Chapter 4 — Calderon-Zygmund theory

We are going to study "singular integral operators", which are operators whose kernel $K(x,y)$ barely fails to be an integrable function near $x=y$.

These arise naturally in various branches of analysis and PDE, e.g. as solution operators to differential equations.

Prototypical examples are pseudo differential operators of order 0 (see Exercise 5.2) or the Hilbert transform, which relates the real and imaginary parts of a holomorphic function and is of relevance to signal processing (mobile communication) and the physics of absorption.

<u>Example:</u> Hilbert transform

DifFun 1 / Chapter 5.6 in GG's book: $PV\frac{1}{x} \in S'(\mathbb{R})$,

$$\langle PV\tfrac{1}{x}, f\rangle := \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{1}{x} f(x)\,dx$$

$$H f(y) := \frac{1}{\pi}\left(PV\tfrac{1}{x} * f\right)(y) = \lim_{\varepsilon\to 0^+} \frac{1}{\pi}\int_{\mathbb{R}\setminus(x-\varepsilon,x+\varepsilon)} \frac{1}{y-x} f(x)\,dx$$

G.G. (5.80): $\widehat{PV\frac{1}{x}} = -\pi i \,\mathrm{sign}(\xi)$

$$\Rightarrow \quad Hf = \frac{1}{\pi}\mathcal{F}^{-1}\left(\widehat{PV\tfrac{1}{x}}\ \hat{f}\right) = -i\left(\mathcal{F}^{-1}\,\mathrm{sign}(\xi)\,\mathcal{F}\right) f$$

As $\mathcal{F}: L^2 \xrightarrow{\sim} L^2$ and $\mathrm{sign}(\xi) \cdot S(\mathbb{R}) \subseteq L^2(\mathbb{R})$, but not
$S(\mathbb{R}) \xrightarrow{\sim} S(\mathbb{R})$ $\subseteq S(\mathbb{R})$

$\Rightarrow$ • $H: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ bounded

• H does not map $S(\mathbb{R})$ to $S(\mathbb{R})$

From the formula $H = -i\, \mathcal{F}^{-1}\, \mathrm{sign}(\xi)\, \mathcal{F}$ $\Rightarrow$

• H commutes w/ dilations and translations

• $H^* = -H$

• $H^2 = -\mathbb{1}$, therefore H is unitary on $L^2(\mathbb{R})$

$\underline{Q}$: $H: L^p(\mathbb{R}) \to L^q(\mathbb{R})$?

$\underline{p \neq q}$: dilation invariance $H f_\lambda(x) = (Hf)_\lambda(x)$

where $G_\lambda(x) := G(\lambda x)$, $\lambda > 0$.

$\Rightarrow$ $\|Hf_\lambda\|_q = \|(Hf)_\lambda\|_q = \lambda^{-1/q}\, \|Hf\|_q$

$\|f_\lambda\|_p = \lambda^{-1/p}\, \|f\|_p$

$\to \|Hf_\lambda\|_q \leq C\, \|f_\lambda\|_p$ $\forall \lambda$ only possible for $p = q$.

$$H\mathbb{1}_{(-1,1)}(x) = \lim_{\varepsilon \to 0^+} \int_{-1}^{x-\varepsilon} \frac{dy}{x-y} + \int_{x+\varepsilon}^{1} \frac{dy}{x-y}$$

$$= \ln|x+1| - \ln|x-1| \notin L^1(\mathbb{R})$$
$$\notin L^\infty(\mathbb{R})$$

$\Rightarrow$ $H: L^1 \not\to L^1$
$L^\infty \not\to L^\infty$

$\underline{\text{Thm}}$: $H: L^p(\mathbb{R}) \to L^p(\mathbb{R})$ continuous $\forall p \in (1,\infty)$.

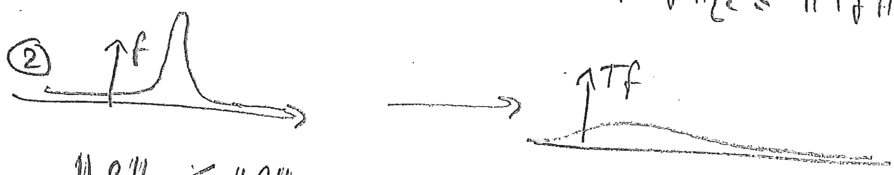
Calderon–Zygmund theory

Extend L^2-boundedness to L^p-boundedness.

Problem: ①

$\|f\|_{L^p} \leq \|f\|_{L^2}$ $\qquad\qquad$ $\|Tf\|_{L^2} \leq \|Tf\|_{L^p}$ $\qquad$ for $p>2$

②



$\|f\|_{L^p} \leq \|f\|_{L^2}$ $\qquad\qquad$ $\|Tf\|_{L^2} \leq \|Tf\|_{L^p}$ $\qquad$ for $p<2$

CZ-decomposition: "bad terms" are localized, of mean 0.

