}_{<\frac{\varepsilon}{4}}<\varepsilon$

and if $x,y\in G\setminus K \Rightarrow |f(x)-f(y)|\le|f(x)|+|f(y)|<2\frac{\varepsilon}{4}<\varepsilon$. ∎

Haar measure

Let X be a locally compact Hausdorff space, $\mathcal{B} = \sigma\text{-algebra}\left\{\begin{smallmatrix}\text{closed}\\ \text{sets}\end{smallmatrix}\right\}$ the Borel σ-algebra.

Def.: $\mu : \mathcal{B} \to \mathbb{C} \cup \{\infty\}$ measure $:\Leftrightarrow$ • $E = \bigcup_{i \in \mathbb{N}} E_i$, $E_i \in \mathcal{B}$, $E_i \cap E_j = \emptyset$ $\forall i \neq j$

$\Rightarrow \mu(E) = \sum \mu(E_i)$

• $E \in \mathcal{B}$, E compact $\Rightarrow \mu(E) < \infty$.

μ non-negative $:\Leftrightarrow \mu : \mathcal{B} \to [0,\infty]$

total variation $|\mu| : \mathcal{B} \to [0,\infty]$ (a measure)

$$E \mapsto \sup_{\substack{E = \bigcup_{i \in \mathbb{N}} E_i \\ E_i \cap E_j = \emptyset}} \sum |\mu(E_i)|$$

μ regular $:\Leftrightarrow \forall E \in \mathcal{B} : |\mu|(E) = \sup_{\substack{K \subset E \\ \text{compact}}} |\mu|(K) = \inf_{\substack{V \supset E \\ \text{open}}} |\mu|(V)$

$M(X) :=$ space of all regular measures on X with $|\mu|(X) < \infty$.

norm: $\|\mu\| = |\mu|(X)$

Riesz representation theorem: $C_0(X)^* \cong M(X)$ isometric isomorphism

$$\begin{array}{ccc} C_0(X)^* & \cong & M(X) \\ \Psi & & \Psi \\ T : C_0(X) \to \mathbb{C} & \longleftrightarrow & \mu \\ f \mapsto \int_X f \, d\mu & & \end{array}$$

In particular, $M(X)$ is a Banach space.

regular, non-negative measures $\leftrightarrow T \in C_0(X)^*$ s.t. $f \geq 0 \Rightarrow Tf \geq 0$.

Facts: • $C_c(X)$ dense in $L^p(m)$, $1 \leq p < \infty$, if m regular, non-negative.

• $L^p(X)^* = L^{p'}(X)$, $\frac{1}{p} + \frac{1}{p'} = 1$, $1 \leq p < \infty$.

Thm: (Haar) G LCA group $\Rightarrow \exists$ non-negative, regular $m \neq 0$ s.t. $\forall x \in G \, \forall E \in \mathcal{B} : m(x+E) = m(E)$.

m is called a Haar measure.

You can find a relatively simple, functional-analytic proof on the homepage of the course.

Exercises: • $\emptyset \neq V \subset G$ open $\Rightarrow m(V) > 0$ • $m(E) = m(-E)$.

__Uniqueness (1.1.3 in Rudin):__

m, m' Haar measures $\Rightarrow \exists \lambda > 0 : m' = \lambda m$.

__Proof:__ Let $g \in C_c(G)$ s.t. $\int_G g\, dm = 1$, $\lambda := \int_G g(-x)\, dm'(x)$.

$$\Rightarrow \forall f \in C_c(G): \int_G f\, dm' = \underbrace{\int_G g(y)\, dm(y)}_{=1} \int_G f(x)\, dm'(x)$$

(translation invariance) $= \int_G g(y)\, dm(y) \int_G f(x+y)\, dm'(x)$

(Fubini) $= \int_G dm'(x) \int_G dm(y)\, \underbrace{g(y) f(x+y)}_{\in C_c(G\times G) \subseteq L^1(m\otimes m')}$

(translation invariance) $= \int_G dm'(x) \int_G dm(y)\, \underbrace{g(y-x) f(y)}_{\in C_c(G\times G)}$

(Fubini) $= \int_G dm(y)\, f(y) \int_G dm'(x)\, g(y-x)$

(translation invariance) $= \lambda \int_G dm(y)\, f(y)$ $\boxed{\text{q.e.d.}}$

__Notation:__ • Write $dx, dy, \ldots$ for $dm(x), dm(y), \ldots$

• $L^p(G) := L^p(m)$

Then also $L^1(G)^* \cong L^\infty(G)$. (A sufficient condition for $L^1(X)^* \cong L^\infty(X)$ is $X = \bigcup_{\alpha\in A} X_\alpha$, X_α σ-compact, pairwise disjoint such that every σ-compact $\tilde{X} \subset X$ is covered by countably many X_α.
For $X = G$, choose $V = -V$ compact 0-neighborhood $\Rightarrow H = \bigcup_n nV$ σ-compact open subgroup, X_α = cosets of H = elements of G/H.
[illegible])

Examples: • $G=\mathbb{R}$: Lebesgue measure , $\int_G f\,dm = \int_{-\infty}^{\infty} f(x)\,dx$

• $G=\mathbb{R}/\mathbb{Z}$: Interpret $\mathbb{R}/\mathbb{Z}$ as the interval $[0,1)$, (identifying the endpoints). Then the Haar measure is just the Lebesgue measure on $[0,1)$. $\int_G f\,dm = \int_0^1 f(x)\,dx$

• $G=\mathbb{Z}$: $m(E) \overset{!}{=}$ number of elements in E, $E\subset\mathbb{Z}$ "counting measure", $\int_G f\,dm = \sum_{x\in E} f(x)$

• $G=\mathbb{Z}/N\mathbb{Z}$: $m(E) = \dfrac{\text{number of Elements in } E}{N}$

so that $m(G)=1$. $\int_G f\,dm = \frac{1}{N}\sum_{x=0}^{N-1} f(x)$

We are going to normalize $m(G)$ to 1 whenever possible.

Convolution

$x\in G$; $f: G\to\mathbb{C}$ $\Rightarrow$ $f_x(y) := f(y-x)$ translate of f

Thm 1.1.5: $X = L^p(G)$, $1\le p<\infty$, or $C_0(G)$, $f\in X \Rightarrow$ $G\to X$, $x\mapsto f_x$ uniformly continuous.

