Symbolic calculus:

Lemma: Let $f \in S(\mathbb{R}^d)$ and $a, (a_k)_{k \in \mathbb{N}_0} \subseteq S^m$ such that $\exists\, C_{\alpha\beta}$ indep. of k:

$$|\partial_x^\alpha \partial_\xi^\beta a_k(x,\xi)| \leq C_{\alpha\beta}\,(1+|\xi|)^{m-|\beta|} \quad \text{and} \quad a_k(x,\xi) \xrightarrow{k\to\infty} a(x,\xi) \text{ pointwise.}$$

$\Rightarrow \forall f \in S: op(a_k) f \xrightarrow{k\to\infty} op(a) f$ in the topology of $S(\mathbb{R}^d)$.

Proof: Apply the dominated convergence theorem to the integral

$$op(a_k) f(x) = \int_{\mathbb{R}^d} (1+|x|^2)^{-N} (1-\Delta_\xi)^N \left(a_k(x,\xi)\, \hat{f}(\xi)\right) e^{ix\xi}\, d\xi$$

$\square$

Cor: Let $\gamma \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, $\gamma(0,0) = 1$. Set $a_\varepsilon(x,\xi) = a(x,\xi)\,\gamma(\varepsilon x, \varepsilon \xi)$.

If $a \in S^m \Rightarrow a_\varepsilon \in S^m$ and $|\partial_x^\alpha \partial_\xi^\beta a_\varepsilon(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\beta|}$ $\forall \varepsilon \in (0,1]$

and $\forall f \in S: op(a_\varepsilon) f \xrightarrow{\varepsilon \to 0^+} op(a) f$ in $S(\mathbb{R}^d)$.

Proof: clear $\square$

Theorem: (Composition of pseudodifferential operators)

Let $a \in S^{m_1}, b \in S^{m_2} \Rightarrow \exists\, c \in S^{m_1+m_2}: op(c) = op(a) \circ op(b)$

and $\forall N > 0$: $c - \sum_{|\alpha|<N} \frac{(i)^{-|\alpha|}}{\alpha!} \underbrace{\partial_\xi^\alpha a\; \partial_x^\alpha b}_{\in S^{m_1+m_2-|\alpha|}} \in S^{m_1+m_2-N}$

Proof: We write a for $a(x,\xi)\gamma(\varepsilon x, \varepsilon\xi)$, b for $b(x,\xi)\gamma(\varepsilon x,\varepsilon\xi)$ as in the Corollary $\Longrightarrow \forall f \in S(\mathbb{R}^d)$:

$$op(b) f(y) = \iint b(y,\xi)\, e^{i\xi(y-z)} f(z)\, dz\, d\xi$$

$$\Rightarrow op(a)(op(b) f)(x) = \iiiint a(x,\eta)\, b(y,\xi)\, e^{i\xi(y-z)} e^{i\eta(x-y)} \cdot f(z)\, dz\, d\xi\, dy\, d\eta$$

$$(*) \quad = \iint \underbrace{\left(\iint a(x,\eta)\, b(y,\xi)\, e^{i(x-y)(\eta-\xi)}\, dy\, d\eta \right)}_{=: c(z,\xi)} e^{i(x-z)\xi} f(z)\, dz\, d\xi$$

Integrating in $y \Rightarrow c(z,\xi) = \int d\eta\, a(x,\xi+\eta)\, \hat{b}(\eta,\xi)\, e^{ix\eta}\, d\eta \quad (**)$

If the original b already had compact support in $x \in \mathbb{R}^d$

$$\Longrightarrow \left|\partial_x^\alpha \partial_\xi^\beta b(x,\xi)\right| \leq C_{\alpha\beta}\, \varphi(x)\, (1+|\xi|)^{m-|\beta|} \qquad \text{for some } \varphi \in C_c^\infty(\mathbb{R}^d)$$

$$\Rightarrow \quad \forall \tilde N > 0: \quad \left|\partial_\eta^\alpha \partial_\xi^\beta \hat b(\eta,\xi)\right| \leq \tilde C_{\alpha\beta,\tilde N}\, (1+|\eta|)^{-\tilde N}\, (1+|\xi|)^{m-|\beta|} \qquad (***)$$

Use Taylor's formula $a(x,\xi+\eta) = \sum_{|\alpha|<N} \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi)\, \eta^\alpha + R_N(x,\xi,\eta)$,

where $|R_N(x,\xi,\eta)| \leq C_N |\eta|^N \sup_{\tau \in \underbrace{[\xi,\xi+\eta]}_{=\text{line segment connecting } \xi \text{ and } \xi+\eta.}} \sup_{|\alpha|=N} |\partial_\tau^\alpha a(x,\tau)|$,

$$(**) = \sum_{|\alpha|<N} \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi) \underbrace{\int_{\mathbb{R}^d} \eta^\alpha\, \hat b(\eta,\xi)\, e^{ix\eta}\, d\eta}_{= i^{-|\alpha|} \partial_x^\alpha b(x,\xi)} + \underbrace{\int_{\mathbb{R}^d} R_N(x,\xi,\eta)\, \hat b(\eta,\xi)\, e^{ix\eta}\, d\eta}_{=:\, R}$$

