Integral operators

(X,μ), (Y,ν) measure spaces, $k : X \times Y \to \mathbb{C}$ measurable

Integral operator $T_k f(y) = \int_X k(x,y) f(x)\, d\mu(x)$ (formally)

Under which conditions is this operator bounded from L^p to L^q?

Schur's test (Ex. 2.2) $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$.

$$\left.\begin{aligned} \| k(x,y) \|_{L^q(Y)} &\le B_0 \quad \text{for a.e. } x \in X \\ \| k(x,y) \|_{L^{p'}(X)} &\le B_1 \quad \text{for a.e. } y \in Y \end{aligned}\right\} \Rightarrow \forall\, 0 \le \Theta \le 1 : T_k : L^{p_\Theta} \to L^{q_\Theta}$$

$$\text{for } \frac{1}{p_\Theta} = 1 - \Theta + \frac{\Theta}{p'}, \qquad \frac{1}{q_\Theta} = \frac{1-\Theta}{q}$$

Example: $q = p = 1 \Rightarrow T_k : L^p \to L^p \ \forall p$.

There's also a weak-type Schur test with a similar proof (using Marcinkiewicz instead of Riesz-Thorin), which allows to treat kernels whose $L^{p',\infty}(X)$ resp $L^{q,\infty}(Y)$-norms are bounded, in particular the convolution operator $f \mapsto \frac{1}{|x|^\alpha} * f$ on $\mathbb{R}^n$, $0 < \alpha < n$.

("Hardy-Littlewood-Sobolev fractional integration inequality").

Kernel truncation: "Smaller kernels should be easier to bound"

Thm 8.1 (Tao's notes 2) Let k, k' be kernels, $|k| \le k'$.

If $T_{k'} : L^p(X) \to L^q(Y)$ is bounded, then so is T_k, and $\|T_k\|_{p\to q} \le \|T_{k'}\|_{p \to q}$. The same holds if L^q is replaced by $L^{q,\infty}$.

Proof: Δ-inequality: $|T_k f(y)| \le \int_X |k(x,y)||f(x)| \le T_{k'} |f|$

$$\Rightarrow \|T_k f\|_{L^q} \le \| T_{k'} |f| \|_{L^q} \le C \|f\|_{L^p}. \quad \blacksquare$$

Statement can fail for signed kernels, such as the Fourier transform, if the boundedness results from subtle cancellations of positive / negative contributions (see exercises).

· Signed kernels are ok if Schur's test can be applied.

Consider $k(x,y) = k'(x,y)\, \mathbb{1}_{\Omega}$.

$\Omega = A \times B \Rightarrow T_k f = \mathbb{1}_B T_{k'}(\mathbb{1}_A f)$ (assume everything is well-defined)

$\Rightarrow \|T_k\|_{p\to q} \leq \|T_{k'}\|_{p\to q}$

Actually, here $T_k : L^p(A) \to L^q(B)$ with the same norm even.

More general block-diagonal restrictions are also ok (at least if $q \geq p$)

$$T_k f = \sum_n \mathbb{1}_{B_n} T_{k'}(\mathbb{1}_{A_n} f)$$

Upper diagonal regions can be treated w/ the following thm. (if $q > p$).

<u>Thm 8.7 (Christ-Kiselev, finite decomposition)</u>

Let $N \geq 1$, $X = \bigcup_{n=1}^{N} X_n$ disjoint, $Y = \bigcup_{n=1}^{N} Y_n$, $1 \leq p < q \leq \infty$, $T : L^p(X) \to L^q(Y)$ bounded, $T' = \sum_{1 \leq n \leq m \leq N} \mathbb{1}_{Y_m} T\, \mathbb{1}_{X_n}$.

$\Rightarrow T' : L^p \to L^q$ bounded, $\exists A = A(p,q) : \|T'\|_{p\to q} \leq A \|T\|_{p \to q}$ (*)

Note that A does not depend on N.

<u>Proof</u>: by Induction on N. Let $A = A(p,q)$ sufficiently large (≥ 1).

<u>Show</u>: (*).

<u>$N=1$</u>: $T' = T \Rightarrow \checkmark$.

<u>general case</u>: Assume (*) holds for $N-1$. Let $f \in L^p(X)$.

Normalization: W.log $\|T\|_{p\to q} = \|f\|_{L^p} = 1$.

The assertion (*) is then $\|T'\|_{p\to q} \leq A$.

