behaves like $\log|\xi|$ as $|\xi| \to \infty$. One can then deduce that, for $n = 1$,

$$f(x) = \varphi(x)\ \log|x|\ \mathrm{sgn}\,|x| \Longrightarrow \hat{f}(\xi) \sim C\,\xi^{-1}\ \log|\xi|, \quad |\xi| \to \infty. \tag{2.12}$$

Thus Proposition 2.5 does not extend to the case $n = 1$, $m = -1$. However, we note that, in this case, $\hat{f}$ belongs to $S_1^{-1+\varepsilon}(\mathbb{R})$, for all $\varepsilon > 0$. In contrast to (2.12), note that, again for $n = 1$,

$$g(x) = \varphi(x)\ \log|x| \Longrightarrow \hat{g}(\xi) \sim C\,|\xi|^{-1}, \quad |\xi| \to \infty. \tag{2.13}$$

In this case, $(d/dx)\log|x| = PV(1/x)$.

Of considerable utility is the classification of $\mathcal{F}(S^m_{cl}(\mathbb{R}^n))$. When $m = -j$ is a negative integer, this was effectively solved in §§8 and 9 of Chapter 3. The following result is what follows from the proof of Proposition 9.2 in Chapter 3.

Proposition 2.6. *Assume $q \in \mathcal{S}'_0(\mathbb{R}^n) \cap L^1_{loc}(\mathbb{R}^n)$. Let $j = 1, 2, 3, \dots$. Then $q = \hat{P}$ for some $P \in S^{-j}_{cl}(\mathbb{R}^n)$ if and only if*

$$q \sim \sum_{\ell \ge 0} \big(q_\ell + p_\ell(x)\log|x|\big), \tag{2.14}$$

where

$$q_\ell \in \mathcal{H}^{\#}_{j+\ell-n}(\mathbb{R}^n), \tag{2.15}$$

and $p_\ell(x)$ is a polynomial homogeneous of degree $j+\ell-n$; these log coefficients appear only for $\ell \ge n - j$.

We recall that $\mathcal{H}^{\#}_\mu(\mathbb{R}^n)$ is the space of distributions on $\mathbb{R}^n$, homogeneous of degree μ, which are smooth on $\mathbb{R}^n \setminus 0$. For $\mu > -n$, $\mathcal{H}^{\#}_\mu(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^n)$. The meaning of the expansion (2.14) is that, for any $k \in \mathbb{Z}^+$, there is an $N < \infty$ such that the difference between q and the sum over $\ell < N$ belongs to $C^k(\mathbb{R}^n)$. Note that, for $n = 1$, the function $g(x)$ in (2.13) is of the form (2.14), but the function $f(x)$ in (2.12) is not.

To go from the proof of Proposition 9.2 of Chapter 3 to the result stated above, it suffices to note explicitly that

$$\varphi(x)x^\alpha \log|x| \in \mathcal{F}\big(S_1^{-n-|\alpha|}(\mathbb{R}^n)\big), \tag{2.16}$$

where φ is the cut-off used before. Since $\mathcal{F}$ intertwines D^α_ξ and multiplication by x^α, it suffices to verify the case $\alpha = 0$, and this follows from the formula (2.11), with x and ξ interchanged.

We can also classify Schwartz kernels of operators in $OPS^m_{1,0}$ and OPS^m_{cl}, if we write the kernel K of (2.2) in the form

$$K(x, y) = L(x, x - y), \tag{2.17}$$

with

$$L(x, z) = (2\pi)^{-n} \int p(x, \xi)e^{iz\cdot\xi}\, d\xi. \tag{2.18}$$

The following two results follow from the arguments given above.

Proposition 2.7. *Assume $-n < m < 0$. Let $L \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ be a smooth function of x with values in $\mathcal{S}'_0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Then (2.17) defines the Schwartz kernel of an operator in $OPS^m_{1,0}$ if and only if, for $z \ne 0$,*

$$|D^\beta_x D^\gamma_z L(x, z)| \le C_{\beta\gamma}\, |z|^{-n-m-|\gamma|}. \tag{2.19}$$

Proposition 2.8. *Assume $L \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ is a smooth function of x with values in $\mathcal{S}'_0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Let $j = 1, 2, 3, \dots$. Then (2.17) defines the Schwartz kernel of an operator in OPS^{-j}_{cl} if and only if*

$$L(x, z) \sim \sum_{\ell \ge 0} \big(q_\ell(x, z) + p_\ell(x, z)\log|z|\big), \tag{2.20}$$

where each $D^\beta_x q_\ell(x, \cdot)$ is a bounded continuous function of x with values in $\mathcal{H}^{\#}_{j+\ell-n}$, and $p_\ell(x, z)$ is a polynomial homogeneous of degree $j + \ell - n$ in z, with coefficients that are bounded, together with all their x-derivatives.