Singular integral operators map them to <u>nearly</u> localized functions (Lemma 2.7)

Prevents scenario ②. The adjoint of such an operator will be of the same kind
$\Rightarrow$ scenario ① excluded by duality.

Let $\Delta := \{(x,x) : x \in \mathbb{R}^d\}$.

<u>Def 2.1:</u> $K : \mathbb{R}^d \times \mathbb{R}^d \setminus \Delta$ <u>singular kernel</u> provided that $\exists\, 0 < \delta \leq 1$

1) $|K(x,y)| \leq C\,|x-y|^{-d}$

2) $|K(x,y') - K(x,y)| \leq C \dfrac{|y-y'|^\delta}{|x-y|^{d+\delta}} \qquad \forall\, |y-y'| \leq \frac{1}{2}|x-y|$

$|K(x',y) - K(x,y)| \leq C \dfrac{|x-x'|^\delta}{|x-y|^{d+\delta}} \qquad \forall\, |x-x'| \leq \frac{1}{2}|x-y|$

<u>Examples:</u>
- <u>1 dimensional:</u> $\dfrac{1}{x-y}$, $\dfrac{1}{|x-y|}$
- <u>>1 dimensional:</u> $\dfrac{\Omega\left(\frac{x-y}{|x-y|}\right)}{|x-y|^d}$ $\quad$ if $\Omega : S^{d-1} \to \mathbb{C}$ Hölder-continuous

- For 2.) $|\partial_x K(x,y)|$, $|\partial_y K(x,y)| \leq C\,|x-y|^{-d-1}$ is sufficient. (Fundamental theorem of calculus).

<u>Def 2.4:</u> A linear operator $T: L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is called <u>Calderón-Zygmund operator</u> (CZO), provided that T is bounded on $L^2(\mathbb{R}^d)$ and $\exists$ singular kernel K s.t. $Tf(y) = \int_{\mathbb{R}^d} K(x,y)\, f(x)\, dx \quad \forall f \in L^2(\mathbb{R}^d)$ with compact support and $y \notin \operatorname{supp} f$.

<u>Remarks:</u>

- K is determined uniquely, $K(x,y) = \frac{1}{x-y}$ for Hilbert trafo
- The operator $f \mapsto mf$, $m \in L^\infty(\mathbb{R}^d)$ has $K = 0$ (it's distributional kernel is $m(x)\,\delta_{x-y}$)
- Not all singular kernels are kernels of a CZO.
- $T \in CZO \Rightarrow T^* \in CZO$.
- We are going to show that all $T \in CZO$ are bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$.
  Marcinkiewicz $\Rightarrow$ $T: L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ bounded $\forall p \in (1,2]$.
  $T^* \in CZO \Rightarrow T: L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ bounded $\forall p \in (1,\infty)$.

Lemma 2.7: $T \in CZO$, $f \in L^1(B_r(x_0)) \Longrightarrow$

- $\forall y \in B_{2r}(x_0)$: $|Tf(y)| \leq C_d \frac{r^\sigma}{|y-x_0|^{d+\sigma}} \int_B |f|$.
- (integrating the above estimate) $\|Tf\|_{L^1(\mathbb{R}^d \setminus B_{2r}(x_0))} \leq C_d \|f\|_{L^1(B)}$

Proof:

$$\begin{aligned}|Tf(y)| &= \left|\int_B K(x,y) f(x)\,dx\right| \\ &= \left|\int_B (K(x,y)-K(x_0,y))\, f(x)\,dx\right| \\ &\overset{\text{CZO 2.}}{\leq} C\int_B \frac{|x-x_0|^\sigma}{|x_0-y|^{d+\sigma}} |f(x)|\,dx \\ &\leq C \frac{r^\sigma}{|x_0-y|^{d+\sigma}} \int_B |f| \qquad \boxtimes\end{aligned}$$

Cor 2.9: $T \in CZO \Longrightarrow T: L^1 \to L^{1,\infty}$ bounded.

Proof: $f \in L^1 \cap L^2$, $\lambda > 0$. Show: $\lambda |\{x : |Tf(x)| \geq \lambda\}| \leq C_d \|f\|_{L^1}$.

As in the proof of HLMI, since we have to show this $\forall f$, we may take $\lambda = 1$. Scaling $x \mapsto \lambda x$, we may also assume $\|f\|_{L^1} = 1$. Therefore:

Show: $|\{x : |Tf(x)| \geq 1\}| \leq C_d \quad \forall f \in L^1 \cap L^2,\ \|f\|_{L^1} = 1$.

