The Hardy-Littlewood maximal inequality

$\mathbb{R}^d$ w/ Lebesgue measure λ, $|E| := \lambda(E)$

$B(x,r) = B_r(x)$ open ball of radius r around x

$cB(x,r) = B(x,cr)$.

$A_r f(x) := \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)\, dy$ average of $f \in L^1_{loc}(\mathbb{R}^d)$ on $B_r(x)$.

Standard measure theory: $A_r f(x)$ continuous fct of $r>0$ and $x \in \mathbb{R}^d$.

$A_r f = T_K f$ for $K(x,y) = \frac{1}{|B_r(y)|} \mathbb{1}_{B_r(y)}(x)$,

Schur's test (Ex. 2.2): $\|A_r\|_{p\to p} \le 1$. uniformly bounded in $r>0$.

Refinement:

Prop 1.1 (Hardy-Littlewood maximal inequality)

The operators $f \mapsto \sup_{r>0} |A_r f|$ are bounded on $L^p(\mathbb{R}^d)$, $1<p\le\infty$,
and $f \mapsto \sup_{r>0} A_r|f|$ and from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$.

Remarks:
- Considering $f \ge 0$, it is clear that one of these operators is bounded if and only if the other one is.
- For $f = \mathbb{1}_{B_1(0)}$, $A_r f(x) = r^{-d} \frac{|B_1(0)|}{|B_1(x)|}$ for $r > 1+\|x\|$
  $= r^{-d}$

  $\to \sup_{r>0} |A_r f| \ge (1+|x|)^{-d} \notin L^1(\mathbb{R}^d) \to$ unbounded on $L^1(\mathbb{R}^d)$.

- L^∞: $\sup_{x\in\mathbb{R}^d} \left|\sup_{r>0} |A_r f(x)|\right| \le \sup_{x\in\mathbb{R}^d} \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f|$
  $\le \sup_{x\in\mathbb{R}^d} \|f\|_\infty = \|f\|_\infty$

  $\Rightarrow$ bounded on L^∞.

- $$\sup_{r>0} |A_r f_\lambda(x)| = \sup_{r>0} \frac{1}{|B(x,r)|} \left| \int_{B(x,r)} f(\lambda y)\, dy \right|$$

$f = \mathbb{1}_{B_1(0)} \Rightarrow \dots \gtrsim (1+|x\lambda|)^{-d}$.

$\Rightarrow \quad \| \dots \|_p \sim \lambda^{+d/p}$

$\quad \| f_\lambda \|_q \sim \lambda^{d/q}$

$\Rightarrow \quad \| \dots \|_p \leq C \| f_\lambda \|_q \quad \forall \lambda \longrightarrow p = q.$

$\Rightarrow$ operator can only be continuous $L^p \to L^q$ if $p=q$.
similarly for weak-L^p-bounds.

- Marcinkiewicz: suffices to show continuity: $L^1 \to L^{1,\infty}$,
  i.e. $\forall f\, \forall \lambda$: $\lambda |\{ \sup_{r>0} |A_r f| \geq \lambda \}| \leq C_d \| f \|_{L^1}$
  i.e. $\forall f$: $|E| := |\{ \sup_{r>0} |A_r f| \geq 1 \}| \leq C_d \| f \|_{L^1}$
  It's enough to show this for $f \geq 0$, compactly supported,
  $r \leq \text{Diameter}(\operatorname{supp} f)$

$K \subset E$ compact $\Rightarrow \forall x \in K\ \exists r = r(x) > 0 : (*)\ |B_{r(x)}(x)| \leq \int_{B_{r(x)}(x)} f$

Want: $|K| \leq \underset{\text{indep. of } K}{C_d} \int_{\mathbb{R}^d} f$

Note: $K \subset \bigcup_x B_{r(x)}(x) \overset{K \text{ compact}}{\Longrightarrow} K \subset \bigcup_{i=1}^{N} B_{r(x_i)}(x_i)$

Extract large subcollection of non-overlapping balls:

Lemma: (Wiener's Vitaly-type covering lemma)

Let $B_1, \dots, B_N \subseteq \mathbb{R}^d$ balls $\Rightarrow \exists\, n_1, \dots, n_k : (B_{n_i})_{i=1}^{k}$ disjoint
and $|B_{n_1} \cup \dots \cup B_{n_k}| \geq 3^{-d} |B_1 \cup \dots \cup B_N|$

Remark: May replace balls by other objects B_j satisfying: $\exists \lambda > 0$:
$\emptyset \neq B_j \cap B_k,\ |B_j| \leq |B_k| \Rightarrow B_j \subseteq \lambda B_k$.