Exercises

1. Using the proof of Proposition 2.2, show that, given $p(x, \xi)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$, then
$$|D^\beta_x D^\alpha_\xi p(x, \xi)| \le C'\langle\xi\rangle^{-|\alpha|+|\beta|}, \quad \text{for } |\beta| \le 1,\ |\alpha| \le n + 1 + |\beta|,$$
implies
$$|K(x, y)| \le C|x - y|^{-n} \quad\text{and}\quad |\nabla_{x,y}K(x, y)| \le C|x - y|^{-n-1}.$$
2. If the map κ is given by (2.2) (i.e., $\kappa(p) = K$) show that we get an isomorphism $\kappa : \mathcal{S}'(\mathbb{R}^{2n}) \to \mathcal{S}'(\mathbb{R}^{2n})$. Reconsider Exercise 3 of §1.
3. Show that κ, defined in Exercise 2, gives an isomorphism (isometric up to a scalar factor) $\kappa : L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^{2n})$. Deduce that $p(x, D)$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^n)$, precisely when $p(x, \xi) \in L^2(\mathbb{R}^{2n})$.

3. Adjoints and products

Given $p(x, \xi) \in S^m_{\rho,\delta}$, we obtain readily from the definition that the adjoint is given by

$$p(x, D)^*v = (2\pi)^{-n} \int p(y, \xi)^* e^{i(x-y)\cdot\xi} v(y)\, dy\, d\xi. \tag{3.1}$$

This is not quite in the form (1.3), as the amplitude $p(y, \xi)^*$ is not a function of (x, ξ). We need to transform (3.1) into such a form.

Before continuing the analysis of (3.1), we are motivated to look at a general class of operators

$$Au(x) = (2\pi)^{-n} \int a(x, y, \xi)e^{i(x-y)\cdot\xi} u(y)\, dy\, d\xi. \tag{3.2}$$

We assume

$$|D_y^\gamma D_x^\beta D_\xi^\alpha a(x,y,\xi)| \le C_{\alpha\beta\gamma} \langle\xi\rangle^{m-\rho|\alpha|+\delta_1|\beta|+\delta_2|\gamma|} \tag{3.3}$$

and then say $a(x,y,\xi) \in S^m_{\rho,\delta_1,\delta_2}$. A brief calculation transforms (3.2) into

$$(2\pi)^{-n} \int q(x,\xi) e^{i(x-y)\cdot\xi} u(y)\, dy\, d\xi, \tag{3.4}$$

with

$$\begin{aligned} q(x,\xi) &= (2\pi)^{-n} \int a(x,y,\eta) e^{i(x-y)\cdot(\eta-\xi)}\, dy\, d\eta \\ &= e^{iD_\xi\cdot D_y} a(x,y,\xi)|_{y=x}. \end{aligned} \tag{3.5}$$

Note that a formal expansion $e^{iD_\xi \cdot D_y} = I + iD_\xi \cdot D_y - (1/2)(D_\xi \cdot D_y)^2 + \cdots$ gives

$$q(x,\xi) \sim \sum_{\alpha\ge 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha a(x,y,\xi)\big|_{y=x}. \tag{3.6}$$

If $a(x,y,\xi) \in S^m_{\rho,\delta_1,\delta_2}$, with $0 \le \delta_2 < \rho \le 1$, then the general term in (3.6) belongs to $S^{m-(\rho-\delta_2)|\alpha|}_{\rho,\delta}$, where $\delta = \max(\delta_1,\delta_2)$, so the sum on the right is formally asymptotic. This suggests the following result:

Proposition 3.1. *If $a(x,y,\xi) \in S^m_{\rho,\delta_1,\delta_2}$, with $0 \le \delta_2 < \rho \le 1$, then (3.2) defines an operator*

$$A \in OPS^m_{\rho,\delta}, \quad \delta = \max(\delta_1,\delta_2).$$

Furthermore, $A = q(x,D)$, where $q(x,\xi)$ has the asymptotic expansion (3.6), in the sense that

$$q(x,\xi) - \sum_{|\alpha|<N} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha a(x,y,\xi)\big|_{y=x} = r_N(x,\xi) \in S^{m-N(\rho-\delta_2)}_{\rho,\delta}.$$