Prop. 4.3: ($\lambda = 1$) $\quad f = g + \sum_Q b_Q$ with: $\|g\|_{L^1} \leq 1$ $\|g\|_\infty \leq 2^d$,

$\mathrm{supp}(b_Q) \subseteq Q$, $\int_Q b_Q = 0$

$\|b_Q\|_{L^1} \leq 2^{d+1} |Q|$

$|\bigcup Q| \leq 1$

Also $\underbrace{|\{x : |Tf(x)| \geq 1\}|}_{\lambda_{Tf}(1)} \leq \underbrace{|\{x : |Tg(x)| \geq \tfrac{1}{2}\}|}_{\lambda_{Tg}(\frac{1}{2})} + \underbrace{|\{x : \sum_Q |Tb_Q| \geq \tfrac{1}{2}\}|}_{\lambda_{\sum_Q |Tb_Q|}(\frac{1}{2})}$.

Chebychev (Ex. 2.4 b): $\lambda_{Tg}(t) \leq t^{-p} \|Tg\|_{L^p}^p$ w/ $t=\frac{1}{2}$, $p=2$

$= 4 \|Tg\|_{L^2}^2$

T L^2-bounded $\leq C \|g\|_{L^2}^2$

log-convexity of L^p-norms $\leq C \|g\|_{L^1} \|g\|_{L^\infty}$

(Ch1, Lemma 2) $\leq C_d$

This bounds the first term in $(*)$.

<u>Concerning $\lambda_{|\sum_Q Tb_Q|}(\frac{1}{2})$:</u> Lemma 2.7: $\|Tb_Q\|_{L^1(\mathbb{R}^d \setminus CQ)} \leq \|b_Q\|_{L^1} \leq 2^{d+1}|Q|$

$\Longrightarrow$ $\|\sum_Q Tb_Q\|_{L^1(\mathbb{R}^d \setminus \cup CQ)} \leq \sum_Q \|Tb_Q\|_{L^1(\mathbb{R}^d \setminus CQ)} \leq 2^{d+1} |\cup Q| \leq 2^{d+1}$

Chebychev $\geq$

$\frac{1}{2}|\{x \in \mathbb{R}^d \setminus \cup CQ : |\sum_Q Tb_Q| \geq \frac{1}{2}\}|$

What about $\mathcal{C} := |\{x \in \cup CQ : |\sum_Q Tb_Q| \geq \frac{1}{2}\}|$?

$\mathcal{C} \leq |\cup CQ| \leq C^d |\cup Q| \leq C^d$

$\Longrightarrow$ $\lambda_{|\sum_Q Tb_Q|}(\frac{1}{2}) \leq 2^{d+2} + C^d$ $\Rightarrow$ second term in $(*)$ bounded $\boxtimes$

<u>Cor. 2.10:</u> $T \in CZO \Longrightarrow T: L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ continuous $\forall p \in (1,\infty)$.

Convergence of 1-dimensional ball multipliers / Fourier inversion

Thm 2.12: $1<p<\infty$, $N>0$, $S_N f(x) := \int_{-N}^{N} \hat{f}(\xi)\, e^{ix\xi}\, d\xi$ $\quad (f\in S(\mathbb{R}))$

$\Rightarrow$ S_N extends uniquely to a bounded operator on $L^p(\mathbb{R})$ and
$\| S_N f - f\|_{L^p} \xrightarrow{N\to\infty} 0 \quad \forall f\in L^p.$ (a.e. convergence: Carleson-Hunt theorem)

Proof:

$$S_N f = \mathcal{F}^{-1}\, \mathbb{1}_{[-N,N]}\, \mathcal{F} f$$
$$= \tfrac{1}{2}\mathcal{F}^{-1}\left(\operatorname{sign}(\xi-N) + \operatorname{sign}(\xi+N)\right)\mathcal{F}f$$

$T_\alpha g(x) = g(x-\alpha)$

$$= \tfrac{1}{2}\mathcal{F}^{-1}\, T_{+N}\operatorname{sign}(\xi)\, T_{-N}\,\mathcal{F}f - \tfrac{1}{2}\mathcal{F}^{-1}\, T_{-N}\operatorname{sign}(\xi)\, T_{+N}\,\mathcal{F}f$$
$$= \tfrac{1}{2} e^{ixN}\, \overbrace{\mathcal{F}^{-1}_{\xi\to x}\operatorname{sign}(\xi)\,\mathcal{F}_{x\to\xi}}^{=iH}\, e^{-ixN} f(x)$$
$$- \tfrac{1}{2} e^{-ixN}\, \underbrace{\mathcal{F}^{-1}_{\xi\to x}\operatorname{sign}(\xi)\,\mathcal{F}_{x\to\xi}}_{=iH}\, e^{ixN} f(x)$$

$=$ Sum of composition of $\underbrace{\text{bounded}}_{\text{uniformly in } N}$ operators on L^p

$\Rightarrow$ $\|S_N\|_{p\to p} \le C \quad \forall N$, but note that $S_N \notin$ CZO.

Know $\|S_N f - f\|_{L^p} \xrightarrow{N\to\infty} 0$ for $f\in S(\mathbb{R})$.