Proof: Wlog $|B_1| \geq |B_2| \geq \dots \geq |B_N|$.

Define n_i recursively, picking the largest possible ball which is disjoint from the previous ones in each step:
Let $n_1 = 1$. For $i = 2, 3, \dots$ let

$$n_i = \min\{k : B_k \cap B_{n_j} = \emptyset \ \forall j \leq i-1\}$$

until there are no balls left.

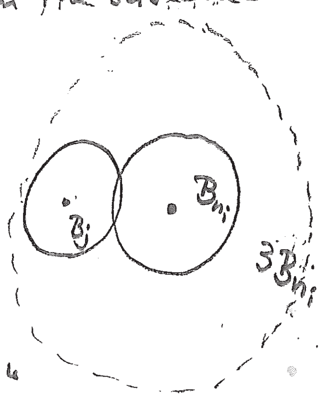
Given a ball B_j, there are 2 possibilities:

- $B_j = B_{n_i}$ (for some $i \leq k$) is element of the subcollection

or

- B_j intersects a ball B_{n_i} from the subcollection s.t. $|B_j| \leq |B_{n_i}|$

In both cases: $B_j \subseteq 3B_{n_i}$



$$\Rightarrow \quad B_1 \cup \dots \cup B_N \subset 3B_{n_1} \cup \dots \cup 3B_{n_k}$$

$$\Rightarrow \quad |B_1 \cup \dots \cup B_N| \leq 3^d |B_{n_1} \cup \dots \cup B_{n_k}| \qquad \square$$

Proof of Proposition:

$$K \subset \bigcup_{i=1}^{N} B_{r(x_i)}(x_i) \overset{\text{Lemma}}{\Longrightarrow} \exists\ B_{r(y_1)}(y_1), \dots, B_{r(y_k)}(y_k) \text{ disjoint}$$

$$\text{s.t. } \left|\bigcup_{i=1}^{k} B_{r(y_i)}(y_i)\right| \geq 3^{-d}|K|$$

$$\Rightarrow \quad |K| \leq 3^d \left|\bigcup_{i=1}^{k} B_{r(y_i)}(y_i)\right| \overset{(*)}{\leq} 3^d \sum_{i=1}^{k} \int_{B_{r(y_i)}(y_i)} f \leq 3^d \int_{\mathbb{R}^d} f$$

Take sup over all $K \subset\subset E$ compact $\Rightarrow$ $|E| \leq 3^d \int_{\mathbb{R}^d} f$ $\square$

We are going to look at alternative proofs later, but first consider an application:

Thm 2.1: (Lebesgue differentiation theorem)

Let $f \in L^1_{loc}(\mathbb{R}^d) \Rightarrow$ • $\lim_{r \to 0^+} A_r f(x) = f(x)$ for a.e. $x \in \mathbb{R}^d$

• $\lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)| \, dy = 0$ for a.e. $x \in \mathbb{R}^d$

Remarks: • $A_r f(x) - f(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} (f(y) - f(x)) \, dy$

$\Rightarrow |A_r f(x) - f(x)| \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)| \, dy$,

so we only have to show the second limit.

• Compare: f is continuous at x

$\Longleftrightarrow \lim_{r \to 0} \sup_{y \in B_r(x)} |f(y) - f(x)| = 0$

"$f \in L^1_{loc}$ is a.e. continuous on average"

• A point $x \in \mathbb{R}^d$ for which $\lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)| \, dy = 0$ is called a "Lebesgue point" of f.

Proof: • The assertion holds for $f \in C_c(\mathbb{R}^d)$, thus a dense subset of L^1.

• Every $x \in \mathbb{R}^d$ is contained in some $B_R(0)$. The assertion holds in x if $f = 0$ on $B_{2R}(0) \Rightarrow f = f \mathbb{1}_{B_{2R}(0)} + f \mathbb{1}_{\mathbb{R}^d \setminus B_{2R}(0)}$

As the assertion holds for the second term, we may restrict to $f \in L^1(\mathbb{R}^d)$.