To prove this proposition, one can first show that the Schwartz kernel

$$K(x,y) = (2\pi)^{-n} \int a(x,y,\xi) e^{i(x-y)\cdot\xi}\, d\xi$$

satisfies the same estimates as established in Proposition 2.1, and hence, altering A only by an operator in $OPS^{-\infty}$, we can assume $a(x,y,\xi)$ is supported on $|x-y| \le 1$. Let

$$\hat{b}(x,\eta,\xi) = (2\pi)^{-n} \int a(x,x+y,\xi) e^{-iy\cdot\eta}\, dy, \tag{3.7}$$

so

$$q(x,\xi) = \int \hat{b}(x,\eta,\xi+\eta)\, d\eta. \tag{3.8}$$

The hypotheses on $a(x,y,\xi)$ imply

$$|D_x^\beta D_\xi^\alpha \hat{b}(x,\eta,\xi)| \le C_{\nu\alpha\beta} \langle\xi\rangle^{m+\delta|\beta|+\delta_2\nu-\rho|\alpha|} \langle\eta\rangle^{-\nu}, \tag{3.9}$$

where $\delta = \max(\delta_1,\delta_2)$. Since $\delta_2 < 1$, it follows that $q(x,\xi)$ and any of its derivatives can be bounded by some power of $\langle\xi\rangle$.

Now a power-series expansion of $\hat{b}(x,\eta,\xi+\eta)$ in the last argument about ξ gives

$$\begin{aligned} &\Big|\hat{b}(x,\eta,\xi+\eta) - \sum_{|\alpha|<N} \frac{1}{\alpha!} (iD_\xi)^\alpha \hat{b}(x,\eta,\xi)\eta^\alpha\Big| \\ &\qquad \le C_\nu |\eta|^N \langle\eta\rangle^{-\nu} \sup_{0\le t\le 1} \langle\xi+t\eta\rangle^{m+\delta_2\nu-\rho N}. \end{aligned} \tag{3.10}$$

Taking $\nu = N$, we get a bound on the left side of (3.10) by

$$C\langle\xi\rangle^{m-(\rho-\delta_2)N} \quad \text{if } |\eta| \le \frac{1}{2}|\xi|, \tag{3.11}$$

while taking ν large, we get a bound by any power of $\langle\eta\rangle^{-1}$ for $|\xi| \le 2|\eta|$. Hence

$$\Big|q(x,\xi) - \sum_{|\alpha|<N} \frac{1}{\alpha!} (iD_\xi)^\alpha D_y^\alpha a(x,x+y,\xi)\big|_{y=0}\Big| \le C\langle\xi\rangle^{m+n-(\rho-\delta_2)N}. \tag{3.12}$$

The proposition follows from this, plus similar estimates on the difference when derivatives are applied.

If we apply Proposition 3.1 to (3.1), we obtain:

Proposition 3.2. *If $p(x,D) \in OPS^m_{\rho,\delta}$, $0 \le \delta < \rho \le 1$, then*

$$p(x,D)^* = p^*(x,D) \in OPS^m_{\rho,\delta}, \tag{3.13}$$

with

$$p^*(x,\xi) \sim \sum_{\alpha\ge 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha p(x,\xi)^*. \tag{3.14}$$

The result for products of pseudodifferential operators is the following.

Proposition 3.3. *Given $p_j(x,D) \in OPS^{m_j}_{\rho_j,\delta_j}$, suppose*

$$0 \le \delta_2 < \rho \le 1, \quad \textit{with } \rho = \min(\rho_1,\rho_2). \tag{3.15}$$

Then

$$p_1(x,D) p_2(x,D) = q(x,D) \in OPS^{m_1+m_2}_{\rho,\delta}, \tag{3.16}$$

with $\delta = \max(\delta_1,\delta_2)$, and

$$q(x,\xi) \sim \sum_{\alpha\ge 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p_1(x,\xi)\, D_x^\alpha p_2(x,\xi). \tag{3.17}$$