For $f\in L^p(\mathbb{R})$, let $f_n \in S(\mathbb{R})$, $f_n \xrightarrow{n\to\infty} f$ in L^p

$$\to \|S_N f - f\|_{L^p} \le \|S_N(f-f_n) - (f-f_n)\|_{L^p} + \|S_N f_n - f_n\|_{L^p}$$
$$\le (C+1)\underbrace{\|f-f_n\|_{L^p}}_{\xrightarrow{n\to\infty} 0} + \underbrace{\|S_N f_n - f_n\|}_{\xrightarrow{N\to\infty} 0}$$

$\to$ $S_N f \to f$ in $L^p(\mathbb{R})$ $\quad\square$

In higher dimensions $\mathcal{F}^{-1}\mathbb{1}_{B_1(0)}\mathcal{F}$ is unbounded on $L^p(\mathbb{R}^d)$ for $p\neq 2$, $d\neq 1$ (Fefferman).

Fourier multipliers

Given $m \in S'(\mathbb{R}^d)$, we have two natural multiplication operators: $S \to S'$

(1) spatial multipliers: $S \ni f \mapsto f m \in S'$.

(2) Fourier multipliers: $S \ni f \mapsto \mathcal{F}^{-1}(m \, \mathcal{F} f) \in S'$, denoted by $m(D)$.

Spatial multipliers are easily analyzed using Hölder's inequality. A spatial multiplier is bounded on $L^p(\mathbb{R}^d) \iff m \in L^\infty(\mathbb{R}^d)$, and its operator norm is $\|m\|_\infty$.
By Plancherel's theorem, the same is true for Fourier multipliers on L^2.

Fourier multipliers $\longleftrightarrow$ convolution operators : $m(D) f = (\mathcal{F}^{-1} m) * f$

Thm: $1 < p < \infty$, $A : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ continuous and
(Hörmander) "translation invariant": $A T_{x_0} f = T_{x_0} A f \quad \forall f \ \forall x_0 \in \mathbb{R}^d$, where
$T_{x_0} g(x) = g(x - x_0)$. $\implies \exists m \in S'(\mathbb{R}^d) : Af = m * f$.

Def: $M^p = M^p(\mathbb{R}^d) = \{ m \in S'(\mathbb{R}^d) : m(D) : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \text{ continuous} \}$

Banach algebra, norm $\|m\|_{M^p}$, $p = 2$: $\|m\|_{M^2} = \|m\|_{L^\infty}$.

adjoint: $m(D)^* = \bar{m}(D) : L^{p'} \to L^{p'} \implies \|m\|_{M^p} = \|\bar{m}\|_{M^p} = \|m\|_{M^{p'}}$

$$\implies m(D) : \begin{matrix} L^p \to L^p \\ L^{p'} \to L^{p'} \end{matrix} \xrightarrow{\text{Riesz-Thorin}} m(D) : L^2 \to L^2$$

$$\|m\|_{L^\infty} = \|m\|_{M^2} \le \|m\|_{M^p}^{1-\theta} \|m\|_{M^{p'}}^{\theta} = \|m\|_{M^p}$$

$\implies$ (again) $m \in L^\infty(\mathbb{R}^d)$.

Same argument: $\|\cdot\|_{M^2} \le \|\cdot\|_{M^p} \le \|\cdot\|_{M^\infty} \quad \forall p$

Furthermore, $\|m\|_{M^\infty} = \|m\|_{M^1} = \|\mathcal{F}^{-1} m\|_{L^1} \equiv \|m\|_{\mathcal{F}L^1}$,

So that $M^1 = M^\infty = \mathcal{F}L^1 \subsetneq C_0(\mathbb{R}^d)$ is the range of the Fourier transform on L^1 (see exercise 1.3).

As mentioned for the Fouriermultipliers with symbol $\mathbb{1}_{B_N(0)}$, it's difficult to decide whether a non-smooth symbol m induces a bounded operator on $L^p(\mathbb{R}^d)$. We are later going to see (Thm 4.4) a sufficient condition if m is smooth outside the point $\{0\}$.

First, we consider the smooth case, and even allow more general symbols:

Def (Exercise 5.2) $\quad S^m := \{ a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) : \forall \alpha, \beta \in \mathbb{N}_0^d \; \exists C_{\alpha\beta} :$
$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} (1+|\xi|)^{m-|\beta|} \}, \quad m \in \mathbb{R}.$$
$$S^{-\infty} := \bigcap_{m \in \mathbb{R}} S^m$$