- According to Hardy-Littlewood, it is enough to know the assertion for a dense subset of $L^1(\mathbb{R}^d)$.

In fact, let $f_n \to f$ in $L^1(\mathbb{R}^d)$ s.t. $\|f_n - f\|_{L^1} \leq 2^{-n}$

We are going to show that $f_n^{(x)} \to f(x)$ a.e.

Chebychev: $\forall \varepsilon > 0: |\{x \in \mathbb{R}^d : |(f_n - f)(x)| \geq \varepsilon\}| \leq \varepsilon^{-1} \|f_n - f\|_{L^1(\mathbb{R}^d)} = \varepsilon^{-1} 2^{-n}$

<u>Note</u>: $f_n^{(x)} \to f(x)$ a.e. $\iff$ $\forall \varepsilon > 0: 0 = |\{x : \forall n \geq 1\ \exists k \geq 1 : |f_{n+k}^{(x)} - f(x)| \geq \varepsilon\}|$

$$= \left| \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \{x : |f_{n+k}(x) - f(x)| \geq \varepsilon\} \right|$$

But this is $\leq \lim_{n\to\infty} \left| \bigcup_{k=1}^{\infty} \{x : |f_{n+k}(x) - f(x)| \geq \varepsilon\} \right|$

$$\leq \lim_{n\to\infty} \varepsilon^{-1} \underbrace{\sum_{k=1}^{\infty} 2^{-n-k}}_{= C 2^{-n}} \xrightarrow{n\to\infty} 0 \quad \forall \varepsilon > 0.$$

Therefore $f_n^{(x)} \to f(x)$ a.e.

Hardy-Littlewood: $\| \sup_{r>0} A_r |f_n - f| \|_{L^{1,\infty}(\mathbb{R}^d)} \xrightarrow{n\to\infty} 0$

$$|\{x : \sup_{r>0} A_r |f_n - f|(x) \geq \varepsilon\}| \xrightarrow{n\to\infty} 0$$

$\xrightarrow{\Delta\text{-ineq.}}$ $|\{x : \sup_{r>0} |A_r(f_n - f)(x)| \geq \varepsilon\}| \xrightarrow{n\to\infty} 0$

$\Longrightarrow$ as above (possibly passing to a subsequence)

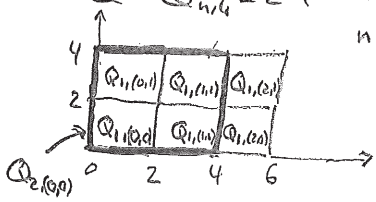
$$\sup_{r>0} |A_r(f_n - f)|(x) \xrightarrow{n\to\infty} 0 \quad \text{a.e.}$$

Therefore we get $A_r f(x) \to f(x)$ from the corresponding fact for f_n.

$\boxtimes$

dyadic Hardy-Littlewood maximal inequality and conditional expectations

Def 1.5: Q = dyadic cube $\subseteq \mathbb{R}^d$ of generation n $\Longleftrightarrow \exists k: Q = Q_{n,k} = 2^n(k + [0,1)^d)$, $n \in \mathbb{N}, k \in \mathbb{Z}^d$



- $\mathbb{R}^d = \bigcup_{k \in \mathbb{Z}^d} Q_{n,k}$
- $Q_{2,(0,0)} = Q_{1,(0,0)} \cup Q_{1,(1,0)} \cup Q_{1,(0,1)} \cup Q_{1,(1,1)}$
- $|Q_{n,k}| = 2^{nd}$
- $Q_{n,k} \cap Q_{m,l} \neq \emptyset$, $n \leq m$ $\Longleftrightarrow$ $Q_{n,k} \subseteq Q_{m,l}$

  "nesting property"

Lemma 1.6: (Covering Lemma for dyadic cubes)

$Q_1, \dots, Q_N$ dyadic cubes $\Rightarrow \exists n_1, \dots, n_k$: $Q_1 \cup \dots \cup Q_N = Q_{n_1} \cup \dots \cup Q_{n_k}$ (*)

and $Q_{n_j} \cap Q_{n_l} = \emptyset$ $\forall j \neq l$.