This can be proved by writing

$$p_1(x,D)p_2(x,D)u = p_1(x,D)p_2^*(x,D)^*u = Au, \tag{3.18}$$

for A as in (3.2), with

$$a(x,y,\xi) = p_1(x,\xi)p_2^*(y,\xi)^*, \tag{3.19}$$

and then applying Propositions 3.1 and 3.2, to obtain (3.16), with

$$q(x,\xi) \sim \sum_{\gamma,\sigma\geq 0} \frac{i^{|\sigma|-|\gamma|}}{\sigma!\gamma!} D_\xi^\sigma D_y^\sigma \Big(p_1(x,\xi) D_\xi^\gamma D_x^\gamma p_2(y,\xi)\Big)\Big|_{y=x}. \tag{3.20}$$

The general term in this sum is equal to

$$\frac{i^{|\sigma|-|\gamma|}}{\sigma!\gamma!} D_\xi^\sigma \Big(p_1(x,\xi) D_\xi^\gamma D_x^{\gamma+\sigma} p_2(x,\xi)\Big).$$

Evaluating this by the product rule

$$D_\xi^\sigma(uv) = \sum_{\alpha+\beta=\sigma} \binom{\sigma}{\alpha} D_\xi^\alpha u \cdot D_\xi^\beta v$$

gives

$$q(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p_1(x,\xi) \sum_{\beta,\gamma} \frac{i^{|\beta|-|\gamma|}}{\beta!\gamma!} D_\xi^{\beta+\gamma} D_x^{\beta+\gamma+\alpha} p_2(x,\xi). \tag{3.21}$$

That this yields (3.17) follows from the fact that, whenever $|\mu| > 0$,

$$\sum_{\beta+\gamma=\mu} \frac{i^{|\beta|-|\gamma|}}{\beta!\gamma!} D_\xi^{\beta+\gamma} D_x^{\beta+\gamma+\alpha} p_2(x,\xi) = 0, \tag{3.22}$$

an identity we leave as an exercise.

An alternative approach to a proof of Proposition 3.3 is to compute directly that $p_1(x,D)p_2(x,D) = q(x,D)$, with

$$\begin{aligned} q(x,\xi) &= (2\pi)^{-n} \int p_1(x,\eta)p_2(y,\xi)e^{i(x-y)\cdot(\eta-\xi)}\, d\eta\, dy \\ &= e^{iD_\eta\cdot D_y} p_1(x,\eta)p_2(y,\xi)\big|_{y=x,\eta=\xi}, \end{aligned} \tag{3.23}$$

and then apply an analysis such as used to prove Proposition 3.1. Carrying out this latter approach has the advantage that the hypothesis (3.15) can be weakened to

$$0 \leq \delta_2 < \rho_1 \leq 1,$$

which is quite natural since the right side of (3.17) is formally asymptotic under such a hypothesis. Also, the symbol expansion (3.17) is more easily seen from (3.23).

Note that if $P_j = p_j(x,D) \in OPS^{m_j}_{\rho,\delta}$ are scalar, and $0 \leq \delta < \rho \leq 1$, then the leading terms in the expansions of the symbols of P_1P_2 and P_2P_1 agree. It follows that the commutator

$$[P_1, P_2] = P_1P_2 - P_2P_1$$

has order lower than $m_1 + m_2$. In fact, the symbol expansion (3.17) implies

$$P_j \in OPS^{m_j}_{\rho,\delta} \text{ scalar} \Longrightarrow [P_1, P_2] \in OPS^{m_1+m_2-(\rho-\delta)}_{\rho,\delta}. \tag{3.24}$$

Also, looking at the sum over $|\alpha| = 1$ in (3.17), we see that the leading term in the expansion of the symbol of $[P_1, P_2]$ is given in terms of the Poisson bracket:

$$[P_1, P_2] = q(x,D), \quad q(x,\xi) = \frac{1}{i}\{p_1, p_2\}(x,\xi) \bmod S^{m_1+m_2-2(\rho-\delta)}_{\rho,\delta}. \tag{3.25}$$

The Poisson bracket $\{p_1, p_2\}$ is defined by

$$\{p_1, p_2\}(x,\xi) = \sum_j \frac{\partial p_1}{\partial \xi_j}\frac{\partial p_2}{\partial x_j} - \frac{\partial p_1}{\partial x_j}\frac{\partial p_2}{\partial \xi_j}, \tag{3.26}$$

as in §10 of Chapter 1.

The result (3.25) plays an important role in the treatment of Egorov's theorem, in §8.