Exercise 5.2 shows that the generalized Fourier multiplication operator

$$op(a) f(x) := a(x,D) f(x) := \underbrace{\int_{\mathbb{R}^d} d\xi \; a(x,\xi) \hat{f}(\xi) e^{ix\xi}}_{= (2\pi)^d \; \mathcal{F}^{-1}_{\xi \to x}(a(x,\xi) \mathcal{F} f)}$$

for $a \in S^m$

- maps $S(\mathbb{R}^d) \to S(\mathbb{R}^d)$.
- is a differential operator if $a(x,\xi)$ is polynomial in ξ.
- The distribution $\mathcal{F}^{-1}_{\xi \to \cdot} a(x,\xi) =: k(x,\cdot)$ is an L^1_{loc}-function away from $\cdot = 0$ and satisfies $|k(x,z)| \leq C_N |z|^{-N} \quad \forall |z| \geq 1 \; \forall N \in \mathbb{N}$ uniformly in $x \in \mathbb{R}^d$. In particular
$$a(x,D) f(x) = \int_{\mathbb{R}^d} k(x, x-y) f(y) \, dy \quad \text{whenever } x \notin \operatorname{supp} f.$$

$a(x,D)$ is called a <u>pseudo differential operator</u> of order m with symbol a.

<u>Thm</u>: Let $a \in S^m(\mathbb{R}^d) \Rightarrow K(x,y) := k(x, x-y)$ is a singular kernel.

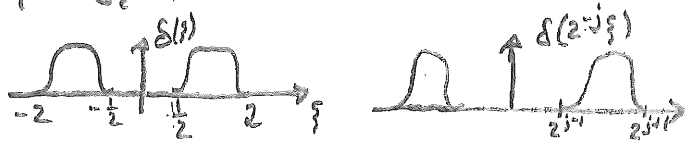
More precisely $k(x,z) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ and

$\forall n+m+|\alpha|+N > 0 \; \forall z \neq 0: \; |\partial_x^\beta \partial_z^\alpha k(x,z)| \leq C_{\alpha,\beta,N} |z|^{-d-m-|\alpha|-N}$.

For the proof, we are going decompose $\mathbb{R}^d$ into dyadic spherical shells:

<u>Littlewood-Paley decomposition:</u> $0 \le \phi \in C_c^\infty(\mathbb{R}^d)$ "bump function":

- $\phi = 1$ on $B_1(0)$, $\operatorname{supp}\phi \subset B_2(0)$
- $\delta(\xi) = \phi(\xi) - \phi(2\xi)$



"dyadic" partitions of unity on $\mathbb{R}^d$

- $1 = \sum_{j=-\infty}^{\infty} \delta(2^{-j}\xi) = \sum_j \{\phi(\xi/2^j) - \phi(\xi/2^{j-1})\} \quad \forall \xi \neq 0$
- $1 = \phi(\xi) + \sum_{j=1}^{\infty} \delta(2^{-j}\xi) \quad \forall \xi$

For every ξ, each of the infinite sums contains at most 2 nonzero terms.

Let $\Phi = \mathcal{F}^{-1}\phi \in S(\mathbb{R}^d)$, $\int \Phi = \mathcal{F}\Phi(0) = \phi(0) = 1$

$\Psi = \mathcal{F}^{-1}\delta \in S(\mathbb{R}^d)$, $\int \Psi = \delta(0) = 0$.

Rescalings: $\Phi_t(x) = t^{-d}\,\Phi(x/t)$, $\Psi_t(x) = t^{-d}\,\Psi(x/t)$

$\Rightarrow \quad \Psi_{2^{-j}} = \Phi_{2^{-j}} - \Phi_{2^{-j+1}}$

Define $\quad S_j(f) := f * \Phi_{2^{-j}} = \phi(2^{-j}D) f$

$\Delta_j(f) := S_j(f) - S_{j-1}(f) = f * \Psi_{2^{-j}} = \delta(2^{-j}D) f$

$\Rightarrow$ Partitions of $\mathbb{1}$:

- $\mathbb{1} = \sum_{j=-\infty}^{\infty} \delta(2^{-j}D) = \sum_{j=-\infty}^{\infty} \Delta_j$
- $\mathbb{1} = \phi(D) + \sum_{j=1}^{\infty} \delta(2^{-j}D) = S_0 + \sum_{j=1}^{\infty} \Delta_j$

Convergence: $f \in S'(\mathbb{R}^d) \rightsquigarrow S_j f \xrightarrow[j\to\infty]{S'} f$

$\rightarrow \quad S_0(f) + \sum_{j=1}^{\ell} \Delta_j(f) = S_\ell(f) \xrightarrow[\ell\to\infty]{S'} f$

$\Rightarrow$ second partition of $\mathbb{1}$ holds on S'.

$f \in S(\mathbb{R}^d) \rightsquigarrow S_j f \xrightarrow[j\to-\infty]{} 0 \quad$ (or just $f \in L^p(\mathbb{R}^d)$) but not for $f = 1$.