Concerning R_N: Note that $\forall\, |\alpha| \geq m_1$

$$\sup_{\tau \in [\xi,\xi+\eta]} |\partial_\tau^\alpha a(x,\tau)| \leq C_\alpha \sup_\tau (1+|\tau|)^{m_1-|\alpha|} \leq \begin{cases} C_\alpha (1+|\xi|)^{m_1-|\alpha|}, & \text{if } |\xi| \geq 2|\eta| \\ C_\alpha, & \text{for all } \xi,\eta. \end{cases}$$

$$\begin{aligned}
\Rightarrow |R| &\leq C\,(1+|\xi|)^{m_1+m_2-\tilde N} \int_{2|\eta|<|\xi|} (1+|\eta|)^{-\tilde N} |\eta|^N\, d\eta \\
&\quad + C\,(1+|\xi|)^{m_2} \int_{2|\eta|\geq|\xi|} (1+|\eta|)^{-\tilde N} |\eta|^N\, d\eta \\
\text{(polar coordinates)} &\leq C\,(1+|\xi|)^{m_1+m_2-\tilde N} \underbrace{\int_{\mathbb{R}^d} (1+|\eta|)^{-\tilde N} |\eta|^N\, d\eta}_{<\infty \text{ for large } \tilde N} \\
&\quad + C\,(1+|\xi|)^{m_2}\, \mathrm{vol}(S^{d-1}) \int_{|\xi|/2}^\infty r^{d-1} \frac{r^N}{(1+r)^{\tilde N}}\, dr \\
&\leq C\,(1+|\xi|)^{m_1+m_2-N}
\end{aligned}$$

$$\int_{|\xi|/2}^\infty r^{d-1} \frac{r^N}{(1+r)^{\tilde N}}\, dr \leq \frac{1}{(1+|\xi|/2)^{\tilde N-N-d-1}} \underbrace{\int_0^\infty \frac{dr}{(1+r)^2}}_{<\infty}$$

for $\tilde N \geq N+d+1-m_2$.

An analogous argument works for $\partial_x^\alpha \partial_\xi^\beta R$.

$\Rightarrow$ Correct estimate (independently of ε) when b already had compact support.

If b does not have compact support, fix $x_0 \in \mathbb{R}^d$, $\eta \in C_c^\infty(\mathbb{R}^d)$ s.t. $\eta = 1$ in $B_1(x_0)$, $\operatorname{supp} \eta \subseteq B_2(x_0)$ and decompose $b = \underbrace{\eta b}_{\text{compact support}} + \underbrace{(1-\eta) b}_{=: b_2}$

Using $(*)$, let

$$c_2(x,\xi) := \iint dy\, d\eta\; a(x,\eta)\, b_2(y,\xi)\, e^{i(x-y)(\eta-\xi)}$$

$$\overset{\text{(integrate by parts)}}{=} \iint dy\, d\eta\; \frac{\Delta_\eta^{N_1} a(x,\eta)}{(-|x-y|^2)^{N_1}}\, b_2(y,\xi)\, e^{i(x-y)(\eta-\xi)}$$

$$\overset{\text{(again, now in } y)}{=} \iint dy\, d\eta\; \Delta_\eta^{N_1} a(x,\eta)\, \frac{(1-\Delta_y)^{N_2}}{(1+|\eta-\xi|^2)^{N_2}} \frac{b_2(y,\xi)}{(-|x-y|^2)^{N_1}} \cdot e^{i(x-y)(\eta-\xi)}$$

This is ok for $x \in B_{1/2}(x_0)$: $|x-y| \geq \frac{1}{2}$ since $\operatorname{supp} b_2 \subset \mathbb{R}^d \setminus B_1(x_0)$

$$\Rightarrow \quad \forall x \in B_{1/2}(x_0): \; |c_2(x,\xi)| \leq C_{N_1,N_2} \iint dy\, d\eta\; \frac{(1+|\eta|)^{m_1-2N_1} (1+|\xi|)^{m_2}}{(1+|\eta-\xi|)^{2N_2} (1+|x-y|)^{2N_1}}$$

Choosing N_1, N_2 large $\Rightarrow$ $\forall x \in B_{1/2}(x_0): |c_2(x,\xi)| \leq C_N (1+|\xi|)^{m_1+m_2-N}$ $\forall N$

$\Rightarrow$ c has the desired form $\forall x \in B_{1/2}(x_0)$

Since x_0 was arbitrary $\Rightarrow$ ✓.

Finally, we let $\varepsilon \to 0^+$. With $a_\varepsilon(x,\xi) = a(x,\xi)\, \gamma(\varepsilon x, \varepsilon \xi)$
$b_\varepsilon(x,\xi) = b(x,\xi)\, \gamma(\varepsilon x, \varepsilon\xi)$

define c_ε by $op(c_\varepsilon) = op(a_\varepsilon) \circ op(b_\varepsilon)$. The above discussion shows $c_\varepsilon \in S^{m_1+m_2}$ uniformly in $\varepsilon \in (0,1]$ and c_ε converges pointwise to some $c \in S^{m_1+m_2}$. By the Corollary, $op(c) = op(b) \circ op(a)$.

$\boxed{\Xi}$

Cor./Lemma 2.16: $a \in S^m,\ b \in S^{m'} \Rightarrow a(x,D)\,b(x,D) - b(x,D)\,a(x,D) = c(X,D)$,
where $c \in S^{m+m'-1}$, $c - i(\partial_\xi a\,\partial_x b - \partial_x a\,\partial_\xi b) \in S^{m+m'-2}$.