Exercises

1. Writing $a_j(x,D)$ in the form (1.10), that is,

$$a_j(x,D) = \int \hat{a}_j(q,p)e^{iq\cdot X}e^{ip\cdot D}\, dq\, dp, \tag{3.27}$$

use the formula (1.11) for $e^{ip\cdot D}e^{iq'\cdot X}$ to express $a_1(x,D)a_2(x,D)$ as a $4n$-fold integral. Show that it gives (3.20).

2. If $Q(x,x)$ is any nondegenerate, symmetric, bilinear form on $\mathbb{R}^n$, calculate the kernel $K_Q(x,y,t)$ for which

$$e^{itQ(D,D)}u(x) = \int_{\mathbb{R}^n} K_Q(x,y,t)\, u(y)\, dy. \tag{3.28}$$

In case $x \in \mathbb{R}^n$ is replaced by $(x,\xi) \in \mathbb{R}^{2n}$, use this to verify (3.5).
(*Hint*: Diagonalize Q and recall the treatment of $e^{it\Delta}$ in (6.42) of Chapter 3, giving

$$e^{-it\Delta}\delta(x) = (-4\pi i t)^{-n/2}\, e^{|x|^2/4it}, \quad x \in \mathbb{R}^n.$$

Compare the treatment of the stationary phase method in Appendix B of Chapter 6.)

3. Establish the identity (3.22), used in the proof of Proposition 3.3.
(*Hint*: The left side of (3.22) is equal to

$$\Bigg(\sum_{\beta+\gamma=\mu} \frac{i^{|\beta|-|\gamma|}}{\beta!\gamma!}\Bigg) D_\xi^\mu D_x^{\mu+\alpha} p_2(x,\xi),$$

so one needs to show that the quantity in parentheses here vanishes if $|\mu| > 0$. To see this, make an expansion of $(z+w)^\mu$, and set $z = (i,\ldots,i)$, $w = (-i,\ldots,-i)$.)

4. Elliptic operators and parametrices

We say $p(x, D) \in OPS^m_{\rho,\delta}$ is elliptic if, for some $r < \infty$,

$$|p(x,\xi)^{-1}| \le C\langle\xi\rangle^{-m}, \quad \text{for } |\xi| \ge r. \tag{4.1}$$

Thus, if $\psi(\xi) \in C^\infty(\mathbb{R}^n)$ is equal to 0 for $|\xi| \le r$, 1 for $|\xi| \ge 2r$, it follows easily from the chain rule that

$$\psi(\xi)p(x,\xi)^{-1} = q_0(x,\xi) \in S^{-m}_{\rho,\delta}. \tag{4.2}$$

As long as $0 \le \delta < \rho \le 1$, we can apply Proposition 3.3 to obtain

$$\begin{aligned} q_0(x,D)p(x,D) &= I + r_0(x,D), \\ p(x,D)q_0(x,D) &= I + \tilde{r}_0(x,D), \end{aligned} \tag{4.3}$$

with

$$r_0(x,\xi),\ \tilde{r}_0(x,\xi) \in S^{-(\rho-\delta)}_{\rho,\delta}. \tag{4.4}$$

Using the formal expansion

$$I - r_0(x,D) + r_0(x,D)^2 - \cdots \sim I + s(x,D) \in OPS^0_{\rho,\delta} \tag{4.5}$$

and setting $q(x,D) = (I + s(x,D))q_0(x,D) \in OPS^{-m}_{\rho,\delta}$, we have

$$q(x,D)p(x,D) = I + r(x,D), \quad r(x,\xi) \in S^{-\infty}. \tag{4.6}$$

Similarly, we obtain $\tilde{q}(x,D) \in OPS^{-m}_{\rho,\delta}$ satisfying

$$p(x,D)\tilde{q}(x,D) = I + \tilde{r}(x,D), \quad \tilde{r}(x,\xi) \in S^{-\infty}. \tag{4.7}$$

But evaluating

$$\big(q(x,D)p(x,D)\big)\tilde{q}(x,D) = q(x,D)\big(p(x,D)\tilde{q}(x,D)\big) \tag{4.8}$$

yields $q(x,D) = \tilde{q}(x,D) \bmod OPS^{-\infty}$, so in fact

$$\begin{aligned} q(x,D)p(x,D) &= I \bmod OPS^{-\infty}, \\ p(x,D)q(x,D) &= I \bmod OPS^{-\infty}. \end{aligned} \tag{4.9}$$