$\Rightarrow$ first partition of $\mathbb{1}$ holds on (e.g.) $S(\mathbb{R}^d)$.

$$|\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi)| \le C \sum_{\beta_1+\beta_2=\beta} |\partial_\xi^{\beta_1}\partial_x^\alpha a|\,|\partial_\xi^{\beta_2}\delta(2^{-j}\xi)|$$
$$\le \tilde{C} \sum_{\beta_1+\beta_2=\beta} (1+|\xi|)^{m-|\beta_1|}\, 2^{-j|\beta_2|} \cdot \sup_\eta |\partial_\eta^{\beta_2}\delta(\eta)|$$
$$\le \tilde{\tilde{C}}\, (1+|\xi|)^{m-|\beta_1|}\, 2^{-j|\beta_2|}$$
but $|\xi| \in (2^{j-1}, 2^{j+1})$
$$\Rightarrow\ \le C_1 (1+|\xi|)^{m-|\beta_1|-|\beta_2|} = C_1 (1+|\xi|)^{m-|\beta|} \quad \forall j$$

operator decomposition: Use 2nd partition of 1:

$$\mathrm{op}(a) = \mathrm{op}(a)\, S_0 + \sum_{j=1}^{\infty} \mathrm{op}(a)\, \Delta_j = \sum_{j=0}^{\infty} \mathrm{op}(a_j),$$

$a_0 = a(x,\xi)\,\phi(\xi) \in S^{-\infty}$

$a_j = a(x,\xi)\,\delta(2^{-j}\xi)$, $j>0$, $\in S^{-\infty}$, uniformly in S^m

kernel of a_j: $k_j(x,z) = \mathcal{F}^{-1}_{\xi\to z} a_j = (2\pi)^{-d}\int a_j(x,\xi)\, e^{i\xi z}\, d\xi \in S_z(\mathbb{R}^d)$

$$\Rightarrow \quad \mathrm{op}(a_j) f(x) = \int k_j(x, x-y)\, f(y)\, dy \qquad \forall x\in\mathbb{R}^d!$$

Lemma: $a\in S^m \Rightarrow |\partial_x^\beta \partial_z^\alpha k_j(x,z)| \le C_{N,\alpha,\beta}\, |z|^{-N}\, 2^{j(d+m-N+|\alpha|)}$

($C_{N,\alpha,\beta}$ indep. of j!)

$\forall \alpha,\beta,\ N\ge 0$

Proof: $|z^\gamma \partial_x^\beta \partial_z^\alpha k_j(x,z)| \simeq \left|\int \underbrace{\partial_\xi^\gamma\left[\xi^\alpha \partial_x^\beta a_j(x,\xi)\right]}_{\substack{\text{support} \subseteq B_{2^{j+1}}(0) \setminus B_{2^{j-1}}(0)\ (j>0) \\ |\text{support}| \le 2^{(j+1)d}}} e^{ix\xi}\, d\xi\right|$

$$\left|\partial_x^\alpha \partial_\xi^\beta \left(a(x,\xi)\,\delta(2^{-j}\xi)\right)\right| \le C \sum_{\beta_1+\beta_2=\beta} \underbrace{|\partial_x^\alpha \partial_\xi^{\beta_1} a(x,\xi)|}_{\lesssim C_{\alpha\beta}\, 2^{(j+1)(m-|\beta_1|)}}\ \underbrace{|\partial_\xi^{\beta_2}\delta(2^{-j}\xi)|}_{\lesssim 2^{-j|\beta_2|}}$$

$$\le C_{\alpha\beta}\, 2^{j(m-|\beta|)}$$

$$\le C\, 2^{j(m-|\gamma|+|\alpha|)}\, 2^{jd} \qquad \forall \alpha,\beta,\gamma.$$

Take $\max_{|\gamma|=N}$ to get the assertion. $\square$

Proof of Thm: $k(x,\cdot) = \sum_{j=0}^{\infty} k_j(x,\cdot)$ where the sum converges in S' for all $x\in\mathbb{R}^d$.

Suffices to show that $\sum_{j=0}^{\infty} |\partial_x^\beta \partial_z^\alpha k_j(x,z)| \le C_{\alpha\beta,N}\, |z|^{-d-m-|\alpha|-N}$:

because then $k(x,z) = \sum k_j(x,z) \in C^\infty(\mathbb{R}^d\times\mathbb{R}^d\setminus\{0\})$ and $\partial_x^\alpha \partial_\xi^\beta k = \sum \partial_x^\alpha \partial_\xi^\beta k_j$. (see the more detailed explaination in Thm. 4.4)

Case 1: $0<|z|\le 1$: $\sum_{j=0}^{\infty} = \underbrace{\sum_{2^j\le |z|^{-1}}}_{①} + \underbrace{\sum_{2^j>|z|^{-1}}}_{②}$

$$① \overset{\text{Lemma w/ } M=0}{\le} C\sum_{2^j\le|z|^{-1}} 2^{j(d+m+|\alpha|)} \le \begin{cases} C\,|z|^{-d-m-|\alpha|} & , d+m+|\alpha|>0 \\ C\,(\log(|z|^{-1})+1) & , d+m+|\alpha|\le 0 \end{cases}$$

$$\le C\,|z|^{-d-m-|\alpha|-N} \qquad \forall |z|\le 1,\ N\ge 0,\ d+m+|\alpha|+N\ge 0.$$