Proof: Let Q_{n_1} = largest cube $\in \{Q_1, \dots, Q_N\}$ containing Q_1

Q_{n_2} = largest cube $\in \{Q_1, \dots, Q_N\}$ containing Q_2 and disjoint from Q_{n_1} (possibly none)

Q_{n_3} = largest cube $\in \{Q_1, \dots, Q_N\}$ containing Q_3 and disjoint from Q_{n_1}, Q_{n_2}.

...

Nesting property assures (*). $\square$

$$M_\Delta f(x) := \sup_{\substack{Q \ni x \\ Q \text{ dyadic cube}}} \frac{1}{|Q|} \int_Q |f| \qquad \text{dyadic maximal fct.}$$

$$\Rightarrow \quad M_\Delta: L^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d) \quad \text{continuous}$$
$$L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d), \quad p > 1$$

The proof is analogous to the proof of Prop. 1.1, using Lemma 1.6 instead of 1.3.

Actually, using a trick due to Christ, the dyadic and the ball-version of the Hardy-Littlewood maximal inequality can be deduced from each other.

Ex. 4.3: <u>Conditional Expectations</u> (see Tao's notes)

$\tilde{\mathcal{B}}$ = Borel σ-algebra on $\mathbb{R}^d$ = σ-algebra generated by all open sets or all closed sets or all intervals of the form $[a_1, b_1) \times \cdots \times [a_d, b_d)$.

$\mathcal{B}$ = σ-finite sub-σ-algebra of $\tilde{\mathcal{B}}$

e.g. $\mathcal{B}_n$ = σ-algebra generated by dyadic cubes of generation n

$L^2(\mathbb{R}^d) \supset L^2(\mathbb{R}^d, \mathcal{B})$ closed subspace $\rightsquigarrow$ orthogonal projection

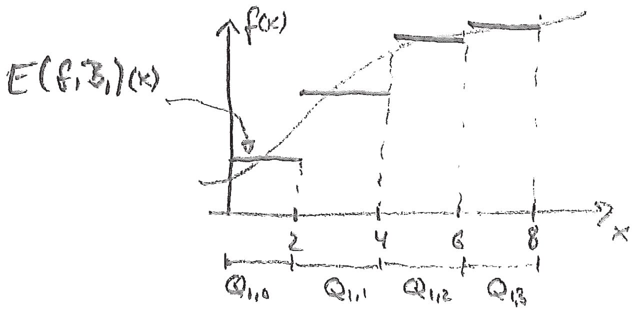
$L^2(\mathbb{R}^d) \ni f \mapsto E(f, \mathcal{B}) \in L^2(\mathbb{R}^d, \mathcal{B})$

$E(\cdot, \mathcal{B})$ "conditional expectation"

extends to contraction on $L^p(\mathbb{R}^d)$

$\mathcal{B} = \mathcal{B}_n$: $E(f, \mathcal{B}_n)(x) = \frac{1}{|Q|} \int_Q f$ for $x \in Q$

note: $L^2(\mathbb{R}^d, \mathcal{B}_n)$ consists of piecewise constant functions



$E(f, \mathcal{B}_n) \xrightarrow{n\to\infty} f$ in L^p, $p<\infty$

Hardy-Littlewood: $\sup_n |E(f, \mathcal{B}_n)|$: $L^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d)$

$L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, $p>1$

continuous.

The TT^*-approach to Hardy-Littlewood:

$\oplus$ Shows L^2-boundedness directly (by interpolation thus L^p-boundedness for $p\geq 2$)

$\oplus$ frequently used trick to show boundedness if you have detailed information about the kernel.

$\ominus$ Restricted to operators $X \to Y$, where X or Y is a Hilbert space.