We say that $q(x,D)$ is a *two-sided parametrix* for $p(x,D)$.
The parametrix can establish the local regularity of a solution to

$$p(x,D)u = f. \tag{4.10}$$

Suppose $u, f \in \mathcal{S}'(\mathbb{R}^n)$ and $p(x,D) \in OPS^m_{\rho,\delta}$ is elliptic, with $0 \le \delta < \rho \le 1$. Constructing $q(x,D) \in OPS^{-m}_{\rho,\delta}$ as in (4.6), we have

$$u = q(x,D)f - r(x,D)u. \tag{4.11}$$

Now a simple analysis parallel to (1.7) implies that

$$R \in OPS^{-\infty} \Longrightarrow R : \mathcal{E}' \longrightarrow \mathcal{S}. \tag{4.12}$$

By duality, since taking adjoints preserves $OPS^{-\infty}$,

$$R \in OPS^{-\infty} \Longrightarrow R : \mathcal{S}' \longrightarrow C^\infty. \tag{4.13}$$

Thus (4.11) implies

$$u = q(x,D)f \bmod C^\infty. \tag{4.14}$$

Applying the pseudolocal property to (4.10) and (4.14), we have the following elliptic regularity result.

Proposition 4.1. *If $p(x,D) \in OPS^m_{\rho,\delta}$ is elliptic and $0 \le \delta < \rho \le 1$, then, for any $u \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\text{sing supp } p(x,D)u = \text{sing supp } u. \tag{4.15}$$

More refined elliptic regularity involves keeping track of Sobolev space regularity. As we have the parametrix, this will follow simply from mapping properties of pseudodifferential operators, to be established in subsequent sections.

Exercises

1. Give the details of the implication (4.1) ⇒ (4.2) when $p(x,\xi) \in S^m_{\rho,\delta}$, $0 \le \delta < \rho \le 1$. Include the case where $p(x,\xi)$ is a $k \times k$ matrix-valued function, using such identities as
$$\frac{\partial}{\partial x_j}\, p(x,\xi)^{-1} = -p(x,\xi)^{-1}\, \frac{\partial p}{\partial x_j}\, p(x,\xi)^{-1}.$$
2. On $\mathbb{R} \times \mathbb{R}^n$, consider the operator $P = \partial/\partial t - L(x, D_x)$, where
$$L(x,D_x) = \sum a_{jk}(x)\, \partial_j \partial_k u + \sum b_j(x)\, \partial_j u + c(x)u.$$
Assume that the coefficients are smooth and bounded, with all their derivatives, and that L satisfies the strong ellipticity condition
$$-L_2(x,\xi) = \sum a_{jk}(x)\xi_j\xi_k \ge C|\xi|^2, \quad C > 0.$$
Show that
$$\big(i\tau - L_2(x,\xi) + 1\big)^{-1} = E(t,x,\tau,\xi) \in S^{-1}_{1/2,0}.$$
Show that $E(t,x,D)P = A_1(t,x,D)$ and $PE(t,x,D) = A_2(t,x,D)$, where $A_j \in OPS^0_{1/2,0}$ are elliptic. Then, using Proposition 4.1, construct a parametrix for P, belonging to $OPS^{-1}_{1/2,0}$.
3. Assume $-n < m < 0$, and suppose $P = p(x,D) \in OPS^m_{cl}$ has Schwartz kernel $K(x,y) = L(x, x-y)$. Suppose that, at $x_0 \in \mathbb{R}^n$,
$$L(x_0, z) \sim a|z|^{-m-n} + \cdots, \quad z \to 0,$$
with $a \ne 0$, the remainder terms being progressively smoother. Show that
$$p_m(x_0,\xi) = b|\xi|^m, \quad b \ne 0,$$
and hence that P is elliptic near x_0.

4. Let $P = (P_{jk})$ be a $K \times K$ matrix of operators in OPS^*. It is said to be "elliptic in the sense of Douglis and Nirenberg" if there are numbers $a_j, b_j, 1 \le j \le K$, such that $P_{jk} \in OPS^{a_j+b_k}$ and the matrix of principal symbols has nonvanishing determinant (homogeneous of order $\sum(a_j + b_j)$), for $\xi \neq 0$. If Λ^s is as in (1.17), let A be a $K \times K$ diagonal matrix with diagonal entries Λ^{-a_j}, and let B be diagonal, with entries Λ^{-b_j}. Show that this "DN-ellipticity" of P is equivalent to the ellipticity of APB in OPS^0.