The finite sequence $\overbrace{\|f \mathbb{1}_{X_1 \cup \dots \cup X_n}\|^p_{L^p(X)}}^{=: c_n}$, $0 \leq n \leq N$, is increasing in n,

$c_0 = 0$, $c_N = 1$. $\Longrightarrow$ $\exists n_0 < N$: $c_{n_0} \leq \frac{1}{2} < c_{n_0+1}$

As the X_j are disjoint: $\|f \mathbb{1}_{X_{n_0+2} \cup \dots \cup X_N}\|^p_{L^p} < \frac{1}{2}$

As $(*)$ holds for n_0, $N-(n_0+2) < N$:

$$\|T'(f \mathbb{1}_{X_1 \cup \dots \cup X_{n_0}})\|_{L^q(Y_1 \cup \dots \cup Y_{n_0})} \leq A \, \|f \mathbb{1}_{X_1 \cup \dots \cup X_{n_0}}\|_{L^p(X)} \leq 2^{-\frac{1}{p}} A$$

and $\|T'(f \mathbb{1}_{X_{n_0+2} \cup \dots \cup X_N})\|_{L^q(Y_{n_0+2} \cup \dots \cup Y_N)} \leq \dots \leq 2^{-\frac{1}{p}} A$

But $\|T'(f \mathbb{1}_{X_1 \cup \dots \cup X_{n_0}})\|_{L^q(Y_{n_0+1} \cup \dots \cup Y_N)}$ by def. of T' equals

$\|T(f \mathbb{1}_{X_1 \cup \dots \cup X_{n_0}})\|_{L^q(Y_{n_0+1} \cup \dots \cup Y_N)}$, which by our normalization is bounded by

$$1 \cdot \|f \mathbb{1}_{X_1 \cup \dots \cup X_{n_0}}\|_{L^p(X)} \leq 1.$$

Similarly $\|T'(f \mathbb{1}_{X_{n_0+1}})\|_{L^q(Y)} \leq 1.$

$$\Rightarrow \quad \|T' f\|_{L^q(Y_1 \cup \dots \cup Y_{n_0})} \leq \|T'(f \mathbb{1}_{X_1 \cup \dots \cup X_{n_0}})\|_{L^q(\dots)} + \|T'(f \mathbb{1}_{X_{n_0+1}})\|_{L^q(\dots)} + \|T'(f \mathbb{1}_{X_{n_0+2} \cup \dots \cup X_N})\|_{L^q(\dots)} \leq 2^{-1/p} A + 2$$

Similarly: $\|T' f\|_{L^q(Y_{n_0+1} \cup \dots \cup Y_N)}$

$$\leq \|T'(f \mathbb{1}_{X_1 \cup \dots \cup X_{n_0}})\|_{L^q(\dots)} + \|T'(f \mathbb{1}_{X_{n_0+1}})\|_{L^q(\dots)} + \|T'(f \mathbb{1}_{X_{n_0+2}})\|_{L^q(\dots)}$$

$$\leq 2^{-1/p} A + 2$$

If A is large enough $2^{-\frac{1}{p}} A + 2 \leq 2^{-1/q} A$ (note $q > p$)

$$\Longrightarrow \|T' f\|_{L^q(Y)} \leq \left(\|T' f\|^q_{L^q(Y_1 \cup \dots \cup Y_{n_0})} + \|T' f\|^q_{L^q(Y_{n_0+1} \cup \dots \cup Y_N)} \right)^{1/q}$$

$$\leq \left((2^{-1/p} A + 2)^q + (2^{-1/p} A + 2)^q \right)^{1/q}$$

$$= 2^{1/q} (2^{-1/p} A + 2) \leq A .$$

$\Longrightarrow$ (*) for N.

$\boxed{\equiv}$

Remark: We will see later that Christ-Kiselev is wrong for $p = q$ unless $p = q \in \{1, \infty\}$.

Corollary 8.8: (maximal operator version of C-K)

Q countable ordered set, $(E_\alpha)_{\alpha \in Q}$ family of subsets of X s.t. $E_\alpha \subset E_{\alpha'}$ $\forall \alpha < \alpha'$. Let $T: L^p(X) \to L^q(Y)$, $1 \leq p < q \leq \infty$ bounded linear operator, $T_* f(y) := \sup_{\alpha \in Q} |T(f \mathbb{1}_{E_\alpha})(y)|$.

$\Longrightarrow T_* : L^p(X) \to L^q(Y)$ bounded, $\|T_*\|_{p \to q} \leq A(p,q) \|T\|_{p \to q}$

Proof: As usual, normalize $\|T\|_{p\to q} = 1 = \|f\|_{L^p}$.