Proof: Let $\varepsilon>0$. $C_c(G)$ dense in X $\Rightarrow$ $\exists g\in C_c : \|g-f\|_X<\varepsilon$

Uniform continuity of g $\Rightarrow$ $\exists$ 0-neighborhood $V\subseteq G$ $\forall x\in V$: $\|g-g_x\|_\infty<\varepsilon$

$$\Rightarrow \forall x\in V: \|f-f_x\|_X \le \underbrace{\|f-g\|_X}_{<\varepsilon} + \underbrace{\|g-g_x\|_X}_{\substack{<\varepsilon,\ X=C_0\\ m(\operatorname{supp}(g-g_x))^{1/p}\varepsilon,\ X=L^p}} + \underbrace{\|g_x-f_x\|_X}_{<\varepsilon}$$

$$\le C(X)\,\varepsilon$$

$$\Rightarrow \forall x-y\in V: \|f_x-f_y\|_X = \|(f-f_{y-x})_x\|_X = \|f-f_{y-x}\|_X < C(X)\varepsilon.$$

$\blacksquare$

Def: For any measurable $f,g: G\to\mathbb{C}$ s.t. $\int_G |f(x-y)\,g(y)|\,dy<\infty$ define $(f*g)(x) := \int_G f(x-y)\,g(y)\,dy$.

Thm 1.1.6/7: a) $f*g = g*f$, $(f*g)*h = f*(g*h)$, $f*(g+h) = f*g + f*h$

$\operatorname{supp} f*g \subseteq \operatorname{supp} f + \operatorname{supp} g \quad \forall f,g,h \in L^1(G)$

b) $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \Rightarrow \|f*g\|_r \leq \|f\|_p \|g\|_q$

c) $(L^1(G), *)$ is a commutative Banach algebra. If G is discrete, $L^1(G)$ is unital with unit $e(x) = \begin{cases} 1 & , x=0 \\ 0 & , x \neq 0 \end{cases}$.

d) $f \in L^p, g \in L^{p'}, \frac{1}{p} + \frac{1}{p'} = 1 \Rightarrow f*g$ bounded, uniformly continuous,

If $p \notin \{1,\infty\} \Rightarrow f*g \in C_0(G)$.

Proof: a) as on $\mathbb{R}$, b) Exercise 2.3 c) $p=q=r=1$ in b) and a).

d) $r=\infty$ in b) $\Rightarrow |f*g(x) - f*g(z)| = |(f_{-x} - f_{-z}) * g(0)|$

$$\leq \|f_{-x} - f_{-z}\|_p \|g\|_{p'}$$

By 1.1.5, $\forall \varepsilon > 0 \ \exists$ 0-neighborhood V s.t. $\|f_{-x} - f_{-z}\|_p < \frac{\varepsilon}{\|g\|_{p'}}$ if $x-z \in V \Rightarrow |f*g(x) - f*g(z)| < \varepsilon \quad \forall x-z \in V$.

If $p \notin \{1,\infty\} \Rightarrow \exists f_n, g_n \in C_c(G) : \|f - f_n\|_p \xrightarrow{n\to\infty} 0$, $\|g - g_n\|_{p'} \xrightarrow{n\to\infty} 0$

$\Rightarrow \sup_{x \in G} |f_n * g_n(x) - f*g(x)| = \sup_x |(f_n - f)*g_n(x) + f*(g-g_n)(x)|$

$$\leq \|f_n - f\|_p \|g_n\|_{p'} + \|f\|_p \|g - g_n\|_{p'}$$

$\xrightarrow{n\to\infty} 0$. As $f_n * g_n \in C_c(G)$ by a)

$\Rightarrow f*g \in C_0(G)$. $\boxtimes$

We always have Dirac sequences (approximate units):

≈Thm 1.1.8: Let $f \in X$, $X = L^p(G)$, $p \in [1,\infty)$, or $C_0(G)$, and $\varepsilon > 0$.

$\Rightarrow \exists$ 0-neighborhood V $\forall 0 \leq u \in C_c(G)$ with $\int_G u = 1$, $\operatorname{supp} u \subset V$:

$\|f - f*u\|_X < \varepsilon$.

Proof: Choose V such that $\| f - f_y \|_X < \varepsilon \quad \forall y \in V$.

$$\Rightarrow \quad f * u(x) - f(x) = \int_G (f(x-y) - f(x))\, u(y)\, dy$$

$$\Rightarrow \quad \| f*u - f \|_X \leq \int_G \| f_y - f \|_X \, u(y)\, dy < \varepsilon . \qquad \square$$

Dual group

Def: $\gamma : G \to \mathbb{C}$ character $:\Leftrightarrow$
- $|\gamma(x)| = 1 \quad \forall x \in G$
- $\gamma(x+y) = \gamma(x)\gamma(y) \quad \forall x,y \in G$

dual group of G $\quad \Gamma :=$ { continuous characters } with addition

$$(\gamma_1 + \gamma_2)(x) := \gamma_1(x)\, \gamma_2(x)$$

Notation: Write $(x, \gamma) := \gamma(x)$.

Remark:
- $(x+y, \gamma) = (x,\gamma)(y,\gamma)$, $\quad (x, \gamma_1 + \gamma_2) = (x, \gamma_1)(x, \gamma_2)$
- $(0, \gamma) = (x, 0) = 1$
- $(-x, \gamma) = (x, -\gamma) = (x,\gamma)^{-1} = \overline{(x,\gamma)}$

Def: "Fourier transform" $\quad L^1(G) \longrightarrow$ Functions on Γ

$$f \longmapsto \hat{f}(\gamma) := \int_G f(x)\, (-x, \gamma)\, dx$$

The range will be denoted by $A(\Gamma)$.

Thm 1.2.2: $\quad$ { complex homomorphisms of $L^1(G)$ } $\setminus \{0\}$ $\xleftrightarrow{\text{bijection}}$ Γ

$$(f \mapsto \hat{f}(\gamma)) \longleftarrow \gamma$$

i.e. the Fourier transform is the Gelfand transform.

We will therefore endow Γ with the Gelfand topology, so that $A(\Gamma) \subset C_0(\Gamma)$.

Corollary / Theorem 1.2.4:

a) $A(\Gamma) \subseteq C_0(\Gamma)$ dense.

b) $\widehat{f*g} = \hat{f}\hat{g}$

c) $a \in A(\Gamma) \Rightarrow a(\cdot - \gamma_0) \in A(\Gamma) \quad \forall \gamma_0 \in \Gamma$

$(x_0, \cdot)a \in A(\Gamma) \quad \forall x_0 \in G$

d) $\wedge : L^1(G) \to C_0(\Gamma)$ continuous and $\|\hat{f}\|_\infty \leq \|f\|_{L^1}$.

e) $(f*\gamma)(x) = (x,\gamma)\hat{f}(\gamma)$, so $\hat{f}(\gamma) = (f*\gamma)(0)$.

Proof: a) $\tilde{f}(x) := \overline{f(-x)}$ has Fourier transform $\overline{\hat{f}}$.

$\Rightarrow$ $A(\Gamma)$ closed under complex conjugation.