5. L^2-estimates

Here we want to obtain L^2-estimates for pseudodifferential operators. The following simple basic estimate will get us started.

Proposition 5.1. *Let (X, μ) be a measure space. Suppose $k(x, y)$ is measurable on $X \times X$ and*

(5.1) $$\int_X |k(x, y)|\, d\mu(x) \le C_1, \quad \int_X |k(x, y)|\, d\mu(y) \le C_2,$$

for all y and x, respectively. Then

(5.2) $$Tu(x) = \int k(x, y)u(y)\, d\mu(y)$$

satisfies

(5.3) $$\|Tu\|_{L^p} \le C_1^{1/p} C_2^{1/q} \|u\|_{L^p},$$

for $p \in [1, \infty]$, with

(5.4) $$\frac{1}{p} + \frac{1}{q} = 1.$$

This is proved in Appendix A on functional analysis; see Proposition 5.1 there. To apply this result when $X = \mathbb{R}^n$ and $k = K$ is the Schwartz kernel of $p(x, D) \in OPS^m_{\rho,\delta}$, note from the proof of Proposition 2.1 that

(5.5) $$|K(x, y)| \le C_N |x - y|^{-N}, \quad \text{for } |x - y| \ge 1$$

as long as $\rho > 0$, while

(5.6) $$|K(x, y)| \le C|x - y|^{-(n-1)}, \quad \text{for } |x - y| \le 1$$

as long as $m < -n + \rho(n - 1)$. (Recall that this last estimate is actually rather crude.) Hence we have the following preliminary result.

Lemma 5.2. *If $p(x, D) \in OPS^m_{\rho,\delta}$, $\rho > 0$, and $m < -n + \rho(n - 1)$, then*

(5.7) $$p(x, D) : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \quad 1 \le p \le \infty.$$

If $p(x, D) \in OPS^m_{1,\delta}$, then (5.7) holds for $m < 0$.

The last observation follows from the improvement of (5.6) given in (2.5). Our main goal in this section is to prove the following.

Theorem 5.3. *If $p(x, D) \in OPS^0_{\rho,\delta}$ and $0 \le \delta < \rho \le 1$, then*

(5.8) $$p(x, D) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n).$$

The proof we give, following [Ho5], begins with the following result.

Lemma 5.4. *If $p(x, D) \in OPS^{-a}_{\rho,\delta}$, $0 \le \delta < \rho \le 1$, and $a > 0$, then (5.8) holds.*

Proof. Since $\|Pu\|^2_{L^2} = (P^*Pu, u)$, it suffices to prove that some power of $p(x, D)^*p(x, D) = Q$ is bounded on L^2. But $Q^k \in OPS^{-2ka}_{\rho,\delta}$, so for k large enough this follows from Lemma 5.2.

To proceed with the proof of Theorem 5.3, set $q(x, D) = p(x, D)^*p(x, D) \in OPS^0_{\rho,\delta}$, and suppose $|q(x, \xi)| \le M - b$, $b > 0$, so

(5.9) $$M - \operatorname{Re} q(x, \xi) \ge b' > 0.$$

In the matrix case, take $\operatorname{Re} q(x, \xi) = (1/2)\big(q(x, \xi) + q(x, \xi)^*\big)$. It follows that

(5.10) $$A(x, \xi) = \big(M - \operatorname{Re} q(x, \xi)\big)^{1/2} \in S^0_{\rho,\delta}$$

and

(5.11) $$A(x, D)^*A(x, D) = M - q(x, D) + r(x, D), \quad r(x, D) \in OPS^{-(\rho-\delta)}_{\rho,\delta}.$$

Applying Lemma 5.4 to $r(x, D)$, we have

(5.12) $$\begin{aligned} M\|u\|^2_{L^2} - \|p(x, D)u\|^2_{L^2} &= \|A(x, D)u\|^2_{L^2} - (r(x, D)u, u) \\ &\ge -C\|u\|^2_{L^2}, \end{aligned}$$

or

(5.13) $$\|p(x, D)u\|^2 \le (M + C)\|u\|^2_{L^2},$$

finishing the proof.