Adjoints: **Lemma 2.18:** $a \in S^m \Rightarrow$ The operator $a(x,D)^*$,

defined by $\langle a(x,D)^* f, g\rangle_{L^2} := \langle f, a(x,D) g\rangle_{L^2}$ $\forall f,g \in S(\mathbb{R}^d)$

is a pseudodifferential operator with symbol $\overline{a(x,\xi)} + r(x,\xi)$,

$r(x,\xi) \in S^{m-1}$. Actually, the symbol has the asymptotic

expansion $$\sum_{|\alpha|\geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha \overline{a(x,\xi)}.$$

Idea of proof: 1.) Replace a by by $a(x,\xi)\chi(\varepsilon\xi)$ for some cut-off function $\chi \in C_c^\infty(\mathbb{R}^d)$, $\chi(0)=1$, and show estimates which hold uniformly in ε. Then let $\varepsilon \to 0^+$ at the end.

2.) $$a(x,D)f(x) = \iint a(x,\xi)\, e^{i(x-y)\xi} f(y)\, d\xi\, dy$$

adjoint given by $x \leftrightarrow y$ + conjugation $\Rightarrow$
$$a(x,D)^* f(x) = \iint \overline{a(y,\xi)\, e^{i(y-x)\xi}} f(y)\, d\xi\, dy$$
$$= \iint \overline{a(y,\xi)}\, e^{i(x-y)\xi} f(y)\, d\xi\, dy$$

Aim: Replace $\overline{a(y,\xi)}$ by $\overline{a(x,\xi)}$.

Taylor: $\overline{a(y,\xi)} = \overline{a(x,\xi)} + (y-x)\,\partial_x \overline{a(x,\xi)} + \dots$

$$\Rightarrow a(x,D)^* f(x) = \overline{a}(x,D)f(x) + \iint \partial_x \overline{a}(x,\xi)\, i\partial_\xi\, e^{i(x-y)\xi} f(y)\, d\xi\, dy$$
$$+ \dots$$
$$= \overline{a}(x,D)f(x) - i \iint \underbrace{\partial_x\partial_\xi \overline{a}(x,\xi)}_{\in S^{m-1}}\, e^{i(x-y)\xi} f(y)\, d\xi\, dy$$
$$+ \dots$$

Show that "..." satisfies the symbol estimates of S^{m-2}. $\boxed{2}$

Remark: Fourier multipliers: $m(D)^* = \overline{m}(D)$

Differential operators: $\left(\sum_{|\alpha|\leq k} a_\alpha D^\alpha\right)^* = \sum_{|\alpha|\leq k} D^\alpha \overline{a_\alpha} = \sum_{|\alpha|\leq k} \overline{a_\alpha} D^\alpha$

+ lower order terms [illegible]

Consequences of the calculus:

Boundedness: Thm: Let $a \in S^0(\mathbb{R}^d) \Longrightarrow a(x,D) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ bounded

Thus $a(x,D)$ is a Calderon-Zygmund operator.

Proof: Recall our estimates for the kernel $K(x,y)$ of $a(x,D)$:

If $a \in S^m$, $m < -d \Rightarrow a \in L^1(\mathbb{R}^d_\xi)$ $\forall x$ and $K(x,y) = \mathcal{F}^{-1}_{\xi \to x-y} a(x,\xi)$

is an honest function $\in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, which satisfies

$$|K(x,y)| \leq C|x-y|^{-d-m} \quad \forall x,y \in \mathbb{R}^d \quad \text{and}$$
$$|K(x,y)| \leq C_N |x-y|^{-N} \quad \forall |x-y| \geq 1 \quad \forall N \in \mathbb{N}$$

(see our Thm. in Ch. 4).

$$\Rightarrow \sup_x \int_{\mathbb{R}^d} |K(x,y)|\,dy < \infty \quad , \sup_y \int_{\mathbb{R}^d} |K(x,y)|\,dx < \infty$$

Schur's test $\Rightarrow A = a(x,D) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ if $m < -d$.

Let $a \in S^m$. Since $\|Au\|^2_{L^2} = \langle A^* A u, u \rangle$, it suffices to show that $B := A^* A \in S^{2m}$ is bounded on L^2. But $\|B^n\| = \|B\|^n$ and $B^n \in S^{2mn}$ for $n \in \mathbb{N}$. If $m<0$, choose $n \in \mathbb{N}$ such that $2mn < -d \Rightarrow B^n$ bounded on $L^2 \Rightarrow A$ bounded on L^2.

Finally, let $a \in S^0 \Rightarrow B = A^* A \in S^0$ w/ symbol $b = |a|^2 + r(x,\xi)$, $r(x,\xi) \in S^{-1}$ and $|b| \leq C$. Write $C = M - \alpha$ for some $\alpha > 0$. $\Rightarrow M - \mathrm{Re}\, b \geq \alpha > 0$ and the chain rule implies $q := (M - \mathrm{Re}\, b)^{1/2} \in S^0$. Consider $q(x,D)$.

$$q(x,D)^* q(x,D) = M - (\mathrm{Re}\, b)(x,D) + r(x,D)$$
$$\overset{!}{=} M - b(x,D) + \tilde{r}(x,D) \quad , \tilde{r}(x,\xi) \in S^{-1}.$$