The assertion now follows from our Corollary to Stone-Weierstrass.

b) seen in the proof of 1.2.2.

c) $g(x) := (x,\gamma_0) f(x) \Rightarrow \hat{g}(\gamma) = \hat{f}(\gamma - \gamma_0)$

$g(x) := f(x - x_0) \Rightarrow \hat{g}(\gamma) = (-x_0, \gamma)\hat{f}(\gamma)$

d), e) clear

$\boxtimes$

Proof of Thm 1.2.2: ① It's a map between the claimed spaces.

$$\Rightarrow (f*g)^\wedge(\gamma) = \int_G (f*g)(x)\,(-x,\gamma)\,dx$$

$$= \int_G dx\,(-x,\gamma) \int_G dy\, f(x-y)\, g(y)$$

$$= \int_G dy\, g(y)\,(-y,\gamma) \int_G dx\; f(x-y)\,(-x+y,\gamma)\,dx$$

(translation invariance) $$= \hat{g}(\gamma)\,\hat{f}(\gamma)$$

$\Rightarrow$ $f \mapsto \hat{f}(\gamma)$ homomorphism (linearity ✓). It's not $\equiv 0$:

Let $0 \leq f \in L^1(G) \Rightarrow g(x) := f(x)\,(x,\gamma) \Rightarrow \hat{g}(\gamma) = \int_G f > 0$.

② <u>Surjectivity:</u> $h \neq 0$ complex homomorphism.

In particular, $h \in L^1(G)^*$, $\|h\| \leq 1 \implies \exists \phi \in L^\infty(G)$, $\|\phi\|_\infty = \|h\| \leq 1$

s.t. $h(f) = \int_G f(x)\,\phi(x)\,dx$.

Note that $h(f)\,\phi(y) = h(f_y)$ for a.e. $y \in G$: Indeed

$$\forall g \in L^1(G): \boxed{\int_G h(f)\,\phi(y)\,g(y)\,dy} = h(f)\,h(g) = h(f*g)$$

$$\boxed{\int_G h(f_y)\,g(y)\,dy} = \int_G dy\, g(y) \int_G f(x-y)\,\phi(x)\,dx = \int_G (f*g)(x)\,\phi(x)\,dx = h(f*g)$$

$$\begin{aligned} G &\longrightarrow L^1(G) \longrightarrow \mathbb{C} \\ y &\longmapsto f_y \longmapsto h(f_y) \end{aligned}$$

continuous as composition of continuous functions.

Choosing f s.t. $h(f) \neq 0$: May assume $\phi(y) = \dfrac{h(f_y)}{h(f)}$ continuous

Compute $h((f_x)_y)$ in 2 ways:

$$h(f)\,\phi(x+y) = h(f_{x+y}) = h((f_x)_y) = h(f_x)\,\phi(y) = h(f)\,\phi(x)\,\phi(y)$$

$\implies \phi(x+y) = \phi(x)\,\phi(y)$.

$\implies \phi(-x) = \phi(x)^{-1}$, $\|\phi\|_\infty \leq 1 \implies |\phi(x)| = 1 \ \forall x$.

$\implies \phi \in \Gamma$.

③ <u>Injectivity:</u> $\hat{f}(\gamma_1) = \hat{f}(\gamma_2) \quad \forall f \in L^1(G)$

$\implies \int_G f(x)\big((-x,\gamma_1) - (-x,\gamma_2)\big)\,dx = 0 \quad \forall f \in L^1(G)$

$\implies (-x,\gamma_1) = (-x,\gamma_2)$ for a.e. $x \in G$

As γ_1, γ_2 are continuous and $m(V) > 0$ for every open $V \subset G$ (Ex 7.3a)

$\implies (-x,\gamma_1) = (-x,\gamma_2) \ \forall x \in G \implies \gamma_1 = \gamma_2$. ▢

If G is discrete $\Rightarrow$ $L^1(G)$ unital $\Rightarrow$ $1 \in$ range of Gelfand transform $\subseteq C_0(\Gamma)$

$\Rightarrow C_0(\Gamma) = C(\Gamma) \Rightarrow \Gamma$ compact.

Conversely, G compact, $m(G)=1$ $\Rightarrow$

$$\int_G (x,\gamma)\,dx = \begin{cases} 1 & , \gamma = 0 \\ 0 & , \gamma \neq 0 \end{cases} =: \delta(\gamma)$$

Indeed, $\int_G (0,\gamma)\,dx = \int_G dx = 1$ and if $\gamma \neq 0 \Rightarrow (x_0,\gamma) \neq 0$ for some x_0

$$\Rightarrow \int_G (x,\gamma)\,dx = (x_0,\gamma) \int_G (x-x_0,\gamma)\,dx \underset{\substack{\text{translation}\\ \text{invariance}}}{=} (x_0,\gamma) \int_G (x,\gamma)\,dx$$

$$\Rightarrow \int_G (x,\gamma)\,dx = 0.$$

G compact $\Rightarrow$ $f(x) \stackrel{!}{=} 1 \in L^1(G) \Rightarrow \hat{f} = \delta \in C_0(\Gamma)$

$\Rightarrow \{0\} \subseteq \Gamma$ open. Analogously $\{\gamma\} \subseteq \Gamma$ open $\forall \gamma \in \Gamma \Rightarrow \Gamma$ discrete

Thus
$$\boxed{\begin{array}{l} G \text{ discrete} \Rightarrow \Gamma \text{ compact} \\ G \text{ compact} \Rightarrow \Gamma \text{ discrete} \end{array}} \qquad (1.2.5).$$

Γ as an LCA group

We have endowed Γ with the Gelfand topology and the structure of an abelian group. We now show that

$$\begin{aligned} \Gamma \times \Gamma &\to \Gamma \\ (\gamma_1, \gamma_2) &\mapsto \gamma_1 - \gamma_2 \end{aligned}$$ is continuous, so that Γ is an LCA group.

To do so, we identify the Gelfand topology as the "compact-open topology" on Γ.

Thm 1.2.6: a) (x,γ) is a continuous function on $G \times \Gamma$.

b) $K \subseteq G$, $C \subseteq \Gamma$ both compact, $B_r = B_r(1) \subseteq \mathbb{C}$ the complex ball around 1 of radius r. Then
$$N(K,r) = \{\gamma \in \Gamma : \forall x \in K: (x,\gamma) \in B_r\} \subseteq \Gamma$$
$$N(C,r) = \{x \in G : \forall \gamma \in C: (x,\gamma) \in B_r\} \subseteq G$$
are open.

c) $\{\gamma + N(K,r) : \gamma \in \Gamma, K \subseteq G \text{ compact}, r > 0\}$ is a base for the topology of Γ.

d) Γ is an LCA group.

