Lecture 1:

$$\mathcal{F}f(\xi) = \int e^{-ix\xi} f(x)\,dx \quad : L^1(\mathbb{R}^n) \longrightarrow C_0(\mathbb{R}^n)$$

$$\mathcal{F}(x^\alpha D_x^\beta f) = (-D_\xi)^\alpha \xi^\beta \hat{f} \quad , \quad \int \hat{\varphi}\psi = \int \varphi \hat{\psi}$$

$$\int \varphi \overline{\psi} = (2\pi)^{-n} \int \hat{\varphi} \overline{\hat{\psi}}$$

$$\mathcal{F}^{-1} = (2\pi)^{-n} \overline{\mathcal{F}}$$

$L^1 \longrightarrow C_0 \subset L^\infty$ injective, continuous

$\mathcal{F} : L^2 \longrightarrow L^2$ isometric isom.,

$S \longrightarrow S$ isom.,

$S' \longrightarrow S'$ isom.,

Restrictions: $\mathcal{F} : L^p \longrightarrow S'$, $\mathcal{E}' \longrightarrow S'$, $C_c^\infty \longrightarrow S$, $\mathcal{E}'$

Recall $L^1_{loc} \hookrightarrow \mathcal{D}'$, $f \longmapsto (\varphi \mapsto \int f\varphi)$

Questions: • Can we say anything about the range? (Paley-Wiener)

• Continuity $\mathcal{F} : L^p \to L^q$ for some q? (Interpolation)

Convolution: $(\varphi * \psi)(x) = \int \varphi(x-y)\psi(y)\,dy = (\psi * \varphi)(x) \quad \forall \varphi, \psi \in C_0^\infty(\mathbb{R}^n)$

$(u*\varphi)(x) = u(y \mapsto \varphi(x-y)), \ \forall \varphi \in C_0^\infty(\mathbb{R}^n) \ \forall u \in \mathcal{D}'(\mathbb{R}^n)$, $\quad \mathrm{supp}(u*\varphi) \subset \mathrm{supp}\, u + \mathrm{supp}\, \varphi$.

$\mathcal{F}(u*\varphi) = \ldots$, $(\varphi*\psi)^\wedge = \hat{\varphi}\hat{\psi}$, $\widehat{\varphi\psi} = (2\pi)^{-n} \hat{\varphi} * \hat{\psi}$

Thm: (1.7.5 in Hörmander) $\quad u \in \mathcal{E}' \Longrightarrow \hat{u}(\xi) = u(x \longmapsto e^{-ix\xi})$ and hol. on $\mathbb{C}$

Proof: $u \in L^1 \cap \mathcal{E}' \Longrightarrow \checkmark$, $\quad$ Show: $u \in L^1 \cap \mathcal{E}' \Longrightarrow \hat{u} \in \mathcal{O}(\mathbb{C}^n)$ (next page)

General case: Mollify! $\quad \varphi \in C_0^\infty, \varphi \geq 0, \int \varphi = 1, \ \varphi_\varepsilon(x) = \varphi(\frac{x}{\varepsilon})\varepsilon$

$\Rightarrow u * \varphi_\varepsilon \to u$ in S' (actually: $\langle u*\varphi_\varepsilon, \psi \rangle \quad \forall \psi \in C^\infty$

$$= \langle u, \check{\varphi}_\varepsilon * \psi \rangle \xrightarrow[\varepsilon \to 0]{} \langle u, \psi \rangle$$

$\Rightarrow$ weakly in $\mathcal{E}' \Rightarrow$ in S')

• $\mathcal{F}$ continuous $\Rightarrow \mathcal{F}(u*\varphi_\varepsilon) \xrightarrow{\varepsilon \to 0} \mathcal{F}u \quad (S')$

hol. on $\mathbb{C} \Rightarrow \underset{L^1}{\mathcal{F}(u*\varphi_\varepsilon)} = \langle u*\varphi_\varepsilon, e^{-ix\xi} \rangle = \langle u, \check{\varphi}_\varepsilon * e^{-ix\xi} \rangle = \underbrace{\hat{\varphi}(\varepsilon\xi)}\, u(e^{-ix\xi})$

$\xrightarrow[\varepsilon \to 0]{} \hat{\varphi}(0) = \int \varphi = 1$ unif. on cpct.