From the above, $|\langle \tilde{r}(x,D) u, u \rangle_{L^2}| \leq C \|u\|^2_{L^2}$, so we obtain

$$M\|u\|_{L^2}^2 - \|a(x,D)u\|_{L^2}^2 = \|q(x,D)u\|_{L^2}^2 - (r(x,D)u,u) \geq -C\|u\|_{L^2}^2$$

or $\|a(x,D)u\|_{L^2}^2 \leq (M+C)\|u\|_{L^2}^2$, i.e. $a(x,D)$ is bounded $\square$

- Sobolev spaces:

Recall, Diffan 1, Def. 6.7: $H^s(\mathbb{R}^d) := \{u \in S'(\mathbb{R}^d) : \langle\xi\rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^d)\}$

L^p-Sobolev spaces: $W^{s,p}(\mathbb{R}^d) := \{u \in S'(\mathbb{R}^d) : \langle D\rangle^s u \in L^2(\mathbb{R}^d)\}$

(as in Diffan 1) $= \overline{S}^{\|\cdot\|_{W^{s,p}}}$

where $\|u\|_{W^{s,p}} = \|\langle D\rangle^s u\|_{L^p}$. For integer $k \geq 0$, $W^{k,p}$ will be shown to coincide with $\{u \in S'(\mathbb{R}^d) : \partial^\alpha u \in L^p(\mathbb{R}^d)\ \forall |\alpha| \leq k\}$.

$(W^{s,p})' = W^{-s,p'}$, $\frac{1}{p}+\frac{1}{p'}=1$. Note $\langle\xi\rangle^{s'-s} \in S^{s'-s} \Rightarrow \forall s'<s$

Prop 3.3: (i) $\|\langle D\rangle^{s'}u\|_{L^p} = \|\underbrace{\langle D\rangle^{s'-s}}_{\text{bounded}} \langle D\rangle^s u\|_{L^p} \leq C\|\langle D\rangle^s u\| \Rightarrow W^{s,p} \hookrightarrow W^{s',p}$

ii) $a \in S^m \Rightarrow \|a(x,D)u\|_{W^{s-m,p}} \leq C\|u\|_{W^{s,p}}$, i.e. $a(x,D): W^{s,p} \to W^{s-m,p}$ continuously

Proof: $\|\underbrace{\langle D\rangle^{s-m} a(x,D)\langle D\rangle^{-s}}_{\text{bounded}} \langle D\rangle^s u\|_{L^p} \leq \|\langle D^s\rangle u\|_{L^p}$.

iii) $|\partial^\alpha m(\xi)| \leq C|\xi|^{-|\alpha|}\ \forall |\alpha| \leq d+2 \Rightarrow m(D): W^{s,p} \to W^{s,p}$ continuously

Proof: $\|\langle D\rangle^s m(D)u\|_{L^p} = \|m(D)\langle D\rangle^s u\|_{L^p} \leq C\|\langle D\rangle^s u\|_{L^p}$

(iv) $f, \partial^\alpha f \in W^{s,p}\ \forall |\alpha|=k \Rightarrow \|f\|_{W^{s+k,p}} \approx \|f\|_{W^{s,p}} + \sum_{|\alpha|=k}\|\partial^\alpha f\|_{W^{s,p}}$

$$\approx \sum_{|\alpha|\leq k}\|\partial^\alpha f\|_{W^{s,p}}$$

Proof: E.g. $k=1$: $\langle D\rangle = (1+\Delta)^{1/2} = \frac{1-\Delta}{\sqrt{1-\Delta}}$

$$= -\underbrace{\frac{\partial x_1}{\sqrt{\cdots}}}_{\text{order }0}\partial x_1 - \cdots - \underbrace{\frac{\partial x_d}{\sqrt{\cdots}}}_{\text{order }0}\partial x_d + \underbrace{\frac{1}{\sqrt{\cdots}}}_{\text{order }0}$$

$\Rightarrow \|\langle D\rangle f\|_{W^{s,p}} \leq C\|\partial_{x_1} f\|_{W^{s,p}} + \cdots + C\|\partial_{x_d} f\|_{W^{s,p}} + C\|f\|_{W^{s,p}}$

Conversely: $\|f\|_{W^{s,p}}, \|\partial_{x_j} f\|_{W^{s,p}} \leq C\|\langle D\rangle f\|_{W^{s,p}} = C\|f\|_{W^{s+1,p}}$ by (i), (ii)

<u>$k>1$</u>: Induction. $\boxed{}$

(v) (Sobolev embedding)

- $s>0$, $p<q<\infty$ s.t. $\frac{d}{q} \geq \frac{d}{p} - s \Rightarrow W^{s,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$

- $s > \frac{d}{p} \Rightarrow W^{s,p}(\mathbb{R}^d) \hookrightarrow C_b(\mathbb{R}^d)$

<u>Proof</u>: From the Hardy-Littlewood-Sobolev inequality (exercises). $\boxed{}$

Littlewood-Paley characterization

$1<p<\infty$, $s \in \mathbb{R}$, ϕ_0 bump function adapted to $B_2(0)$
ψ_j bump functions adapted to $B_{2^{j+1}}(0) \setminus B_{2^{j-1}}(0)$ s.t.

$c \leq \phi_0^2 + \sum \psi_j^2 \leq C \Longrightarrow$ The $W^{s,p}(\mathbb{R}^d)$-norm is

equivalent to $|||f||| := \|\phi_0(D) f\|_{L^p} + \|(\sum_{j=1}^{\infty} 2^{2js} |\psi_j(D) f|^2)^{1/2}\|_{L^p(\mathbb{R}^d)}$

<u>Proof</u>: $2^{js}\psi_j(D) f \simeq \langle D\rangle^s \psi_j(D) f = \psi_j(D)\langle D\rangle^s f$

because ψ_j is supported in $|\xi| \in [2^{j-1}, 2^{j+1}]$.