Proof: We have seen (1.2.2 - or check again) that $\hat{f_x}(\gamma) = \hat{f}(\gamma)(x,\gamma)$.
Once we know that (*) $\forall f \in L^1(G)$: $(x,\gamma) \mapsto \hat{f_x}(\gamma)$ continuous then a) will follow by choosing f s.t. $\hat{f}(\gamma) \neq 0$ and
$$(x,\gamma) = \frac{\hat{f_x}(\gamma)}{\hat{f}(\gamma)}.$$

To show (*), Let $x_0 \in G$, $\gamma_0 \in \Gamma$, $\varepsilon > 0$. Continuity of $x \mapsto f_x$ (1.1.5)
$\Rightarrow \exists$ x_0-neighborhood V $\forall x \in V$: $\|f_x - f_{x_0}\|_{L^1} < \varepsilon$
Continuity of $\hat{f_{x_0}}$ $\Rightarrow \exists$ γ_0-neighborhood W $\forall \gamma \in W$: $|\hat{f_{x_0}}(\gamma) - \hat{f_{x_0}}(\gamma_0)| < \varepsilon$

As $\|\hat{g}\|_\infty \le \|g\|_{L^1}$ $\forall g \in L^1 \Rightarrow |\hat{f_x}(\gamma) - \hat{f}_{x_0}(\gamma)| \le \|f_x - f_{x_0}\|_{L^1} < \varepsilon$

$\Rightarrow \forall x \in V\ \forall \gamma \in W: |\hat{f_x}(\gamma) - \hat{f}_{x_0}(\gamma_0)| \le$

$$\le \underbrace{|\hat{f}_x(\gamma) - \hat{f}_{x_0}(\gamma)|}_{<\varepsilon} + \underbrace{|\hat{f}_{x_0}(\gamma) - \hat{f}_{x_0}(\gamma_0)|}_{<\varepsilon}$$

$< 2\varepsilon$

$\Rightarrow (x,\gamma) \mapsto \hat{f_x}(\gamma)$ continuous. $\Rightarrow$ a).

Given K, r, let $\gamma_0 \in N(K,r)$. As (x,γ) is continuous on $G \times \Gamma$ by a)

$\forall x_0 \in K$ $\exists x_0$-neighborhood V_{x_0}, γ_0-neighborhood W_{x_0} $\forall x \in V_{x_0}, \gamma \in W_{x_0}: (x,\gamma) \in B_r$.

K compact $\Rightarrow K \subset \bigcup_{i=1}^N V_{x_i}$ and $W^* := \bigcap_{i=1}^N W_{x_i} \subseteq N(K,r)$ is an open γ_0-neighborhood. As $\gamma_0 \in N(K,r)$ was arbitrary, $N(K,r)$ is open.

Analogous: $N(C,r)$ open. $\Rightarrow$ b)

<u>Show</u>: $N(K,r)$ and translates are a base for the Gelfand topology.

i.e. <u>Show</u>: $\forall \gamma_0$-neighborhood V $\exists K,r: \gamma_0 + N(K,r) \subset V$.

Wlog $\gamma_0 = 0$. Gelfand topology = weak*-top. As $C_c(G) \subset L^1(G)$ dense $\Rightarrow \exists f_1, \dots, f_N \in C_c(G)\ \exists \varepsilon > 0: \bigcap_{i=1}^N \{\gamma \in \Gamma : |\hat{f}_i(\gamma) - \hat{f}_i(0)| < \varepsilon\}$

$\cap\!|$

V

Let $K = \bigcup_{i=1}^N \operatorname{supp} f_i$, $r < \frac{\varepsilon}{\max_i \|f_i\|_{L^1}} \Rightarrow \forall \gamma \in N(K,r)\ \forall i \in \{1,\dots,N\}$

$$|\hat{f}_i(\gamma) - \hat{f}_i(0)| = \left|\int_G f_i(x)((-x,\gamma) - 1)\,dx\right|$$

$$\le \int_K |f_i(x)|\ \underbrace{|(-x,\gamma) - 1|}_{<r}\,dx$$

$< r\,\|f_i\|_{L^1} < \varepsilon \Rightarrow \gamma \in V$

Thus $N(K,r) \subset V$. $\Rightarrow$ c).

The continuity of $(\gamma_1, \gamma_2) \longrightarrow \gamma_1 - \gamma_2$ follows, since

$\gamma_1 + N(K, \frac{r}{2})$ is a γ_1-neighborhood, $\gamma_2 + N(K, \frac{r}{2})$ is a γ_2 neighborhood whose difference is contained in a given $\gamma_1 - \gamma_2$-neighborhood $\gamma_1 - \gamma_2 + N(K,r)$.

Examples (See Exercise 4)

- $G = \mathbb{R} \Rightarrow \Gamma = \mathbb{R}$, $\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-ixy} dx$
- $G = \mathbb{R}/\mathbb{Z} \rightarrow \Gamma = \mathbb{Z}$, $\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx$
- $G = \mathbb{Z} \rightarrow \Gamma = \mathbb{R}/\mathbb{Z}$, $\hat{f}(y) = \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n y}$
- $G = \mathbb{Z}/N\mathbb{Z} \Rightarrow \Gamma = \mathbb{Z}/N\mathbb{Z}$ $\hat{f}(m) = \frac{1}{N}\sum_{n=0}^{N-1} f(n) e^{-\frac{2\pi i n m}{N}}$

Def.: (Fourier transform of measures:)

$$\begin{aligned} M(G) &\longrightarrow \text{Functions on } \Gamma \\ \mu &\longmapsto \hat{\mu}(\gamma) = \int_G (-x,\gamma)\, d\mu(x) \end{aligned}$$

Fourier–Stieltjes transform

The range will be denoted by B(Γ).

Injectivity of the inverse transform:

Thm 1.3.6: $\mu \in M(\Gamma)$, $\int_\Gamma (x,\gamma)\, d\mu(\gamma) = 0 \quad \forall x \in G \Rightarrow \mu = 0.$

Proof: $\forall f \in L^1(G)$: $\int_\Gamma \hat{f}(\gamma)\, d\mu(\gamma) = \int_\Gamma \int_G f(x)(-x,\gamma)\, dx\, d\mu(\gamma)$

$$= \int_G dx\, f(x) \underbrace{\int_\Gamma (-x,\gamma)\, d\mu(\gamma)}_{} = 0$$

As the set of all $\hat{f}$, $A(\Gamma)$, is dense in $C_0(\Gamma)$ $\Rightarrow \mu = 0$. ∎

Positive-definite functions and Bochner's theorem

Def: $\phi : G \to \mathbb{C}$ positive-definite $:\Longleftrightarrow$ $\sum_{n,m=1}^{N} c_n \bar{c}_m \phi(x_n - x_m) \geq 0$

$\forall x_1, \ldots, x_N \in G \ \forall c_1, \ldots, c_N \in \mathbb{C}$

Remark: Basic properties:

$N=2,\ x_1=0,\ x_2=x,\ c_1=1,\ c_2=c \Rightarrow$

$$(\underset{\substack{\uparrow\\ n=m=0}}{1}+\underset{\substack{\uparrow\\ n=m=1}}{|c|^2})\,\phi(0) + \underset{\substack{n=2\\ m=1}}{c\,\phi(x)} + \underset{\substack{n=1\\ m=2}}{\bar{c}\,\phi(-x)} \geq 0$$

Choose
$c=0 \Rightarrow \boxed{\phi(0) \geq 0}$
$c=1 \Rightarrow \phi(x)+\phi(-x)$ real
$c=i \Rightarrow i(\phi(x)-\phi(-x))$ ireal $\Big\} \Rightarrow \boxed{\phi(x) = \overline{\phi(-x)}}$
$c = -\frac{\phi(x)}{|\phi(x)|} \Rightarrow \boxed{|\phi(x)| \leq \phi(0)} \Rightarrow \phi$ bounded

$N=3 \rightsquigarrow |\phi(x)-\phi(y)|^2 \leq 2\phi(0)\ \mathrm{Re}[\phi(0)-\phi(x-y)]$

$\Rightarrow \phi$ uniformly continuous if ϕ continuous at 0.

Examples:

- $f \in L^2(G)$, $\tilde{f}(x) = \overline{f(-x)} \Rightarrow \phi := f * \tilde{f}$ is pos. def., continuous

(Exercise 7.5)

- characters are pos. def., more generally:
$0 \leq \mu \in M(\Gamma) \Rightarrow \phi(x) = \int_\Gamma (x,\gamma)\, d\mu(\gamma)$ pos. def., continuous

Proof: $\sum c_n \bar{c}_m \phi(x_n - x_m) = \int_\Gamma d\mu \sum_{n,m} c_n \bar{c}_m (x_n,\gamma)(-x_m,\gamma)$

$= \int_\Gamma d\mu \left|\sum_n c_n (x_n,\gamma)\right|^2 \geq 0.$

Continuity: Let $\delta > 0$. $|\mu|$ regular $\Rightarrow \exists C \subset \Gamma$ compact: $|\mu|(\Gamma \setminus C) < \delta \Rightarrow \forall x_1 - x_2 \in N(C,\delta)$:

$$|\phi(x_1) - \phi(x_2)| \leq \int_\Gamma |1 - (x_1 - x_2, \gamma)|\, d|\mu|(\gamma)$$

$= \int_{\Gamma \setminus C} \cdots + \int_C \cdots < 2\delta + \delta \|\mu\|$ ☐

Thm 1.4.3: (**Bochner**)

$\phi: G \to \mathbb{C}$ positive definite and continuous $\Leftrightarrow \exists! \, 0 \le \mu \in M(\Gamma): \phi(x) = \int_\Gamma (x,\gamma)\, d\mu(\gamma)$

Proof: „$\Leftarrow$": above example.

„$\Rightarrow$": Uniqueness follows from the injectivity of the inverse transform, Thm 1.3.6.

Normalization: $|\phi(x)| \le \phi(0) \Rightarrow$ May assume $\phi(0) = 1$.

Above remark: ϕ is even uniformly continuous.

For $f \in C_c(G)$:

$$\underbrace{\int_G dx \int_G dy \; f(x)\overline{f(y)}\,\phi(x-y)}_{\geq 0} = \lim_{N\to\infty} \underbrace{\sum_{i,j=1}^N f(x_i)\overline{f(x_j)}\,\phi(x_i - x_j) \cdot m(E_i)\, m(E_j)}_{\geq 0}$$

$\Rightarrow \; \geq 0 \;\; \forall f \in L^1(G)$. Define $[f,g] := \int_G dx \int_G dy \; f(x)\, g(y)\, \phi(x-y)$

$\Rightarrow [f,f] \geq 0$, $[g,f] = \overline{[f,g]}$ (recall $\overline{\phi(x)} = \phi(-x)$), $[f,g]$ linear in f

$$\overset{\text{Ex. 7.1a}}{\Longrightarrow} \quad |[f,g]|^2 \le [f,f][g,g].$$

Write $[f,g] = \underbrace{\int_G (f * \tilde{g})(x)\, \phi(x)\, dx}_{=:\, T_\phi(f*\tilde g)}$, $\tilde{g}(x) = \overline{g(-x)}$.

Let $V = -V$ 0-neighborhood. $g := m(V)^{-1} \mathbb{1}_V$

$\Rightarrow \forall f \in L^1: \left|[f,g] - T_\phi(f)\right| = \left| \int_G dx\; f(x) \frac{1}{m(V)} \int_V [\phi(x-y) - \phi(x)]\, dy \right|$

Let $\delta > 0$. ϕ unif cont. $\Rightarrow \exists\, V = -V$ small enough $\forall x, y \in V$:

$$|\phi(x-y) - \phi(x)| < \delta / \|f\|_{L^1}$$

$$\forall \delta > 0 \; \forall f \in L^1: \left|[f, m(V)^{-1}\mathbb{1}_V] - T_\phi(f)\right| < \int_G dx\, |f(x)|\, \delta / \|f\|_{L^1} = \delta$$

$$\Rightarrow \forall \delta > 0: \quad \underset{\geq\, |T_\phi(f)|^2 - \delta}{|[f,g]|^2} \le [f,f]\, \underset{\le\, \phi(0) + \delta \,=\, 1 + \delta}{[g,g]} \le (1+\delta)\, T_\phi(f * \tilde{f})$$

$$\Rightarrow \quad |T_\phi(f)|^2 \le T_\phi(\underbrace{f * \tilde{f}}_{=:h}) \overset{\text{iterate}}{\le} T_\phi(h*h)^{1/2} \le \dots \le T_\phi(\overbrace{h * \dots * h}^{2^n \text{ factors}})^{2^{-n}}$$

Note $|T_\phi(\alpha)| \leq \underbrace{\sup|\phi|}_{=1} \int_G |\alpha| = \|\alpha\|_{L^1}$

$\Rightarrow |T_\phi(f)|^2 \leq \|h * \cdots * h\|_{L^1}^{2^{-n}} \xrightarrow[\substack{\text{spectral}\\\text{radius}\\\text{(Exercise 7.1.d)}}]{n\to\infty} r(h) = \|\hat{h}\|_\infty = \|\hat{f}\|_\infty^2.$

$\Rightarrow T_\phi \in A(\Gamma)^*$, norm ≤ 1

Hahn-Banach: Extend to $C_0(\Gamma)^*$ with same norm.

Riesz representation thm: $\exists \mu \in M(\Gamma) : \|\mu\| \leq 1 \; \forall f \in L^1(G)$:

$$\int_G f(x)\phi(x)\,dx = T_\phi(f) = \int_G dx\, f(x) \int_\Gamma d\mu(\gamma)\,(x,\gamma)$$

$$\Rightarrow \phi(x) = \int_\Gamma d\mu(\gamma)\,(x,\gamma) \quad \text{for a.e. } x \in G$$

Both sides continuous $\Rightarrow$ $\forall x \in G$.