TT^*-method: Lemma 1.12 / Exercise 4.1: H Hilbert space, X normed space; $T: H \to X$ continuous, $T^*: X^* \to H^* = H$ adjoint of T

$$\Rightarrow \|T\|_{H\to X} \overset{①}{=} \|T^*\|_{X^*\to H} \overset{②}{=} \|TT^*\|^{1/2}_{X^*\to X}$$

Proof: ①: $\|T\|_{H\to X} = \sup_{\|x^*\|_{X^*}\leq 1} \sup_{\|h\|_H \leq 1} |\langle x^*, Th\rangle|$

$$= \sup_{\|h\|_H\leq 1} \sup_{\|x^*\|_{X^*}\leq 1} |\langle T^*x^*, h\rangle| = \|T^*\|_{X^*\to H}$$

②: $\|TT^*\|_{X^*\to X} \leq \|T\|_{H\to X} \|T^*\|_{X^*\to H} \overset{①}{=} \|T^*\|^2_{X^*\to H} \Rightarrow$ „$\geq$"

„$\leq$": $\forall x^* \in X^*: \|T^*x^*\|^2_H = \langle x^*, TT^*x^*\rangle \leq \|TT^*\|_{X^*\to X} \|x^*\|^2_{X^*}$ ∎

To show the Hardy-Littlewood max. inequality directly for $L^2(\mathbb{R}^d)$, we have to show

$$(*) \quad \|\sup_{r>0} A_r f\|_{L^2(\mathbb{R}^d)} \leq C_d \|f\|_{L^2(\mathbb{R}^d)} \quad \forall\, 0\leq f \in L^2(\mathbb{R}^d).$$

- Replace $\{r>0\}$ by an arbitrary finite subset R. If we can find a constant C_d which does not depend on R, then we can just take the supremum over all finite subsets R at the end and recover $(*)$.

• Remember that Schur's test gave $\|A_r\| \le 1 \quad \forall r$

$$\Rightarrow \|\sup_{r\in R} A_r f\|_{L^2} \le \||A_{r_1} f| + \dots + |A_{r_{|R|}} f|\|_{L^2}$$
$$\le |R| \, \|f\|_{L^2}$$
, so we know the operator is bounded for finite R.

Our task is to improve the upper bound for the operator norm from $|R|$ to some constant independent of R.

As in the proof of the maximal operator version of Christ-Kiselev, we use the linearization trick and consider the linear operator

$T_r f(x) = A_{r(x)} f(x)$ for given $r: \mathbb{R}^d \to R$ measurable.

If $\sup_{\substack{r:\mathbb{R}^d \to R \\ \text{measurable}}} \|T_r f(x)\|_{L^2} \le D \|f\|_{L^2} \Rightarrow \|\sup_{r\in R} A_r f\|_{L^2} \le D\|f\|_{L^2}$.

TT^*-Lemma $\Rightarrow \quad \sup_r \|T_r\|_{L^2\to L^2} = \sup_r \|T_r T_r^*\|^{1/2}_{L^2\to L^2}$

What's T_r^*? Integral kernel of T_r: $K(x,y)$

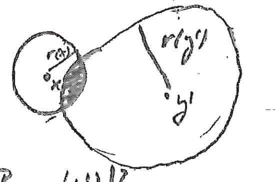
Integral kernel of T_r^*: $K(y,x)$

$$T_r f(y) = \int_{\mathbb{R}^d} \frac{1}{|B_{r(y)}(y)|} 1\!\!1_{\{|x-y| \le r(y)\}} f(x) \, dx$$
$$T_r^* g(y) = \int_{\mathbb{R}^d} \frac{1}{|B_{r(x)}(x)|} 1\!\!1_{\{|y-x| \le r(x)\}} g(x) \, dx$$

$$\Rightarrow T_r T_r^* g(y') = \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dx \frac{1}{|B_{r(y')}(y')|} \frac{1}{|B_{r(x)}(x)|} 1\!\!1_{\{|y-x| \le r(x)\}} 1\!\!1_{\{|y'-y| \le r(y')\}} g(x)$$

$|y-x| \le r(x)$ and $|y'-y| \le r(y') \Rightarrow |x-y'| \overset{\Delta\text{-inequality}}{\le} r(x) + r(y')$ (**)

$\Rightarrow \int dy \, 1\!\!1_{\{\dots\}} 1\!\!1_{\{\dots\}}$ = Volume of:



(black area only!)