$\Rightarrow u(e^{-ix\xi})$ hol. on $\mathbb{C}$

restriction to $\mathbb{R}$ is $\mathcal{F}$. $\square$

Thm: (measure theory) When is the integral of a holomorphic function holomorphic?

Let $G \subset \mathbb{C}^n$ open, $f: G \times X \to \mathbb{C}$ s.t.
(X, μ) a measure space

a) $x \mapsto f(z,x) \in L^1(X)$ $\forall z \in G$

b) $z \mapsto f(z,x) \in \mathcal{O}(G)$ $\forall x \in X$

c) For every compact disk $K \subset G$: $\exists g_K \in L^1(X)$ $\forall z \in K$ $\forall x \in X$: $|f(z,x)| \leq g_K(x)$

$$\Longrightarrow \quad F(z) := \int_X f(z,x)\, d\mu(x) \quad \text{holomorphic on } G$$

$$\text{and} \quad \partial_z^k F(z) = \int_X \partial_z^k f(z,x)\, d\mu(x) \quad \forall n \in \mathbb{N}_0.$$

Proof: Let $a \in G$, r small s.t. $K := \overline{B_{2r}(a)} \subset G$. Wlog $n = 1 = k$.

Cauchy's integral thm says $f(z,x) = \frac{1}{2\pi i} \int_{\partial K} \frac{f(\zeta, x)}{\zeta - z}\, d\zeta$ $\quad (z \in \mathring{K})$

$$\Rightarrow \forall z, w \in B_r(a),\ z \neq w: \quad \frac{F(z) - F(w)}{z - w} = \int_X \frac{1}{2\pi i} \int_{\partial K} \frac{f(\zeta, x)}{(\zeta - z)(\zeta - w)}\, d\zeta\, d\mu$$

Let $(w_k)_k \subseteq B_r(a)$, $w_k \to z$, $w_k \neq z$ $\forall k$ and

$$\varphi_k(z,x) := \frac{1}{2\pi i} \int_{\partial K} \frac{f(\zeta, x)}{(\zeta - z)(\zeta - w_k)}\, d\zeta$$

$$\Rightarrow \quad |\varphi_k(z,x)| \leq \frac{1}{2\pi} |\partial K| \frac{|g_K(x)|}{r \cdot r} = \frac{2}{r} g_K(x)$$

$$\text{and} \quad \lim_{k \to \infty} \varphi_k(z,x) \underset{\substack{\uparrow \\ \text{uniform} \\ \text{convergence}}}{=} \frac{1}{2\pi i} \int_{\partial K} \frac{f(\zeta, x)}{(\zeta - z)^2}\, d\zeta \underset{\substack{\uparrow \\ \text{Cauchy}}}{=} \partial_z f(z,x)$$

$$\Rightarrow \text{(dominated convergence)} \quad \lim_{k \to \infty} \frac{F(z) - F(w_k)}{z - w_k} \text{ exists}$$

$$\text{and} \quad = \int_X \partial_z f(z,x)\, d\mu(x). \qquad \boxtimes$$

To see $u \in L^1 \cap \mathcal{E}' \Rightarrow \hat{u} \in \mathcal{O}(\mathbb{C}^n)$, let $f(z,x) = u(x) e^{-ixz}$.

Thm (Paley-Wiener):

a) $\mathcal{F}\, C^\infty_{B_0(R)}(\mathbb{R}^n) = \{U \in \mathcal{O}(\mathbb{C}^n) : \forall N \in \mathbb{N}\ \exists C_N :$
$|U(\zeta)| \leq C_N (1+|\zeta|)^{-N} e^{R|\operatorname{Im}\zeta|}\}$

b) $\mathcal{F}\, \mathcal{E}'_{B_0(R)}(\mathbb{R}^n) = \{U \in \mathcal{O}(\mathbb{C}^n) : \exists N, C : |U(\zeta)| \leq C(1+|\zeta|)^N e^{R|\operatorname{Im}\zeta|}\}$

Proof: a) "$\subseteq$" $u \in C^\infty_c(\mathbb{R}^n)$, $\operatorname{supp} u \subseteq B_0(R)$

$$|\zeta^\beta \hat{u}(\zeta)| \overset{\text{int. by parts}}{=} \left|\int_{\mathbb{R}^n} e^{-ix\zeta} D^\beta u(x)\,dx\right| , \quad \zeta - \eta$$

$$= \left|\int_{|x|\leq R} e^{-ix\zeta} D^\beta u(x)\,dx\right|$$

$$\leq e^{R|\operatorname{Im}\zeta|} \underbrace{\int_{|x|\leq R} |D^\beta u(x)|\,dx}_{=:\,C_\beta} \qquad \forall \beta$$

"$\supseteq$" Let U as above

$\Rightarrow$ $u(x) := (2\pi)^{-n} \int U(\xi) e^{ix\xi}\,d\xi$ satisfies $\hat{u} = U$

as distributions and

$D^\alpha u(x) = (2\pi)^{-n} \int U(\xi)(-\xi)^\alpha e^{ix\xi}\,d\xi$ exists

since this integral is uniformly convergent.