Littlewood-Paley characterization of $L^p \Rightarrow$

$|||f||| \asymp \|\langle D\rangle^s f\|_{L^p}$, which is $= \|f\|_{W^{s,p}}$ $\boxed{}$

Astrid will say something about interpolation of Sobolev spaces.
In particular, if $A: W^{s_1,p} \to W^{\tilde{s}_1,q}$
$W^{s_2,p} \to W^{\tilde{s}_2,q}$ continuous

$\Rightarrow A: W^{s,p} \to W^{\tilde{s},q}$ $\forall\ s = (1-\theta)s_1 + \theta s_2 \in [s_1, s_2]$
$\tilde{s} = (1-\theta)\tilde{s}_1 + \theta\tilde{s}_2 \in [\tilde{s}_1, \tilde{s}_2]$
$\forall \theta \in [0,1]$.

Gårding's inequality:

Theorem: $a \in S^m$, $\mathrm{Re}\, a \geq C |\xi|^m$ for $|\xi| \geq R$, $C > 0$.

$\Rightarrow \forall s \leq \frac{m}{2} \ \exists C_0, C_1 \ \forall u \in W^{\frac{m}{2},2} = H^{\frac{m}{2}}$:

$$\mathrm{Re}\, \langle a(x,D)u, u\rangle \geq C_0 \|u\|^2_{H^{m/2}} - C_1 \|u\|^2_{H^s} \quad \text{(coercivity)}$$

Proof: Replacing $a(x,D)$ by $\langle D\rangle^{-\frac{m}{2}} a(x,D) \langle D\rangle^{-\frac{m}{2}}$, we may assume $m = 0$. As in the proof of the L^2-boundedness, consider $A(x,\xi) = \left(\mathrm{Re}\, a(x,\xi) - \frac{1}{2}C\right)^{1/2} \in S^0$

$$\Rightarrow \quad A(x,D)^* A(x,D) = \mathrm{Re}\, a(x,D) - \tfrac{1}{2}C + r(x,D), \quad r \in S^{-1}$$

$$\Rightarrow \quad \mathrm{Re}\, \langle a(x,D)u,u\rangle = \|A(x,D)u\|^2_{L^2} + \frac{C}{2}\|u\|^2_{L^2} + \langle r(x,D)u,u\rangle$$

$$\geq \frac{1}{2}C\|u\|^2_{L^2} - C_1 \|u\|^2_{H^{-1/2}}$$

If $s \leq -\frac{1}{2}$, use $\|u\|^2_{H^s} \leq \varepsilon \|u\|^2_{L^2} + C(\varepsilon)\|u\|^2_{H^{-1/2}}$. $\square$

Remark: The sharp Gårding inequality improves this to
$\mathrm{Re}\, \langle a(x,D)u,u\rangle \geq -C\|u\|^2_{L^2}$ when $\mathrm{Re}\, a(x,\xi) \geq 0$.

Parametrix construction and microlocal elliptic regularity

Let $a \in S^m$ such that $|a(x,\xi)| \geq C\langle\xi\rangle^m \ \forall |\xi|$

$\Rightarrow$ (Exercises 6)

$$\exists\, b \in S^{-m}: \quad b(x,D)a(x,D) - 1 \in S^{-\infty}$$
$$a(x,D)\, b(x,D) - 1 \in S^{-\infty}$$

Therefore, if $a(x,D)u = f$, $u \in S'$, $f \in H^s$

$$\Rightarrow \quad b(x,D)a(x,D)u = b(x,D)f$$

$$u + \underbrace{r(x,D)u}_{\in C^\infty}$$

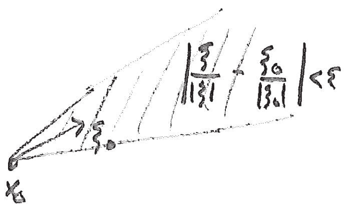
so $u \in H^{s+m}(\mathbb{R}^d) + C^\infty(\mathbb{R}^d) \subseteq H^{s+m}_{loc}(\mathbb{R}^d)$.

More generally, $a \in S^m$ is said to be elliptic in (x_0, ξ_0),

if $\exists \varepsilon>0, \exists R>0$: $|a(x,\xi)| \gtrsim C\langle\xi\rangle^m \quad \forall (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$ w/

$|x-x_0|<\varepsilon$, $\left|\frac{\xi}{|\xi|}-\frac{\xi_0}{|\xi_0|}\right|<\varepsilon$

and $|\xi| \geq R$.



Let $\psi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ s.t. $\psi(x,\xi)=1$ if $|x-x_0|<\frac{\varepsilon}{2}$,

$(*)$ $\left|\frac{\xi}{|\xi|}-\frac{\xi_0}{|\xi_0|}\right|<\frac{\varepsilon}{2}$

and $|\xi| \gtrsim 2R$

and $\psi(x,\xi)=0$ if $|x-x_0| \geq \varepsilon$ or $\left|\frac{\xi}{|\xi|}-\frac{\xi_0}{|\xi_0|}\right| \geq \varepsilon$

or $|\xi| \leq R$.