$$\le \begin{cases} \min\{\underbrace{|B_{r(x)}(x)|}_{=: B_x}, |B_{r(y')}(y')|\} & \text{, (**) holds} \\ 0 & \text{, otherwise} \end{cases}$$

$$\Rightarrow \quad |T_r T_r^* g(y')| \leq \int_{\mathbb{R}^d} \mathbb{1}_{\{|x-y'| \leq r(x)+r(y')\}} \underbrace{\frac{\min\{B_x, B_{y'}\}}{B_x\, B_{y'}}}_{=\frac{1}{\max\{B_x, B_{y'}\}}} |g(x)|\, dx$$

$$\leq \underbrace{\int_{\mathbb{R}^d} \mathbb{1}_{\{|x-y'| \leq 2r(x)\}} \frac{|g(x)|}{B_x}\, dx}_{r(x) \geq r(y')}$$

$$+ \underbrace{\int_{\mathbb{R}^d} \mathbb{1}_{\{|x-y'| \leq 2r(y')\}} \frac{g(x)}{B_{y'}}\, dx}_{r(x) \leq r(y')}$$

$$= T_{2r}|g|(y') + T_{2r}^*|g|(y') \qquad \forall y' \in \mathbb{R}^d$$

But $\sup_r \|T_r\|_{L^2 \to L^2} = \sup_r \|T_{2r}\|_{L^2 \to L^2}$

$$\Rightarrow \quad \underset{\|T_r\|^2}{\underset{\|}{\|T_r T_r^*\|}} \leq 2 \sup_r \|T_r\| \quad \Rightarrow \quad \left(\sup_r \|T_r\|\right)^2 \leq 2 \sup_r \|T_r\|$$

$$\Rightarrow \quad \sup_r \|T_r\| \leq 2. \qquad \boxed{}$$

Calderon-Zygmund decomposition

Recall that in the proof of the Marcinkiewicz interpolation theorem it was useful to decompose $f = \sum f_n$, $f_n = f \, \mathbb{1}_{\{2^n \le |f| \le 2^{n+1}\}}$, into pieces of controlled size of the function.

The basic decomposition is $f = \underbrace{f \, \mathbb{1}_{\{|f| \le \lambda\}}}_{=: g} + \underbrace{f \, \mathbb{1}_{\{|f| > \lambda\}}}_{=: b}$

into a "good" bounded part g and a "bad" large part depending on a parameter $\lambda > 0$.

If $f \in L^1(X)$, we have the basic estimates

- $\|g\|_{L^1} \le \|f\|_{L^1}$, $\|g\|_{L^\infty} \le \lambda$ $\leftarrow$ bounded by λ
- $\lambda |\mathrm{supp}(b)| \le \int_X |b| = \|b\|_{L^1} \le \|f\|_{L^1}$ $\leftarrow$ $|\text{support}| \lesssim \frac{1}{\lambda}$

Aim: More control over "bad" part for special X.

Lemma 4.1 / Exercise 4.4: (1-dim rising sun Lemma) $I \subset \mathbb{R}$ bounded interval, $f: I \to \mathbb{R}$ integrable, $\lambda \ge \frac{1}{|I|} \int_I f \Rightarrow \exists$ family $(I_n)_{n \in \mathbb{N}}$ disjoint open intervals

s.t.
- $\|f \, \mathbb{1}_{I \setminus \bigcup_n I_n}\|_\infty \le \lambda$
- $\frac{1}{|I_n|} \int_{I_n} f = \lambda$ $\forall n$. bad part "locally oscillating"

Cor 4.2 / Exercise 4.4: $f = \underbrace{\left(f \, \mathbb{1}_{I \setminus \bigcup_n I_n} + \lambda \mathbb{1}_{\bigcup_n I_n}\right)}_{g} + \underbrace{\sum_n \underbrace{(f - \lambda) \mathbb{1}_{I_n}}_{b_n}}_{b}$,

- $\|g\|_\infty \le \lambda$, $\int_I g = \int_I f$
- $\int b_n = 0$. If $f \ge 0$: $\lambda |\mathrm{supp}(b)| \le \|f\|_{L^1}$

  $\|b_n\|_{L^1} \le 2\lambda |I_n|$

Prop 4.3: (dyadic Calderon-Zygmund decomposition)

$f \in L^1(\mathbb{R}^d)$, $\lambda > 0$ $\Rightarrow$ $f = g + \sum_Q b_Q$, Q disjoint dyadic cubes

- $\|g\|_{L^1} \le \|f\|_{L^1}$, $\|g\|_\infty \le 2^d \lambda$
- $\mathrm{supp}(b_Q) \subset Q$, $\int b_Q = 0$, $\|b_Q\|_{L^1} \le 2^{d+1} \lambda |Q|$
- $\bigcup Q = \{ M_\Delta f > \lambda \} \; (\subseteq \{ Mf \gtrsim \lambda \})$