② Lemma w/ $M>d+m+|\alpha|$:

$$② \le C\,|z|^{-M}\sum_{2^j>|z|^{-1}} 2^{j(d+m+|\alpha|-M)} \le C|z|^{-d-m-|\alpha|}$$

$$\le C|z|^{-d-m-|\alpha|-N} \qquad \forall N\ge 0 \quad (\text{since } |z|\le 1)$$

Case 2: $|z|\ge 1$: Lemma or Ex. 5.2 shows that

$$|\text{kernel}| \le C\,|z|^{-M} \quad \text{for some } M > d+m+|\alpha|+N$$

$$\le C\,|z|^{-d-m-|\alpha|-N} \qquad \forall N \quad (\text{since } |z|\ge 1).$$

$\boxtimes$

Many natural operators have a singularity at $\xi=0$, e.g. the Hilbert transform.

Thm 4.4: (Hörmander – Mikhlin multiplier theorem)

$m\in L^\infty(\mathbb{R}^d)$ s.t. $|\partial_\xi^\alpha m(\xi)| \le C_\alpha |\xi|^{-\alpha}$ $\forall \xi\ne 0$ $\forall\, 0\le|\alpha|\le d+2$

$\Longrightarrow$ $m(D)\in CZO$ and, in particular, $m(D): L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)$ bounded $\forall p\in(1,\infty)$.

Remark: The hypothesis may be relaxed to fewer derivatives and weaker decay: $|\xi^\alpha \partial_\xi^\alpha m(\xi)| \le C$ $\forall\xi$ $\forall \alpha\in\{0,1\}^d$.

<u>Proof:</u> $m \in L^\infty \Rightarrow m(D)$ bounded on L^2. We have to show that $K(x,y) = k(x-y) = \mathcal{F}^{-1}_{\xi \to x-y} m(\xi)$ is a singular kernel.

We use the Littlewood-Paley decomposition

$$1 = \sum_{j \in \mathbb{Z}} \delta(2^{-j}\xi) \qquad \forall \xi \neq 0$$

$$m(\xi) = \underbrace{\sum_{j \in \mathbb{Z}} \underbrace{m(\xi)\,\delta(2^{-j}\xi)}_{=: m_j(\xi)}}_{\text{converges is } S'} \quad , \quad k_j(z) = \mathcal{F}^{-1}_{\xi \to z} m_j(\xi)$$

$k = \sum_{j \in \mathbb{Z}} k_j$ converges in S' $\Rightarrow$ suffices to estimate

$$\sum_{j \in \mathbb{Z}} |\partial_z^\alpha k_j(z)| \qquad \text{for } z \neq 0$$

Taking a bit more care than in the previous Lemma:

$$|\partial_\xi^\alpha m_j(\xi)| \leq C_\alpha 2^{-j|\alpha|} \quad \text{by assumption on } m + \text{Leibniz rule}$$
$$\forall\, 0 \leq |\alpha| \leq d+2$$

$$|k_j(z)| = \left|(2\pi)^{-d}\int_{\mathbb{R}^d} m_j(\xi)\, e^{iz\xi}\, d\xi\right| \overset{\|m\|_\infty \leq C_0}{\leq} C_d 2^{dj}$$

$$|k_j(z)| = \left|(2\pi)^{-d}\int_{\mathbb{R}^d} e^{iz\xi}\, z^{-\alpha}\, \partial_\xi^\alpha m_j(\xi)\right| \leq C_d\, 2^{dj}\, 2^{-j|\alpha|}\, z^{-\alpha}$$

$$\Rightarrow \quad |k_j(z)| \leq C_d \min\{2^{dj},\ 2^{dj}\, 2^{-(d+2)j}\, z^{-(d+2)}\}$$
$$= C_d |z|^{-d} \min\{(2^j|z|)^d,\ (2^j|z|)^{-2}\}$$

Similar for $\partial k_j(z)$: $\quad |\partial k_j(z)| \leq C_d \min(2^{dj},\ 2^{dj}\, 2^{-j(d+2)}\, z^{-(d+2)})$
$$= C_d |z|^{-d-1} \min\left((2^j|z|)^{d+1},\ (2^j|z|)^{-1}\right)$$

$\Rightarrow \forall |z| > \varepsilon$ $\quad \sum_j k_j$ converges uniformly by Weierstrass test
$\qquad\qquad\quad \sum_j \partial k_j$ — " —— " —— " —— " —— "

$\Rightarrow k \in C^1(\mathbb{R}^d \setminus \{0\})$ satisfying $|k(z)| \leq C|z|^{-d}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; |\partial k(z)| \leq C|z|^{-d-1}$

By the remark after the definition of singular kernels, this suffices to show that $K(x,y) = k(x-y)$ defines a singular kernel.