Then $b(x,\xi)=\psi(x,\xi)\, a(x,\xi)^{-1} \in S^{-m}$ and as in the exercises

$$b(x,\xi) \# a(x,\xi) = 1 + r(x,\xi) \qquad \text{for } (x,\xi) \text{ satisfying } (*).$$
$$a(x,\xi) \# b(x,\xi) = 1 + \tilde{r}(x,\xi)$$

Here, $r, \tilde{r}$ satisfy the symbol estimates of order -1 for (x,ξ) satisfying $(*)$.

Neumann series:

$$b_1(x,\xi) = \Big(1+\sum_{j=1}^{\infty} r^{\# j}\Big) \# b(x,\xi) \qquad \text{satisfies}$$

$b_1 \# a = 1 + s$, $a \# b_1 = 1 + \tilde{s}$, where $s, \tilde{s}$ satisfy the symbol estimates of order m $\forall m \in \mathbb{R}$ and (x,ξ) satisfying $(*)$.

In a similar way, all the results in this chapter can be transferred to open subsets of $\mathbb{R}^d$ and manifolds.

$\Longrightarrow$ If $a(x,D)u = f$, $u, f \in S'$ and f is smooth in the direction ξ_0 at x_0 (i.e. can be differentiated to all orders in this direction)

$\Longrightarrow$ u is also smooth in this direction.

Littlewood-Paley characterizations of function spaces:

$0<p\le\infty$, $0<q\le\infty$, $-\infty<s<\infty$:

$$B^s_{pq}(\mathbb{R}^n) := \{ f\in S'(\mathbb{R}^n) : \| (\| 2^{sj} \varphi_j(D) f \|_{L^p})_j \|_{\ell^q} < \infty \}$$

$0<p<\infty$:

$$F^s_{pq}(\mathbb{R}^n) := \{ f\in S'(\mathbb{R}^n) : \| \| (2^{sj} \varphi_j(D) f)_j \|_{\ell^q} \|_{L^p} < \infty \}$$

($p=\infty$ more subtle, $p=q=\infty$: all natural definitions give $F^s_{\infty,\infty} = B^s_{\infty,\infty}$)
Pseudodifferential operators have natural mapping properties in these spaces.

$\Rightarrow$ • $B^s_{\infty,\infty} = \mathcal{C}^s$ $(s>0)$ <u>Hölder-Zygmund spaces</u>

For $s\notin\mathbb{N}$, write $s = k+\tilde{s}$, $k\in\mathbb{N}$, $\tilde{s}\in(0,1)$.

Then $f\in\mathcal{C}^s \Leftrightarrow f\in C^k(\mathbb{R}^n)$ and all derivatives of f of order k are Hölder continuous of exponent $\tilde{s}$.

• <u>Sobolev spaces</u> as we know them: $W^{s,p} = F^s_{p,2}$ $(1<p<\infty, s\in\mathbb{R})$

In the literature on function spaces, these are often called <u>Bessel-potential spaces</u> and denoted by H^s_p.

• <u>Slobodeckij W^s_p and Besov spaces Λ^s_{pq}</u>: An extension of the integer-order Sobolev spaces analogous to how Hölder spaces extend C^k-functions: $W^s_p = \Lambda^s_{pp}$ $(0<s\notin\mathbb{N}, p<\infty)$

W^k_p = usual Sobolev spaces for $k\in\mathbb{N}_0$.

$\Lambda^s_{pq} = B^s_{pq}$ for $s>0$, $1\le p,q<\infty$.

• <u>local Hardy spaces h_p</u>: refinement of L^p such that Calderón-Zygmund theory extends to $0<p<\infty$.

$h_p = L^p$ for $1<p<\infty$, $h_p = F^0_{p,2}$ for $0<p<\infty$.

• <u>bmo</u> (bounded mean oscillation): analogous refinement of L^∞.
$bmo = F^0_{\infty,2}$.

First we introduce $C(R_n)$ as the set of all complex-valued bounded and uniformly continuous functions on R_n. If $m=1, 2, 3, \ldots$, we define

$$C^m(R_n)=\{f \mid D^\alpha f\in C(R_n) \quad \text{for all} \quad |\alpha|\leqq m\}\,. \tag{1}$$

Let $C^0(R_n)=C(R_n)$. Here D^α are classical derivatives. Of course, $C^m(R_n)$ endowed with the norm

$$\|f \mid C^m(R_n)\|=\sum_{|\alpha|\leqq m}\|D^\alpha f \mid L_\infty(R_n)\|$$

is a Banach space. Concerning the space $C(R_n)$ (and consequently also for the spaces $C^m(R_n)$) one has at least two other candidates: (i) the space of all bounded and continuous (but not necessarily uniformly continuous) functions on R_n and (ii) the completion of $S(R_n)$ in $C(R_n)$. The first space is larger and the second one is smaller than $C(R_n)$. There are some reasons (which will not be explained in detail at this moment) that $C(R_n)$ is the best choice. After some "basic" spaces in the sense of 2.2.1. have been defined, we come to the constructive spaces and the Hölder spaces. First we describe the spaces and the (in Remark 1, below) we give the necessary explanations.