Can assume $f \geq 0$. Then by the monotone convergence thm we may assume $\mathbb{Q} = \{1, \dots, N\}$ finite.

Can take $E_N = X$. Define $E_0 = \emptyset$. $\Rightarrow$

$$\forall y \in Y \,\exists \alpha(y): T_* f(y) = \sup_{\alpha \in \mathbb{Q}} |T(f 1\!\!1_{E_\alpha})(y)| = |T(f 1\!\!1_{E_{\alpha(y)}})(y)|$$

To show: $\| T(f 1\!\!1_{E_{\alpha(y)}})(y) \|_{L^q_y(Y)} \leq A(p,q)$

Note that unlike T_*, $f \longmapsto T(f 1\!\!1_{E_{\alpha(y)}})(y)$ is <u>linear</u>.

Construct suitable partitions:

$\forall n \in \{1, \dots, N\}$: $X_n := E_n \setminus E_{n-1}$, $Y_n := \{y \in Y : \alpha(y) = n\}$

Claim: $$T(f 1\!\!1_{E_{\alpha(y)}}) = \sum_{1 \leq n \leq m \leq N} 1\!\!1_{Y_m} T(1\!\!1_{X_n} f)$$

Apply C-K to this operator! $\boxtimes$

Application: Menshov-Paley-Zygmund

$$\mathcal{F}_* f(\xi) := \sup_{\substack{I \subseteq \mathbb{R} \\ \text{compact interval}}} \left| \int_I f(x) e^{-ix\xi} \, dx \right| = \sup_I |\widehat{f 1\!\!1_I}(\xi)|$$

is bounded as an operator $L^p(\mathbb{R}) \to L^{p'}(\mathbb{R})$ for $p \in [1,2)$.

Proof: May restrict to $I = [0,q]$ for some $q \in \mathbb{Q}$, $q > 0$.

Have to show $\| \sup_{q \in \mathbb{Q}_+} |\widehat{f 1\!\!1_{[0,q]}}| \|_{L^{p'}} \leq C_p \|f\|_{L^p}$.

This follows from Hausdorff-Young and Christ-Kiselev. $\boxtimes$

Cor: $f \in L^p(\mathbb{R})$, $1 \leq p < 2 \Rightarrow$ For a.e. $\xi \in \mathbb{R}$:

$$\lim_{N_+, N_- \to \infty} \int_{-N_-}^{N_+} f(x) e^{-ix\xi} \, dx = \hat{f}(\xi).$$

Proof of Corollary: (I created some confusion in the lecture!) Here are all details.

Note that $\lim_{N_+, N_- \to \infty} \left| \int_{-N_-}^{N_+} f(x) e^{-ix\xi} dx - \hat{f}(\xi) \right| = 0$ for a.e. ξ

$\iff$ this quantity $= 0$ in $L^q(\mathbb{R})$ for any q, choose q s.t. $\frac{1}{q} + \frac{1}{p} = 1$

So we show $\left\| \lim_{N_+, N_- \to \infty} \left| \int_{-N_-}^{N_+} f(x) e^{-ix\xi} dx - \hat{f}(\xi) \right| \right\|_{L^q(\mathbb{R})} = 0.$

For $f \in L^1 \cap L^p$, this is true for all ξ by definition of the Fourier transform on L^1 and the dominated convergence theorem.

If $f \in L^p(\mathbb{R})$, choose a sequence $(f_n)_{n \in \mathbb{N}} \subseteq L^1 \cap L^p$ s.t. $f_n \to f$ in L^p. As $\mathcal{F}: L^p \to L^q$ is continuous, $\hat{f}_n \to \hat{f}$ in L^q.

$$\Rightarrow \left\| \lim_{N_+, N_- \to \infty} \left| \int_{-N_-}^{N_+} f(x) e^{-ix\xi} dx - \hat{f}(\xi) \right| \right\|_{L^q(\mathbb{R})}$$

$$\leq \left\| \lim_{N_+, N_- \to \infty} \left| \int_{-N_-}^{N_+} (f(x) - f_n(x)) e^{-ix\xi} dx - (\hat{f}(\xi) - \hat{f}_n(\xi)) \right| \right\|_{L^q}$$

$$+ \underbrace{\left\| \lim_{N_+, N_- \to \infty} \left| \int_{-N_-}^{N_+} f_n(x) e^{-ix\xi} dx - \hat{f}_n(\xi) \right| \right\|_{L^q}}_{=0}$$

$$\leq \underbrace{\left\| \lim_{N_+, N_- \to \infty} \left| \int_{-N_-}^{N_+} (f(x) - f_n(x)) e^{-ix\xi} dx \right| \right\|_{L^q}}_{\leq \| \mathcal{F}_* \|_{p \to q} \underbrace{\| f - f_n \|_{L^p}}_{\xrightarrow[n\to\infty]{} 0}} + \underbrace{\| \hat{f}(\xi) - \hat{f}_n(\xi) \|_{L^q}}_{\xrightarrow[n\to\infty]{} 0}$$

$$\Rightarrow \left\| \lim_{N_+, N_- \to \infty} \left| \int_{-N_-}^{N_+} f(x) e^{-ix\xi} dx - \hat{f}(\xi) \right| \right\|_{L^q} = 0$$