$\Rightarrow$ $u \in C^\infty(\mathbb{R}^n)$.

To see that $\operatorname{supp} u \subset B_0(R)$, note that by Cauchy's thm

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} U(\xi + i\eta) e^{ix(\xi+i\eta)}\,d\xi \quad \forall \eta.$$

But $\left|\int_{\mathbb{R}^n} U(\xi+i\eta) e^{ix(\xi+i\eta)}\,d\xi\right|$

$$\leq C_N e^{-x\eta} e^{R|\eta|} \underbrace{\int \frac{d\xi}{(1+|\xi+i\eta|)^N}}_{\leq C < \infty \text{ for } N \text{ large } \forall \eta}$$

Let $\eta = tx$, $t \to \infty$.
$\Rightarrow$ $\operatorname{supp} u \subseteq B_0(R)$. ∎

Part b) will be discussed in the exercises.

<u>Interpolation:</u>

$\mathcal{F}: L^1 \longrightarrow L^\infty$ cts, i.e. $\mathcal{F}: L^1 + L^2 \longrightarrow L^\infty + L^2$

$L^2 \longrightarrow L^2$

$\|\mathcal{F}\|_{L^1 \to L^\infty} = 1$

$\|\mathcal{F}\|_{L^2 \to L^2} = (2\pi)^{n/2}$

<u>Q:</u> $\mathcal{F}: L^p \longrightarrow L^q$ for $p \in (1,2)$, $q = q(p) = ?$

Exercises: $\frac{1}{p} + \frac{1}{q} = 1$, $p \le 2$; necessary condition.

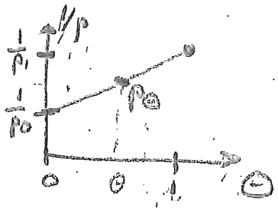
Want to establish $\|\mathcal{F}\|_{L^p \to L^q} \le C(n,p)$ for $\frac{1}{p} + \frac{1}{q} = 1$, $p \le 2$.

<u>Lemma 2 (in Tao)</u> (Log-convexity of L^p-norms)

Let $(X, \mathcal{A}, \mu)$ measure space, $1 \le p_0 < p_1 \le \infty$, $f \in L^{p_0}(X) \cap L^{p_1}(X)$

$\Longrightarrow$ • $f \in L^p(X)$ $\forall\, p_0 \le p \le p_1$

• $\|f\|_{L^{p_\theta}(X)} \le \|f\|^{1-\theta}_{L^{p_0}(X)} \|f\|^{\theta}_{L^{p_1}(X)}$ $\quad \forall \theta \in [0,1]$, $\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$



In particular $\frac{1}{p} \longmapsto \|f\|_{L^p(X)}$ is log-convex.

<u>Proof 1:</u> Hölder: (say $p_1 < \infty$)

$$\|f\|^{p_\theta}_{L^{p_\theta}} = \int_X |f|^{(1-\theta)p_\theta} |f|^{\theta p_\theta}$$

$$\le \left\| |f|^{(1-\theta)p_\theta} \right\|_{L^{\frac{p_0}{(1-\theta)p_\theta}}} \left\| |f|^{\theta p_\theta} \right\|_{L^{\frac{p_1}{\theta p_\theta}}}$$

$$= \|f\|^{(1-\theta)p_\theta}_{L^{p_0}} \|f\|^{\theta p_\theta}_{L^{p_1}}$$

Thm (Lindelöf) (Thm 7 in Tao)

Let $S := \{\sigma+it : 0\le\sigma\le 1, t\in\mathbb{R}\}$, $f\in\mathcal{O}(S)\cap C(\bar{S})$ s.t.