(i) *The Hölder spaces* $C^s(R_n)$. If s is a real number, then

$$s=[s]+\{s\} \text{ with } [s] \text{ integer and } 0\leqq\{s\}<1\,. \tag{2}$$

If $s>0$ is not an integer, then

$$C^s(R_n)=\Big\{f \mid f\in C^{[s]}(R_n)\,, \tag{3}$$
$$\|f \mid C^s(R_n)\|=\|f \mid C^{[s]}(R_n)\|+\sum_{|\alpha|=[s]}\sup_{\substack{x\neq y\\ x\in R_n, y\in R_n}}\frac{|D^\alpha f(x)-D^\alpha f(y)|}{|x-y|^{\{s\}}}<\infty\Big\}\,.$$

(ii) *The Zygmund spaces* $\mathscr{C}^s(R_n)$. If s is a real number, then

$$s=[s]^-+\{s\}^+ \text{ with } [s]^- \text{ integer and } 0<\{s\}^+\leqq 1. \tag{4}$$

Furthermore, if $f(x)$ is an arbitrary function on R_n and $h\in R_n$, then

$$(\Delta_h^1 f)(x)=f(x+h)-f(x),\quad (\Delta_h^l f)(x)=\Delta_h^1(\Delta_h^{l-1}f)(x) \tag{5}$$

where $l=2, 3, \ldots$ and $x\in R_n$. If $s>0$ then

$$\mathscr{C}^s(R_n)=\{f \mid f\in C^{[s]^-}(R_n)\,,\ \|f \mid \mathscr{C}^s(R_n)\|=\|f \mid C^{[s]^-}(R_n)\| \tag{6}$$
$$+\sum_{|\alpha|=[s]^-}\sup_{0\neq h\in R_n}|h|^{-\{s\}^+}\|\Delta_h^2 D^\alpha f \mid C(R_n)\|<\infty\}\,.$$

(iii) *The Sobolev spaces* $W_p^m(R_n)$. If $1<p<\infty$ and $m=1, 2, 3, \ldots$, then

$$W_p^m(R_n)=\{f \mid f\in L_p(R_n), \|f \mid W_p^m(R_n)\|=\sum_{|\alpha|\leqq m}\|D^\alpha f \mid L_p(R_n)\|<\infty\}\,. \tag{7}$$

Furthermore, $W_p^0(R_n)=L_p(R_n)$.

(iv) *The Slobodeckij spaces* $W_p^s(R_n)$. If $1\leqq p<\infty$ and $0<s\neq$ integer, then

$$W_p^s(R_n)=\Big\{f \mid f\in W_p^{[s]}(R_n), \|f \mid W_p^s(R_n)\|=\|f \mid W_p^{[s]}(R_n)\| \tag{8}$$
$$+\sum_{|\alpha|=[s]}\Big(\int_{R_n\times R_n}\frac{|D^\alpha f(x)-D^\alpha f(y)|^p}{|x-y|^{n+\{s\}p}}\,dx dy\Big)^{\frac{1}{p}}<\infty\Big\}.$$

(v) *The Besov (or Lipschitz) spaces* $\Lambda_{p,q}^s(R_n)$. If $s>0$, $1\leqq p<\infty$ and $1\leqq q<\infty$, then

$$\Lambda_{p,q}^s(R_n)=\Big\{f \mid f\in W_p^{[s]^-}(R_n), \|f \mid \Lambda_{p,q}^s(R_n)\|=\|f \mid W_p^{[s]^-}(R_n)\| \tag{9}$$
$$+\sum_{|\alpha|=[s]^-}\Big(\int_{R_n}|h|^{-\{s\}^+q}\|\Delta_h^2 D^\alpha f \mid L_p(R_n)\|^q\frac{dh}{|h|^n}\Big)^{\frac{1}{q}}<\infty\Big\}.$$

If $s>0$ and $1\leqq p<\infty$, then

$$\Lambda_{p,\infty}^s(R_n)=\{f \mid f\in W_p^{[s]^-}(R_n), \|f \mid \Lambda_{p,\infty}^s(R_n)\|=\|f \mid W_p^{[s]^-}(R_n)\| \tag{10}$$
$$+\sum_{|\alpha|=[s]^-}\sup_{0\neq h\in R_n}|h|^{-\{s\}^+}\|\Delta_h^2 D^\alpha f \mid L_p(R_n)\|<\infty\}\,.$$

(vi) *The Bessel-potential (or Lebesgue or Liouville) spaces* $H_p^s(R_n)$. If s is a real number and $1<p<\infty$, then

$$H_p^s(R_n)=\{f \mid f\in S'(R_n), \|f \mid H_p^s(R_n)\|=\|F^{-1}(1+|x|^2)^{\frac{s}{2}}Ff \mid L_p(R_n)\|<\infty\}. \tag{11}$$

(vii) *The local Hardy spaces* $h_p(R_n)$. If $\varphi(x)$ is a test function on R_n (i.e. an infinitely differentiable function on R_n with compact support) with $\varphi(0)=1$, then we put $\varphi_t(x)=\varphi(tx)$ with $x\in R_n$ and $t>0$. If $0<p<\infty$, then

$$h_p(R_n)=\{f \mid f\in S'(R_n), \|f \mid h_p(R_n)\|^\varphi=\|\sup_{0<t<1}|F^{-1}\varphi_t Ff| \mid L_p(R_n)\|<\infty\}\,. \tag{12}$$

(viii) *The space* $bmo(R_n)$. If $f(x)$ is a locally Lebesgue-integrable function on R_n and if Q is a cube in R_n, then

$$f_Q=\frac{1}{|Q|}\int_Q f(x)\,dx \tag{13}$$

is the mean value of $f(x)$ with respect to Q.