$x=0$: $1 = \phi(0) = \int_\Gamma d\mu = \mu(\Gamma) \leq |\mu|(\Gamma) = \|\mu\| \leq 1 \Rightarrow \mu(\Gamma) = |\mu|(\Gamma)$

$\Rightarrow \mu \geq 0.$ ∎

The basic theorems

1) Fourier Inversion

Thm 1.5.1: a) $f \in \overbrace{L^1(G) \cap B(G)}^{=:B^1} \Rightarrow \hat{f} \in L^1(G)$.

b) $\exists\, C>0\ \forall f \in L^1(G) \cap B(G):\quad f(x) = C\int_\Gamma \hat{f}(\gamma)(x,\gamma)\,d\gamma$

Remark: With our previous normalizations of the Haar measure:

$G = \mathbb{R}$: $C = \frac{1}{2\pi}$

$G = \mathbb{R}/\mathbb{Z}$ and $\mathbb{Z}$: $C = 1$

$G = \mathbb{Z}/N\mathbb{Z}$: $C = N$.

Note that one only has to check this for a single $f \in L^1(G)$.
From now on, we choose the Haarmeasure on Γ s.t. $C=1$.

Proof of Thm: $f \in B(G) \Leftrightarrow \exists \mu \in M(\Gamma): f(x) = \int_\Gamma (x,\gamma)\,d\mu(\gamma)$

1.3.6 $\Rightarrow$ μ unique, write μ_f.

Note: $\forall f,g \in B^1\ \forall h \in L^1(G)$: $\cdot\ (h*f)(0) = \int_G h(-x)f(x)\,dx = \int_G dx\, h(x)\int_\Gamma (-x,\gamma)\,d\mu_f(\gamma)$

$= \int_\Gamma \hat{h}(\gamma)\,d\mu_f(\gamma)$

$\cdot\ \int_\Gamma \hat{h}\hat{g}\,d\mu_f = ((h*g)*f)(0) = ((h*f)*g)(0) = \int_\Gamma \hat{h}\hat{f}\,d\mu_g$

$A(G) = \{\hat{h}\}$ dense in $C_0(\Gamma) \Rightarrow \hat{g}\,d\mu_f = \hat{f}\,d\mu_g$ $(*)$

Show: $d\mu_f = \hat{f}\,d\gamma$, γ = Haar measure on Γ.

Define a positive, translation invariant linear functional $\neq 0$ on $C_c(\Gamma)$:
(it has to be a multiple of $\int_\Gamma d\gamma$!)

$\psi \in C_c(\Gamma)$, $\operatorname{supp}\psi =: K$. Let $\gamma_0 \in K$. Choose $u_0 \in C_c(G)$: $\hat{u}_0(\gamma_0) \neq 0$.
u_0 exists because $C_c(G)$ is dense in $L^1(G)$, thus $C_c(G)^\wedge$ dense in $L^1(G)^\wedge = A(G)$ which is dense in $C_0(\Gamma)$. $\Rightarrow$ $(u_0 * \tilde{u}_0)^\wedge = |\hat{u}_0|^2 \geq 0$, >0 in γ_0-neighborhood U_{γ_0}. K compact $\Rightarrow$ $\exists\, \gamma_0, \dots, \gamma_N\ \exists\, u_0, \dots, u_N$: $g := u_1 * \tilde{u}_1 + \dots + u_N * \tilde{u}_N$ has $\hat{g} > 0$ on K. Ex. 7.5: g pos. def. $\Rightarrow$ (Bochner) $g \in B^1$, $\mu_g \geq 0$

$\underline{\text{Define:}}$ $T\psi := \int_{\Gamma} \frac{\psi}{\hat{g}}\, d\mu_g$: linear, $\psi \geq 0 \Rightarrow T\psi \geq 0$ $(\mu_g \geq 0!)$, $\not\equiv 0$.

$$(*) \Rightarrow T \text{ indep. of } g: \quad \int_{\Gamma} \frac{\psi}{\hat{g}} \underbrace{\frac{\hat{f}}{\hat{f}}\, d\mu_g}_{=\hat{g}\, d\mu_f} = \int_{\Gamma} \frac{\psi}{\hat{g}} \frac{\hat{g}}{\hat{f}}\, d\mu_f \quad (**)$$

$\underline{\text{Translation invariance:}}$ $\gamma_0 \in \Gamma$, Replace g by function g_1 s.t. $\hat{g}_1 > 0$ on $K \cup (\gamma_0 + K)$. $G(x) := (-x, \gamma_0)\, g_1(x)$

$$\Rightarrow \hat{G}(\gamma) = \hat{g}_1(\gamma + \gamma_0), \quad \mu_G(E) = \mu_{g_1}(E - \gamma_0) \quad \left(\text{uniqueness of } \mu_f!\right)$$

$$\Rightarrow T\psi_{\gamma_0} \overset{(**)}{=} \int_{\Gamma} \frac{\psi(\gamma - \gamma_0)}{\hat{g}_1(\gamma)}\, d\mu_{g_1}(\gamma)$$

$$= \int_{\Gamma} \frac{\psi(\gamma)}{\hat{G}(\gamma)}\, d\mu_G(\gamma) \overset{(**)}{=} T\psi$$

$$\Longrightarrow \quad T\psi = c \int_{\Gamma} \psi(\gamma)\, d\gamma \quad (\text{"uniqueness" of Haar measure}).$$

$$f \in B^1 \Rightarrow T(\psi \hat{f}) = \int_{\Gamma} \psi \frac{\hat{f}}{\hat{g}}\, d\mu_g = \int_{\Gamma} \psi\, d\mu_f \quad \forall \psi \in C_c(\Gamma)$$

$$\parallel$$

$$c \int_{\Gamma} \psi(\gamma) \hat{f}(\gamma)\, d\gamma$$

$$\Longrightarrow c \hat{f}\, d\gamma = d\mu_f, \quad |\mu_f|(\Gamma) < \infty \Rightarrow \hat{f} \in L^1(\Gamma)$$

and $$\boxed{f(x) = \int_{\Gamma} (x, \gamma)\, d\mu_f(\gamma) = c \int_{\Gamma} (x, \gamma)\, \hat{f}(\gamma)\, d\gamma}$$ $\square$

Consequences:

Corollary: $\{x+N(C,r) : C \subseteq \Gamma \text{ compact}, r>0, x\in G\}$ is a base for the topology of G

Proof: $V \subset G$ 0-neighborhood, W compact 0-neighborhood s.t. $W-W \subset V$
Show: $\exists N(C,r) \subseteq V$.