- $\exists A,\delta>0: |f(\sigma+it)|\le A\exp(\exp((\pi-\delta)t))$ $\forall\, \sigma+it\in S$
- $|f(0+it)|\le B_0$, $|f(1+it)|\le B_1$ $\forall\, t\in\mathbb{R}$

$$\Longrightarrow \quad |f(\theta+it)|\le \underbrace{B_0^{1-\theta}B_1^{\theta}}_{=:B_\theta} \quad \forall\,\theta\in[0,1],\ \forall\, t\in\mathbb{R}.$$

"The growth bound allows functions" of super exponential growth $\sim |t|^{|t|}$, $e^{|t|^\alpha}$, but not $e^{-ie^{i\pi z}}$.

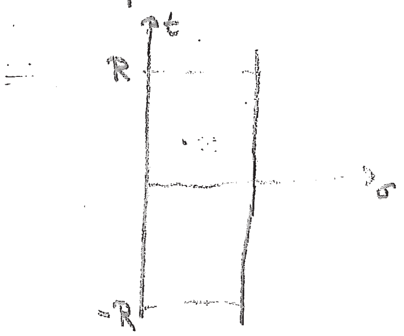
Proof: $z\mapsto B_0^{1-z}B_1^{z}\in\mathcal{O}^*(S)$, $|B_0^{1-z}B_1^{z}| = B_{\mathrm{Re}\, z}$

Consider $\dfrac{f(z)}{B_0^{1-z}B_1^{z}}$ $\Rightarrow$ • $B_0=B_1=B_\theta=1$.
• $\delta \rightsquigarrow \delta-\varepsilon$ for any $\varepsilon>0$.

Assume first that $f(\sigma+it)\xrightarrow{|t|\to\infty}0$

$\Rightarrow \exists R: |f(\sigma+it)|\le\frac{1}{2}\sup_S|f|$ $\forall\, t\ge R$

$\Rightarrow$



Maximum modulus principle: $\sup_S|f|$ is assumed on either $[0-iR, 0+iR]$ or $[1-iR, 1+iR]$

But $|f|\le 1$ there $\Rightarrow$ $\sup_S|f|\le 1$.

If $f \xrightarrow{|t|\to\infty} 0$, consider $F_\varepsilon(z) := f(z)\, g_\varepsilon(z)\, g_\varepsilon(1-z)$ for some ε:

$g_\varepsilon(z) := \exp\left(\varepsilon i \exp\left(i\left((\pi - \frac{\varepsilon}{2})z + \frac{\varepsilon}{4}\right)\right)\right)$

$\Rightarrow$ $f(z)\, g_\varepsilon(z) \xrightarrow{\operatorname{Im} z \to -\infty} 0$, $g_\varepsilon(1-z)$ bdd there

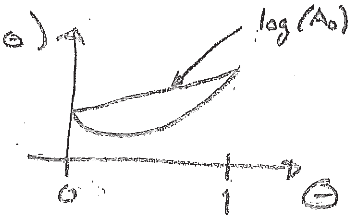
$f(z)\, g_\varepsilon(1-z) \xrightarrow{\operatorname{Im} z \to \infty} 0$, $g_\varepsilon(z)$ bdd there

$\Rightarrow$ $F_\varepsilon(z)$ satisfies hypotheses of previous step.

Take $\varepsilon \to 0$. $\boxtimes$

<u>Remark:</u> $\Theta \longmapsto \sup_{t\in\mathbb{R}} |f(\Theta + it)|$ is log-convex on $[0,1]$.

(OK) $A(\Theta)$ log-convex $\Leftrightarrow$ $\log(A_\Theta)$ $\log(A_0)(1-\Theta) + \log(A_1)\Theta$
$\Theta \in [0,1]$



$\Rightarrow$ $A_\Theta \leq A_0^{1-\Theta} A_1^{\Theta}$

Get <u>upper</u> bounds for intermediate values.

Cannot:
- extrapolate: $\Theta > 1$ or $\Theta < 0$
- get lower bounds for $\Theta \in [0,1]$.

<u>Proof 2 of Lemma 2:</u>
- Consider $|f|$ instead of f $\Rightarrow$ $f \geq 0$.
- Express as monotone limit of simple fcts

$\Rightarrow$ Wlog f simple. Since $f \in L^p \Rightarrow \mu(\operatorname{supp} f) < \infty$.