$$bmo(R_n)=\Big\{f \mid f(x) \text{ locally Lebesgue-integrable on } R_n,\ \|f \mid bmo(R_n)\| \tag{14}$$
$$=\sup_{|Q|\leqq 1}\frac{1}{|Q|}\int_Q|f(x)-f_Q|\,dx+\sup_{|Q|>1}\frac{1}{|Q|}\int_Q|f(x)|dx<\infty\Big\}\,.$$

Remark 1 (Technical explanations). $D^\alpha f$ in (3) and (6) are classical derivations. On the other hand $D^\alpha f$ in (7)–(10) must be interpreted in the sense of distributions. In other words, $f\in W_p^m(R_n)$ means that the distribution f and all its derivatives $D^\alpha f$ with $|\alpha|\leqq m$ are regular and that they belong (interpreted as functions) to $L_p(R_n)$. Similarly in (8)–10). We recall that $F^{-1}(1+|x|^2)^{\frac{s}{2}}Ff$ in (11) means $F^{-1}[(1+|x|^2)^{\frac{s}{2}}Ff(x)](\cdot)$, cf. 1.5.2. and 1.5.1. If $f\in S'(R_n)$ then $F^{-1}(1+|x|^2)^{\frac{s}{2}}Ff\in S'(R_n)$ and one asks in (11) whether this is a regular distribution belonging to $L_p(R_n)$. Similarly, $F^{-1}\varphi_t Ff$ in (12) means $(F^{-1}[\varphi_t\cdot Ff])(\xi)$. If $\xi\in R_n$ is fixed, then one takes in (12) the supremum of $|(F^{-1}[\varphi_t\cdot Ff])(\xi)|$ with respect to t. Then one asks whether the result, considered as a function of $\xi\in R_n$, belongs to $L_p(R_n)$. Finally, $\sup\limits_{|Q|\leqq 1}$ in (14) means that the supremum is taken over all cubes in R_n with $|Q|\leqq 1$, where $|Q|$ is the volume of Q (similarly for $\sup\limits_{|Q|>1}$).

Remark 2. The definition of $h_p(R_n)$ is somewhat complicated. If $f\in S'(R_n)$ then $F^{-1}\varphi_t Ff$ satisfies the hypotheses of the Paley-Wiener-Schwartz theorem, cf. Theorem 1.2.1/2. Hence, $(F^{-1}\varphi_t Ff)(\xi)$ is an analytic function with respect to $\xi\in R_n$ for every fixed $t>0$. Hence $\sup\limits_{0<t<1}|(F^{-1}\varphi_t Ff)(\xi)|$ makes sense for any $\xi\in R_n$ and any $f\in S'(R_n)$. Furthermore, because

Traces of integral operators

Lidskii's theorem: (see Julie's presentation)

Let A be a trace class operator on a Hilbert space H and $\{e_n\}$ an orthonormal basis of H. Then $\operatorname{Tr} A = \sum_n \langle A e_n, e_n \rangle = \sum_n \lambda_n(A)$.

Let $T_k : L^2(K) \to L^2(K)$, $T_k f(y) = \int_K k(x,y) f(x)\, dx$.

Assume K compact and that $k(x,y) = \sum_{j,m=1}^{\infty} k_{jm} \overline{e_j(x)}\, e_m(y)$ uniformly for some ONB $\{e_n\}_{n\in\mathbb{N}}$ of $L^2(K)$. For example $K=[0,1]$, $\{e_n\}$ = orthogonal polynomials, k continuous (or $\{e_n\}$ = trigonometric polynomials and k Hölder-continuous).

$$\Rightarrow \operatorname{tr} T_k = \sum_n \langle T e_n, e_n \rangle = \sum_n \sum_{j,m} k_{jm} \int dx \int dy\, \overline{e_j(x)}\, e_m(y)\, e_n(x) \overline{e_n(y)}$$

$$= \sum_n k_{nn} = \int dx \sum_n k_{nn} |e_n(x)|^2 = \int_K dx\, k(x,x)$$

Pseudo differential operators: Recall that $A \in L(H)$ is called a Hilbert-Schmidt operator if $\sum_n \|Ae_n\|_H^2 < \infty$ for some ONB $\{e_n\}$.

For an integral operator $\sum_n \|Ae_n\|_H^2 = \int dx \int dy\, |k(x,y)|^2$.

Let $a \in S^m(\mathbb{R}^d) \Rightarrow k(x,y) = (2\pi)^{d} \int e^{i(x-y)\xi} a(x,\xi)\, d\xi$

$$\Rightarrow \int dx \int dy\, |k(x,y)|^2 = (2\pi)^{2d} \int dx \int dy \left| \int d\xi\, e^{iy\xi} a(x,\xi) \right|^2$$

$$\underset{\text{(Plancherel)}}{=} (2\pi)^{2d} \int dx \int d\xi\, |a(x,\xi)|^2$$

$\Rightarrow$ $a(x,D)$ Hilbert-Schmidt if $m < -\frac{d}{2}$ and the symbol is L^2 in the x-variable.

Let $a \in S^m(\mathbb{R}^d)$, $m<-d$, compactly supported. Using the composition formula, one can write $a(x,D)$ as sum of products of Hilbert-Schmidt operators. As products of Hilbert-Schmidt operators are trace class $\Rightarrow$ $a(x,D)$ is trace class and

$$\operatorname{tr} a(x,D) = \int dx\, k(x,x) = \int dx \int d\xi\, a(x,\xi).$$