$g := \mathbb{1}_W * \tilde{\mathbb{1}}_W / m(W)$ continuous, pos. def. by Exercise 7.5

$\operatorname{supp} g \subseteq W-W \Rightarrow g \in L^1 \cap B(G)$

$\Rightarrow 1 = g(0) = \int_\Gamma \hat{g}\, d\gamma$, $\hat{g} = |\hat{f}|^2 \geq 0$

$\Rightarrow \exists C \subset \Gamma$ compact: $\int_C \hat{g}\, d\gamma > \frac{2}{3}$

$x \in N(C, \frac{1}{3}) \Rightarrow |1 - \underbrace{(x,\gamma)}_{\hat{C}}| \leq \frac{1}{3} \Rightarrow \operatorname{Re}(x,\gamma) > \frac{2}{3}$

$$\Rightarrow |g(x)| = \left|\int_\Gamma (x,\gamma)\,\hat{g}(\gamma)\,d\gamma\right| \geq \underbrace{\int_C |(x,\gamma)||\hat{g}|}_{\geq \frac{2}{3}\cdot\frac{2}{3}} - \underbrace{\int_{C'} \underbrace{|(x,\gamma)|}_{\leq 1}|\hat{g}|}_{< \frac{1}{3}}$$
$$\geq \frac{4}{9} - \frac{1}{3} = \frac{1}{9} \Rightarrow x \in W-W \subseteq V \Rightarrow N(C,r) \subset V.$$

Conversely:
As $N(C,r)$ open (Thm 1.2.6 b) $\Rightarrow \exists$ 0-neighborhood $V \subset N(C,r)$

$\Rightarrow$ conclusion $\boxed{\text{Q}}$

The previous argument shows that $(x_0,\gamma) \neq 1$ for some $\gamma \in \Gamma$ if $x_0 \notin V$. Given $0 \neq x_0 \in G$, we may choose V small enough s.t. $x_0 \notin V$

$\Rightarrow \forall x_0 \neq 0\ \exists \gamma \in \Gamma: (x_0,\gamma) \neq 1 \Rightarrow$

Corollary: Γ separates points on G, i.e. $x_1 \neq x_2 \Rightarrow \exists \gamma \in \Gamma: (x_1,\gamma) \neq (x_2,\gamma)$

2.) Planchersel Theorem:

Thm 1.6.1: $\wedge : L^1(G) \cap L^2(G) \to L^2(\Gamma)$

- is an isometry: $\int_G |f|^2 = \int_\Gamma |\hat{f}|^2$
- has dense range.

It therefore extends to an isometric isomorphism $\wedge : L^2(G) \to L^2(\Gamma)$.

Proof: $f \in L^1 \cap L^2$, $g = f * \tilde{f} \in L^1$ continuous and positive definite (Ex. 7.5)

$$\hat{g} = |\hat{f}|^2 \Rightarrow \int_G |f|^2 = \int_G f(x)\,\tilde{f}(-x) = g(0) = \int_\Gamma \hat{g} = \int_\Gamma |\hat{f}|^2.$$

dense range: $\Phi := (L^1 \cap L^2)^\wedge \subseteq L^{1\wedge} = A(G)$.

$$f \in L^1 \cap L^2 \Rightarrow f_x \in L^1 \cap L^2 \Rightarrow (x,\gamma)\hat{f}(\gamma) \in \Phi \quad \forall x \in G \ (*)$$

$$(x,\gamma) f \in L^1 \cap L^2 \Rightarrow \hat{f}_\gamma \in \Phi \quad \forall \gamma \in \Gamma \ (**)$$

Let $\psi \in L^2(\Gamma)$ s.t. $\int_\Gamma \phi \overline{\psi} = 0 \quad \forall \phi \in \Phi$.

$$\overset{(*)}{\Longrightarrow} \int_\Gamma (x,\gamma) \underbrace{\phi \overline{\psi}}_{\in L^1(\Gamma)} = 0 \quad \forall \phi \in \Phi \ \forall x \in G$$

Injectivity of the dual transform (1.3.6) $\Rightarrow \forall \phi \in \Phi : \phi \overline{\psi} = 0$ a.e.

(**) $\Rightarrow \forall \gamma \in \Gamma$: can find $\phi \in \Phi$ s.t. $\phi(\gamma) \neq 0$

$\Rightarrow \overline{\psi} = 0$ a.e.

$\Rightarrow \{\psi \in L^2 : \int_\Gamma \phi \overline{\psi} = 0 \quad \forall \phi \in \Phi\} = \{0\}$

$\overset{\text{Ex. 7.1(b)}}{\Longrightarrow} \Phi \subseteq L^2(\Gamma)$ dense. $\boxed{E}$

Polarization: $4 f\overline{g} = |f+g|^2 - |f-g|^2 + i\,|f+ig|^2 - i\,|f-ig|^2$

$$\Rightarrow \int_G f\overline{g} = \int_\Gamma \hat{f}\,\overline{\hat{g}} \qquad (\text{Parseval}).$$

Consequences

Thm 1.6.3: $A(\Gamma) = \{ F_1 * F_2 : F_1, F_2 \in L^2(\Gamma) \}$

Proof: „$\supseteq$": $f, g \in L^2(G) \Rightarrow \int_G fg = \int \hat{f}(\gamma)\hat{g}(-\gamma)$

$$\Rightarrow \int_G f(x)\, g(x)\, (x, \gamma_0)\, dx = \int \hat{f}(\gamma)\, \hat{g}(\gamma_0 - \gamma)\, d\gamma = (\hat{f} * \hat{g})(\gamma_0)$$

$\Rightarrow \hat{f} * \hat{g} = \widehat{fg} \in A(\Gamma)$.

„$\subseteq$": Write $h \in L^1(G)$ as: $h = \underbrace{|h|^{1/2} \frac{h}{|h|}}_{\in L^2} \underbrace{|h|^{1/2}}_{\in L^2} =: fg$

$$\Rightarrow \hat{h} = \hat{f} * \hat{g}.$$

$\boxed{}$

Thm 1.6.4: $\emptyset \neq E \subset \Gamma$ open $\Rightarrow \exists 0 \neq \hat{f} \in A(\Gamma) : \hat{f}(\gamma) = 0$ outside E.

Proof: $K \subset E$ compact s.t. $m(K) > 0$, V compact 0-neighborhood s.t. $K + V \subset E$. $\hat{f} := \mathbb{1}_K * \mathbb{1}_V \Rightarrow \hat{f} = 0$ outside $K + V \subset E$

$\hat{f} \in A(\Gamma)$ by 1.6.3, $\hat{f} \not\equiv 0$. $\boxed{}$

Thm (Hausdorff-Young): The Fourier transform induces a continuous transformation $\wedge : L^p(G) \to L^{p'}(G)$, $1 \leq p \leq 2$.