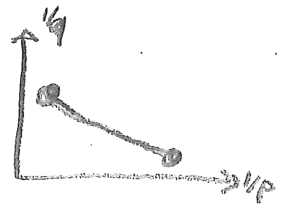
$F: z \longmapsto \int_X |f|^{(1-z)p_0 + z p_1}$ holomorphic of exponential growth

$z = 0 + it \Rightarrow |F(z)| \leq \int_X |f|^{p_0}$, $z = 1 + it \Rightarrow |F(z)| \leq \int_X |f|^{p_1}$

Thm 1 w/ Θ s.t. $(1-\Theta)p_0 + \Theta p_1 = p_\Theta$ yields Lemma 2 ∎

$(X, \mathcal{X}, \mu)$, $(Y, \mathcal{Y}, \nu)$ (σ-finite) measure spaces.

- $T: L^{p_0}(X) + L^{p_1}(X) \longrightarrow L^{q_0}(Y) + L^{q_1}(Y)$
- $\|T\|_{L^{p_0}(X) \to L^{q_0}(Y)} \leq B_0$
- $\|T\|_{L^{p_1}(X) \to L^{q_1}(Y)} \leq B_1$



Thm (Riesz-Thorin) (Thm 3 in Tao)

$1 \leq p_0, p_1, q_0, q_1 \leq \infty$, $0 < \Theta < 1$, $\frac{1}{p_\Theta} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$, $\frac{1}{q_\Theta} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$

$\longrightarrow$ $T: L^{p_\Theta}(X) \longrightarrow L^{q_\Theta}(Y)$ is bdd w/ norm $\leq B_0^{1-\Theta} B_1^{\Theta}$.

Remark: Lemma 2 $\rightsquigarrow$ $X = \{pt\}$

Corollary (Hausdorff-Young inequality)

The Fourier transform is bounded as a map

$$\mathcal{F}: L^p(\mathbb{R}^n) \longrightarrow L^q(\mathbb{R}^n), \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p \leq 2$$

w/ $\|\mathcal{F}\|_{p \to q} \leq (2\pi)^{n(\frac{1}{p} - 1)}$

Proof of R-T: $p_0 = p_1$ follows from Lemma 2, so let $p_0 < p_1$.

$\Rightarrow$ $p_\Theta, p_0 < \infty$. Wlog $B_0 = B_1 = B_\Theta = 1$ (normalize measures)

$\forall f \in L^{p_\alpha}(X)\ \forall g \in L^{q'_\alpha}(Y)$:

Hölder: $+\left(T : L^{p_0} \to L^{q_0} \text{ bdd}\right)$

$$\left| \int_Y (Tf)\, g \, d\nu \right| \leq \|Tf\|_{L^{q_{\alpha_1}}} \|g\|_{L^{q'_{\alpha_1}}} \leq \|f\|_{L^{p_{\alpha_1}}} \|g\|_{L^{q'_{\alpha_1}}}$$

$$\frac{1}{q_{\alpha_1}} + \frac{1}{q'_{\alpha_1}} = 1 \qquad (*)$$

<u>Show:</u> $\left| \int_Y (Tf)\, g \, d\nu \right| \leq \|f\|_{L^{p_\theta}} \|g\|_{L^{q'_\theta}}$

for a $\underbrace{\text{rich set}}$ of f's in L^{p_θ}, g's in $L^{q'_\theta}$

$S := \{$simple fcts whose support has finite measure$\}$

Normalize $\|f\|_{L^{p_\theta}} = \|g\|_{L^{q'_\theta}} = 1$, $f = |f| \operatorname{sgn}(f)$, $g = |g| \operatorname{sgn}(g)$

Define
$$F(z) := \int_Y \left(T\, |f|^{(1-z)\frac{p_\theta}{p_0} + z\frac{p_\theta}{p_1}} \operatorname{sgn}(f) \right) \cdot \left(|g|^{(1-z)\frac{q'_\theta}{q'_0} + z\frac{q'_\theta}{q'_1}} \operatorname{sgn}(g) \right) d\nu$$

$F \in \mathcal{O}(S) \cap C(\bar{S})$ of exponential growth

$$F(\theta) = \int_Y (Tf)\, g \, d\nu$$

$$(*) \implies |F(0+it)|,\ |F(1+it)| \leq 1$$

Lindelöf's thm $\implies |F(\theta)| \leq 1$.

approx. in $L^{p_0} \cap L^{p_1} \cap S$

<u>General f, g:</u> g as above $f =$ bdd + finite measure support

approximate in $L^{p_0} \cap L^{p_1} \cap S$

$(*)$ allows to take limits.
Monotone convergence allows to consider general f. ∎