Also $\langle m(D)f, g\rangle = \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dx \; k(y-x) f(x) g(y)$

whenever $f, g \in S$, $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset$.

$\Rightarrow$ also for $f, g \in L^2$ w/ compact, disjoint supports.

$\Rightarrow$ $m(D) \in CZO$. $\boxed{\ddot\smile}$

<u>Examples:</u>

- Hilbert transform $m(\xi) = -i \operatorname{sign}(\xi)$
- fractional derivatives $|D|^{it}$: $m = |\xi|^{it}$, $t \in \mathbb{R}$.

  $\langle D\rangle^{it}$: $m = (1+|\xi|^2)^{\frac{it}{2}}$, $t \in \mathbb{R}$.
- Riesz-transform $\operatorname{sign}(D) = \frac{D}{|D|} =: R$

  $m(\xi) = \frac{i\xi}{|\xi|}$

Application: $\partial_{x_j} \partial_{x_k} f = - R_j R_k \Delta f \quad \forall f \in S(\mathbb{R}^d)$

(Fourier transform both sides!)

$$\Rightarrow \sum_{j,k} \| \partial_{x_j} \partial_{x_k} f \|_{L^p} \leq \sum_{j,k} \| R_j R_k \Delta f \|_{L^p} \leq C_{d,p} \| \Delta f \|_{L^p}$$

$$\Rightarrow \Big(f \in L^p,\ \Delta f \in L^p \Rightarrow \underbrace{f \in W^{2,p}}_{L^p\text{-Sobolev space}} \Big)$$

"elliptic regularity on L^p"

More general: $\sum_{|\alpha|=k} \| \partial^\alpha f \|_{L^p} \leq C_{p,d,k} \| |D|^k f \|_{L^p}$

$|D|^k$ = multiplier w/ symbol $|\xi|^k$.

Vector-valued singular integral operators:

The above definitions and results readily extend to kernels $k: \mathbb{R}^d \times \mathbb{R}^d \setminus \Delta \to \mathcal{B}(H)$ taking values in the bounded linear operators on a Hilbert space.

One main application of the vector-valued theory is the Littlewood-Paley inequality.

Prop. 5.3/Cor. 5.4: Let $\psi_j : \mathbb{R}^d \to \mathbb{C}$ be bump functions adapted to the annulus $\{\xi : |\xi| \in (2^{j-1}, 2^{j+1})\}$ s.t. $0 < c \leq \sum_j |\psi_j|^2 \leq C < \infty$.

$\Rightarrow$ $\forall\ 1 < p < \infty$: $\| (\sum_{j\in\mathbb{Z}} |\psi_j(D) f|^2)^{1/2} \|_{L^p(\mathbb{R}^d)}$ is equivalent to the norm $\|f\|_{L^p(\mathbb{R}^d)}$ and $\forall (f_j)_{j\in\mathbb{Z}} \subseteq L^p(\mathbb{R}^d)$

$$\| \sum_{j\in\mathbb{Z}} \psi_j(D) f_j \|_{L^p(\mathbb{R}^d)} \lesssim C_{p,d} \| (\sum_{j\in\mathbb{Z}} |f_j|^2)^{1/2} \|_{L^p(\mathbb{R}^d)} .$$

Idea of proof: The operator
$$T: L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d, \ell^2(\mathbb{Z})), \quad f \longmapsto (\psi_j(D) f)_{j\in\mathbb{Z}}$$
is an $\ell^2(\mathbb{Z})$-valued CZ operator. $\Longrightarrow$ „$\lesssim$".

To get $\| (\sum_j |\psi_j(D) f|^2)^{1/2} \|_{L^p} \geq c \|f\|_{L^p}$ in the first part of the assertion, let $\tilde{\psi}_j$ bump functions $\equiv 1$ on supp ψ_j,

$$\varphi_j := \tilde{\psi}_j \frac{\psi_j}{\sum_j |\psi_j|^2} \Rightarrow 1 = \sum_j \varphi_j \psi_j \quad \text{or} \quad f = \sum_j \varphi_j(D) \psi_j(D) f$$

for $f \in S(\mathbb{R}^d)$ $\xRightarrow{1^{st}\text{ part of proof}}$ $\|f\|_{L^p(\mathbb{R}^d)} \lesssim C_{p,d} \| (\sum_j |\psi_j(D) f|^2)^{1/2} \|_{L^p(\mathbb{R}^d)}$ $\square$

Remark: $\psi_j(D) f$ = component of f of frequency $\sim 2^j$

LP-inequality: $\psi_j(D)$ behave as if they are orthogonal, no cancellations.

$\| (\sum_{j\in\mathbb{Z}} |\psi_j(D) f|^2)^{1/2} \|_{L^p}$ "square function" of f.