Proof: Riesz-Thorin $\boxed{}$

3.) Pontryagin duality:

G LCA $\rightarrow$ Γ LCA $\rightsquigarrow$ $\hat{\Gamma}$ dual group of Γ (LCA)
Dual pairing (x, γ) $\qquad$ $(\gamma, \hat{\gamma})$
<u>Aim:</u> $G \cong \hat{\Gamma}$.

Thm. 1.2.6: (x, γ) continuous on $G \times \Gamma$

$$\Rightarrow \alpha : G \to \hat{\Gamma} \quad \text{s.t.} \quad (\gamma, \alpha(x)) := (x, \gamma).$$

<u>Note:</u> α homomorphism:

$$\begin{aligned}(\gamma, \alpha(x+y)) &= (x+y, \gamma)\\ &\overset{!}{=} (x,\gamma)(y,\gamma)\\ &\overset{!}{=} (\gamma, \alpha(x))(\gamma, \alpha(y))\\ &\overset{!}{=} (\gamma, \alpha(x)+\alpha(y)) \quad \forall \gamma.\end{aligned}$$

Fourier inversion $\Rightarrow$ Γ separates points on G

$$\Rightarrow \alpha \text{ injective.}$$

<u>Thm 1.7.2:</u> α homeomorphism onto $\hat{\Gamma}$.

<u>Proof:</u> <u>1.) α homeomorphism onto its range</u>

$C \subset \Gamma$ compact, $r > 0$ $\rightsquigarrow$
$$V := \{x \in G : |1 - (x,\gamma)| < r \quad \forall \gamma \in C\} \subseteq G$$
$$W := \{\hat{\gamma} \in \hat{\Gamma} : |1 - (\gamma, \hat{\gamma})| < r \quad \forall \gamma \in C\} \subseteq \hat{\Gamma}$$

These sets and their translates form a base of the topology of G (Fourier inversion) resp. $\hat{\Gamma}$ (Thm. 1.2.6c).
(Gelfand top = compact open top)

Since $\alpha(V) = W \cap \alpha(G)$, both α and $\alpha^{-1} : \alpha(G) \to G$ are continuous in 0. By translation also at any other point.

<u>2.) $\alpha(G)$ closed:</u> Will be shown later.

3.) $\alpha(G)$ dense in $\hat{\hat{\Gamma}}$. Assume $\overline{\alpha(G)} \neq \hat{\hat{\Gamma}}$.

Thm. 1.6.4 (Corollary of Plancherel) $\Rightarrow \exists 0 \neq F \in A(\hat{\hat{\Gamma}})$: $F = 0$ on $\alpha(G)$

$F \in A(\hat{\hat{\Gamma}}) \Leftrightarrow \exists \phi \in L^1(\Gamma)$: $F(\hat{\gamma}) = \int_\Gamma \phi(\gamma)\,(-\gamma, \hat{\gamma})\,d\gamma$

$0 = F(\alpha(x)) = \int_\Gamma \phi(\gamma)\,(-x, \gamma)\,d\gamma$

$\Longrightarrow \phi = 0$ by injectivity of the dual transform Thm 1.3.6

$\Rightarrow F = 0$ $\lightning$ contradiction ! $\boxed{}$

Consequences of the duality theorem:

1.2.5: G compact $\Rightarrow$ Γ discrete

G discrete $\Rightarrow$ Γ compact

Pontryagin $\Rightarrow$ An LCA group is discrete $\Leftrightarrow$ it is the dual of a compact group.

An LCA group is compact $\Leftrightarrow$ it is the dual of a discrete group.

1.3.6: (Injectivity of dual transform)

$\Longrightarrow$ $\mu \in M(G)$, $\hat{\mu} \equiv 0 \Rightarrow \mu \equiv 0$, (injectivity of Fourier transform)

Therefore, the Gelfand transform on $L^1(G)$ / $M(G)$ is injective.

1.1.7: G discrete $\Rightarrow$ $L^1(G)$ unital

Pontryagin $\Rightarrow$ G discrete $\Leftrightarrow$ $L^1(G)$ unital $\Leftrightarrow$ $L^1(G) = M(G)$.

Proof: G not discrete $\Rightarrow$ Γ not compact $\Rightarrow$ $A(\Gamma) \subset C_0(\Gamma)$ not unital

But $A(\Gamma) \overset{\wedge}{\cong} L^1(G)$. $\boxed{}$

Another Consequence of Fourier inversion: G compact.

"Trigonometric polynomials" $\left\{ \sum_{j=1}^{n} a_j (x, \gamma_j) \right\}_{\mathbb{C}} \subseteq C(G)$, closed under complex conjugation.

As Γ separates points on $G \Rightarrow$ Trig. polynomials do so too $\Rightarrow$ dense in $C(G)$.

We still have to prove that $\alpha(G)$ is closed in $\hat{\Gamma}$. This follows from the following general Lemma:

<u>Lemma:</u> Let $\hat{\Gamma}$ be a locally compact (abelian) group and H $(=\alpha(G))$ a subgroup, which is locally compact in the subspace topology. Then H is closed in $\hat{\Gamma}$.

<u>Proof:</u> H locally compact $\Rightarrow$ $\exists$ open 0-neighborhood $U \subseteq \hat{\Gamma}$ s.t. the closure in H of $U \cap H$ $(=: K)$ is compact in H. But then K is also compact, thus closed in $\hat{\Gamma}$, so K is the closure of $U \cap H$ in $\hat{\Gamma}$.

Now suppose $\hat{\gamma}_0 \in \overline{H}$ and let $\{\hat{\gamma}_\alpha\} \to \hat{\gamma}_0$. Let $V = -V$ 0-neighborhood in $\hat{\Gamma}$ s.t. $V + V \subseteq U$ $\Rightarrow -\hat{\gamma}_0 \in \overline{H}$, since $\overline{H}$ is a subgroup $\Rightarrow$ $(V - \hat{\gamma}_0) \cap H \neq \emptyset$. Let $\eta \in (V - \hat{\gamma}_0) \cap H$.
As $\hat{\gamma}_\alpha$ is eventually $\in \hat{\gamma}_0 + V$ $\Rightarrow \eta + \hat{\gamma}_\alpha$ eventually $\in (V - \hat{\gamma}_0) + (\hat{\gamma}_0 + V) = V + V \subseteq U$. Moreover $\eta + \hat{\gamma}_\alpha \in H$ and $\eta + \hat{\gamma}_\alpha \to \eta + \hat{\gamma}_0 \Rightarrow \eta + \hat{\gamma}_0 \in K \subset H$
$\Rightarrow \hat{\gamma}_0 = -\eta + (\eta + \hat{\gamma}_0) \in H \Rightarrow H$ closed. $\blacksquare$