Real interpolation:

Sublinear operators: $T : \{ f = \sum_{j=1}^{N} f_j \chi_{A_j} : \forall j : \mu(A_j) < \infty \} \to \left\{ \begin{array}{c} \text{measurable} \\ [0,\infty]\text{-valued} \\ \text{functions} \end{array} \right\}$

s.t. • $|T(cf)| = |c| |Tf| \quad \forall c \in \mathbb{C} \ \forall f$

• $|T(f+g)| \leq |Tf| + |Tg| \quad \forall f, g$

Examples: $(T_\alpha)_{\alpha \in A}$ linear $\Rightarrow$ $f \mapsto |T_\alpha f|$, $f \mapsto \sup_\alpha |T_\alpha f|$

$f \mapsto \left(\sum_{\alpha \in A} |T_\alpha f|^p \right)^{1/p}$ all sublinear

weak-L^p spaces: Exercise 2.4: (X, μ) measure space

$f : X \to \mathbb{C}$ measurable

$\Rightarrow \quad \lambda_f : [0,\infty) \to [0,\infty]$ distribution function

$t \longmapsto \mu(\{ x \in X : |f(x)| \geq t \})$

monotonously $\downarrow$

$L^{p,\infty}(X) := \{ f : X \to \mathbb{C}$ measurable $: \underbrace{\sup_{t>0} t \lambda_f(t)^{1/p}}_{=: \|f\|_{L^{p,\infty}(X)}} < \infty \}$

Properties: • $L^p(X) \subseteq L^{p,\infty}(X)$, $f(x) = |x|^{-n/p} \in L^{p,\infty}(\mathbb{R}^n) \setminus L^p(\mathbb{R}^n)$

• $\|\cdot\|_{p,\infty}$ not quite a norm (only generalised Δ-inequality), $\boxed{\text{but } \exists \text{ equivalent norm s.t. } L^{p,\infty} \text{ is a Banach space}}$

Riesz-Thorin (linear operators)

• T sublinear, $1 \leq p, q \leq \infty$.

(strong type (p,q)) 1.) $T : L^p(X) \to L^q(Y)$ continuous $\Leftrightarrow \exists B \forall f : \|Tf\|_{L^q} \leq B \|f\|_{L^p}$

(weak type (p,q)) 2.) $T : L^p(X) \to L^{q,\infty}(Y)$ continuous $\Leftrightarrow \exists B \forall f : \|Tf\|_{L^{q,\infty}} \leq B \|f\|_{L^p}$

Marcinkiewicz

$\Leftrightarrow t^q \lambda_{Tf}(t) \leq B^q \|f\|^q_{L^p} \quad (*)$

• Chebyshev's inequality (2.4.6):

$\|f\|_{L^{q,\infty}} \leq \|f\|_{L^q}$

says that "strong type" $\Rightarrow$ "weak type"

• $\|f\|_{L^p} \asymp \left(\sum_{n \in \mathbb{Z}} \lambda_f(2^n) 2^{np} \right)^{1/p} = \| (2^n \lambda_f(2^n)^{1/p}) \|_{\ell^p(\mathbb{Z})} \quad (**)$

Marcinkiewicz Interpolation Thm: (Thm 4 in Tao's notes)

$1 \le p_0, p_1, q_0 \ne q_1 \le \infty$, $0 < \Theta < 1$, $T: L^{p_0}(X) + L^{p_1}(X) \to L^{q_0,\infty}(Y) + L^{q_1,\infty}(Y)$,
$p_j \le q_j$

sublinear (or linear) s.t. $\|T\|_{L^{p_0} \to L^{q_0,\infty}} \le B_0$

$\|T\|_{L^{p_1} \to L^{q_1,\infty}} \le B_1$

$\Longrightarrow \exists\, C_{\Theta, p_j, q_j, \dots} : \|T\|_{L^{p_\Theta} \to L^{q_\Theta}} \le C_{\Theta, p_j, q_j, \dots} B_0^{1-\Theta} B_1^{\Theta}$

for $\frac{1}{p_\Theta} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$, $\frac{1}{q_\Theta} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$.