  Hardy-Littlewood $\Rightarrow \lambda |\bigcup Q| \lesssim \|f\|_{L^1}$

Proof: A cube Q is "bad" $:\Leftrightarrow \frac{1}{|Q|} \int_Q |f| > \lambda$ $\quad$ ($\Rightarrow M_\Delta f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f| > \lambda$ for x in a bad cube)

"good" $\Leftrightarrow \frac{1}{|Q|} \int_Q |f| \le \lambda$

A bad cube is "maximal" if $\forall \tilde{Q} \supsetneq Q$: $\tilde{Q}$ good.

$\mathcal{Q} := \{ Q : Q \text{ maximal bad cube} \}$.

$f \in L^1 \Rightarrow \frac{1}{|Q|} \int_Q |f| \le \frac{\|f\|_{L^1}}{|Q|} \longrightarrow 0$ for $|Q|$ large

$\Rightarrow$ large cubes are good $\Rightarrow$ Every bad cube is contained in a maximal bad cube.

$Q \in \mathcal{Q} \Rightarrow Q' =$  satisfies $\underset{\substack{\vee\!\! \mid \\ \int_Q |f|}}{\int_{Q'} |f|} \le \lambda |Q'| = 2^d \lambda |Q|$

So $\frac{1}{|Q|} \int_Q |f| \le 2^d \lambda$.

$$f = \underbrace{\left(f \mathbb{1}_{\mathbb{R}^d \setminus \bigcup Q} + \sum_Q \left(\frac{1}{|Q|} \int_Q f \right) \mathbb{1}_Q \right)}_{g} + \sum_Q \underbrace{\left(f - \frac{1}{|Q|} \int_Q f \right) \mathbb{1}_Q}_{b_Q}$$

does the job. $\square$

Remark: $\mathcal{B} = \sigma(\{Q : Q \in \mathcal{Q}\} \cup \mathrm{Borel}(\mathbb{R}^d \setminus \bigcup Q))$

$\Longrightarrow g = E(f, \mathcal{B})$

Instead of building the bad set $\bigcup Q$ from specifically chosen cubes, we can also start with the level set $\{M_\Delta f > \lambda\}$ $(= \bigcup Q)$ and then subdivide it appropriately.

Prop 4.5: (Dyadic Whitney decomposition)

$\Omega \subsetneq \mathbb{R}^d_+$ open $\Longrightarrow$ $\Omega = \bigcup_{Q \in \mathcal{Q}} Q$, Q: disjoint dyadic cubes
$Q \in \mathcal{Q} \Rightarrow Q' \notin \mathcal{Q}$.

Proof: Note that every $x \in \Omega$ is contained in a maximal dyadic cube which contains this point (note $\Omega \neq \mathbb{R}^d_+$).
Let $\mathcal{Q} := \{$ maximal dyadic cubes $\subset \Omega \}$. $\square$

In particular, $0 \leq \mathrm{dist}(Q, \mathbb{R}^d_+ \setminus \Omega) \leq \sqrt{d}\, \mathrm{diam}(Q)$ because $Q' \not\subset \Omega$.
Can get lower bound:

Prop. 4.6: (Whitney decomposition)

$\Omega \subsetneq \mathbb{R}^d$ open, $K \geq 1$ $\Longrightarrow$ $\Omega = \bigcup_{Q \in \mathcal{Q}} Q$, Q disjoint dyadic cubes
$C_1 K \mathrm{diam}(Q) \leq \mathrm{dist}(Q, \mathbb{R}^d \setminus \Omega) \leq C_2 K \mathrm{diam}(Q)$.

Proof: $\mathcal{Q}' := \{Q : Q \subseteq \Omega$ dyadic cube, $K \mathrm{diam}(Q) \leq \mathrm{dist}(Q, \mathbb{R}^d \setminus \Omega) \leq 5K \mathrm{diam}(Q)\}$

then $\bigcup_{Q \in \mathcal{Q}'} Q = \Omega$ $\square$

/As cubes are contained in balls of $\approx$ same volume, one can deduce a similar decomposition into nearly disjoint balls (i.e. only finitely many contain a given point).