$\boxtimes$

Smooth cutoffs:

Def: $\Omega \subset \mathbb{R}^d$ bounded open, $L: \mathbb{R}^d \to \mathbb{R}^d$ invertible, $x \mapsto Ax+b$, $\phi: \mathbb{R}^d \to \mathbb{C}$. Then ϕ is a <u>bump function adapted to $L(\Omega)$</u> if $\phi \in C^\infty(\mathbb{R}^d)$, $\operatorname{supp}\phi \subset L(\Omega)$ and

$$\forall k \in \mathbb{N}\ \exists C_{k,\Omega} \geq 0: \sup_{x\in\mathbb{R}^d} |\partial^k \phi(L(x))| \leq C_{k,\Omega}$$

Remark: This definition is empty, unless we consider families $(L_\alpha)_{\alpha\in A}$ of transformations and require $C_{k,\Omega}$ to be independent of $\alpha \in A$.

Prop 8.16: $K \in L^1_{loc}(\mathbb{R}^d \times \mathbb{R}^{d'})$, $p,q \in [1,\infty]$ s.t. $T_K: L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^{d'})$ bounded. $\Omega \subset \mathbb{R}^d$, $\Omega' \subset \mathbb{R}^{d'}$ bounded open, L, L' affine, $\phi: \mathbb{R}^d \times \mathbb{R}^{d'} \to \mathbb{C}$ bump function adapted to $L(\Omega) \times L'(\Omega')$

$$\Longrightarrow T_{K\phi}: L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^{d'}) \text{ bounded and}$$

$$\|T_{K\phi}\|_{p\to q} \lesssim C(\Omega, \Omega', d, d')\, \|T_K\|_{p\to q}.$$

Remark: Unlike the characteristic functions considered above, smooth truncations thus do not affect the boundedness of the integral operator.

Proof:

- Considering $L(\Omega)$, $L'(\Omega')$ instead of Ω, Ω', we may take $L = \mathrm{id}_{\mathbb{R}^d}$, $L' = \mathrm{id}_{\mathbb{R}^{d'}}$.
- Wlog $\Omega \times \Omega' \subseteq [-\frac{1}{4}, \frac{1}{4}]^{d+d'}$.
- $\operatorname{supp}\phi \subseteq [-\frac{1}{4}, \frac{1}{4}]^{d+d'} \Rightarrow$ may extend ϕ periodically with period 1 along each coordinate axis.

This smooth periodic function $\tilde{\phi}$ has a convergent Fourier series:

$$\tilde{\phi}(x,y) = \sum_{n\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^{d'}} c_{nm}\, e^{2\pi i(nx+my)}$$

Standard theory: $\tilde{\phi}$ smooth $\Rightarrow \begin{cases} c_{nm} \text{ decay rapidly} \\ \text{series converges uniformly} \end{cases}$

$$\Rightarrow K\phi = \sum_{n,m} c_{nm} K(x,y)\, e^{2\pi i nx}\, 1\!\!1_{[-\frac14,\frac14]^d}(x)\, e^{2\pi i my}\, 1\!\!1_{[-\frac14,\frac14]^{d'}}(y)$$

$$\Rightarrow T_{K\phi} = \sum c_{nm}\, C'_m\, T_K\, C_n$$

where $C_n f(x) = e^{2\pi i n x} \mathbb{1}_{[-\frac{1}{4},\frac{1}{4}]^d}(x)\, f(x)$ are bounded (norm 1)

$C'_m f(y) = e^{2\pi i m y} \mathbb{1}_{[-\frac{1}{4},\frac{1}{4}]^{d'}}(y)\, f(y)$ on all L^p-spaces,

$$\Rightarrow \quad \| T_{K\phi} \|_{p\to q} \leq \underbrace{\Big(\sum_{n,m} |c_{nm}| \Big)}_{<\infty} \| T_K \|_{p\to q} \qquad \boxtimes$$