(i.e. weak type (p_j, q_j), $j=0,1$ $\Rightarrow$ strong type (p_Θ, q_Θ))

Proof: Assumptions $\Rightarrow t^{q_j} \lambda_{Tf}(t) \le B_j^{q_j} \|f\|_{L^{p_j}}^{q_j}$ $\forall f$

To show: (***) $\|Tf\|_{L^{q_\Theta}}^{q_\Theta} \lesssim \|f\|_{L^{p_\Theta}}^{q_\Theta}$ $\forall f = \sum_{j=1}^N f_j \chi_{A_j}$, $\mu(A_j) < \infty$

$\asymp$ (**)

$\sum_{m \in \mathbb{Z}} \lambda_{Tf}(2^m) 2^{q_\Theta m}$

Wlog $\|f\|_{L^{p_\Theta}} = 1$.

$\left(\sum_{n \in \mathbb{Z}} \lambda_f(2^n) 2^{p_\Theta n}\right)^{1/p_\Theta} \asymp$

$p_0 = p_1$: simple – try yourself! ((*)!)

$p_0 \ne p_1 \Rightarrow$ Wlog. $p_0 < p_\Theta < p_1$

Key idea: Decompose f! Dyadic decomposition:

$$f = \sum_{m \in \mathbb{Z}} f_m, \quad f_m = f\, \mathbb{1}_{\{2^m \le |f| \le 2^{m+1}\}}$$

Sublinearity $\Rightarrow$ $Tf \le \sum Tf_m$

$\Rightarrow \lambda_{Tf}(2^n) \le \sum_m \lambda_{Tf_m}(c_{n,m} 2^n)$, $\sum_m c_{n,m} = 1$

<u>Weak-type:</u> $\lambda_{Tf_m}(t) \le B_j \, \|f_m\|_{L^{p_j}}^{q_j} / t^{q_j}$

$\Downarrow$

$$\lambda_{Tf_m}(c_{n,m} 2^n) \lesssim \|f_m\|_{L^{p_j}}^{q_j} \, 2^{-nq_j} \, c_{n,m}^{-q_j}$$

$$2^m \le |f_m| \le 2^{m+1} \xrightarrow[\lambda_f \text{ mon.} \downarrow]{} \|f_m\|_{L^{p_j}}^{p_j} = p_j \int_0^\infty \lambda_{f_m}(t)\, t^{p_j} \frac{dt}{t} \le p_j \, \lambda_{f_m}(2^m) \int_0^{2^{m+1}} t^{p_j} \frac{dt}{t}$$

$$\Rightarrow \|f_m\|_{L^{p_j}}^{p_j} \lesssim 2^{mp_j} \lambda_f(2^m)$$

$$\lesssim c_{n,m}^{-q_j} \, 2^{-nq_j} \, 2^{mq_j} \, \lambda_f(2^m)^{q_j/p_j} =: K_{n,m}^j$$

Thus (***) follows if

$$\left(\sum_n \lambda_{Tf}(2^n)\, 2^{nq_\theta} \lesssim \right) \sum_{n,m} 2^{nq_\theta} \min\{K_{n,m}^0, K_{n,m}^1\} \lesssim 1$$

Since $\sum_{m \in \mathbb{Z}} \lambda_f(2^m)\, 2^{mp_\theta} \lesssim 1$ by normalization $\|f\|_{L^{p_\theta}} = 1$

$$\Longrightarrow \lambda_f(2^m)\, 2^{mp_\theta} \lesssim 1 \Rightarrow \left(\lambda_f(2^m)\, 2^{mp_\theta}\right)^{q_j/p_j} \lesssim \lambda_f(2^m)\, 2^{mp_\theta}$$

Some simple algebra shows that for $\frac{1}{p_i} = \frac{1}{p_\theta} + x_i$,
$\alpha = \frac{1}{x_i}\left(\frac{1}{q_i} - \frac{1}{q_\theta}\right)$ the choice

$c_{n,m} \sim 2^{-|n\alpha q_\theta - mp_\theta| \min(|x_0|,|x_1|)/2}$ will do the job. $\blacksquare$

Remarks on interpolation theorems:

	"complex"	"real"
basic theorem	Riesz–Thorin	Marcinkiewicz
key ideas	Maximum modulus principle from complex analysis	decompose functions into several pieces, e.g "large" and "small" parts, estimate independently, optimize.
assumptions:	T bounded in extreme cases	T ~~weakly~~ bounded in extreme cases
numerical value for $\lVert T\rVert$	sharper by a constant than real method	
generalizations	• holomorphic families of T's (Stein's theorem) • simple extension to general Banach spaces, multilinear operators,...	• nonlinear operators • certain non-Banach spaces of functions