From these L^2-estimates easily follow L^2-Sobolev space estimates. Recall from Chapter 4 that the Sobolev space $H^s(\mathbb{R}^n)$ is defined as

(5.14) $$H^s(\mathbb{R}^n) = \{u \in S'(\mathbb{R}^n) : \langle\xi\rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^n)\}.$$

Equivalently, with

(5.15) $$\Lambda^s u = \int \langle\xi\rangle^s \hat{u}(\xi) e^{ix\cdot\xi}\, d\xi; \quad \Lambda^s \in OPS^s,$$

we have

(5.16) $$H^s(\mathbb{R}^n) = \Lambda^{-s}L^2(\mathbb{R}^n).$$

The operator calculus easily gives the next proposition:

Proposition 5.5. *If* $p(x, D) \in OPS^m_{\rho,\delta}$, $0 \le \delta < \rho \le 1$, $m, s \in \mathbb{R}$, *then*

$$p(x, D) : H^s(\mathbb{R}^n) \longrightarrow H^{s-m}(\mathbb{R}^m). \tag{5.17}$$

Given Proposition 5.5, one easily obtains the Sobolev regularity of solutions to the elliptic equations studied in §4.

Calderon and Vaillancourt sharpened Theorem 5.3, showing that

$$p(x, \xi) \in S^0_{\rho,\rho},\ 0 \le \rho < 1 \Longrightarrow p(x, D) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n). \tag{5.18}$$

This result, particularly for $\rho = 1/2$, has played an important role in linear PDE, especially in the study of subelliptic operators, but it will not be used in this book. The case $\rho = 0$ is treated in the exercises below.

Another important extension of Theorem 5.3 is that $p(x, D)$ is bounded on $L^p(\mathbb{R}^n)$, for $1 < p < \infty$, when $p(x, \xi) \in S^0_{1,\delta}$. Similarly, Proposition 5.5 extends to a result on L^p-Sobolev spaces, in the case $\rho = 1$. This is important for applications to nonlinear PDE, and will be proved in Chapter 13.

Exercises

Exercises 1–7 present an approach to a proof of the Calderon-Vaillancourt theorem, (5.18), in the case $\rho = 0$. This approach is due to H. O. Cordes [Cor]; see also T. Kato [K] and R. Howe [How]. In these exercises, we assume that $U(y)$ is a (measurable) unitary, operator-valued function on a measure space Y, operating on a Hilbert space $\mathcal{H}$. Assume that, for $f, g \in \mathcal{V}$, a dense subset of $\mathcal{H}$,

$$\int_Y |(U(y)f, g)|^2\, dm(y) = C_0\|f\|^2\,\|g\|^2. \tag{5.19}$$

1. Let $\varphi_0 \in \mathcal{H}$ be a unit vector, and set $\varphi_y = U(y)\varphi_0$. Show that, for any $T \in \mathcal{L}(\mathcal{H})$,

$$C_0^2(Tf_1, f_2) = \int_Y\int_Y L_T(y, y')\,(f_1, \varphi_{y'})\,(\varphi_y, f_2)\, dm(y)\, dm(y'), \tag{5.20}$$

where

$$L_T(y, y') = (T\varphi_{y'}, \varphi_y). \tag{5.21}$$

(*Hint*: Start by showing that $\int (f_1, \varphi_y)(\varphi_y, f_2)\, dm(y) = C_0(f_1, f_2)$.)
A statement equivalent to (5.20) is

$$T = \iint L_T(y, y')\, U(y)\Phi_0 U(y')\, dm(y)\, dm(y'), \tag{5.22}$$

where Φ_0 is the orthogonal projection of $\mathcal{H}$ onto the span of φ_0.

2. For a partial converse, suppose L is measurable on $Y \times Y$ and

$$\int |L(y, y')|\, dm(y) \le C_1, \quad \int |L(y, y')|\, dm(y') \le C_1. \tag{5.23}$$

Define

$$T_L = \iint L(y, y')\, U(y)\Phi_0 U(y')^*\, dm(y)\, dm(y'). \tag{5.24}$$

Show that the operator norm of T_L on $\mathcal{H}$ has the estimate

$$\|T_L\| \le C_0^2 C_1.$$

3. If G is a trace class operator, and we set

$$T_{L,G} = \iint L(y, y')\, U(y)GU(y')^*\, dm(y)\, dm(y'), \tag{5.25}$$

show that

$$\|T_{L,G}\| \le C_0^2 C_1\, \|G\|_{\mathrm{TR}}. \tag{5.26}$$

(*Hint*: In case $G = G^*$, diagonalize G and use Exercise 2.)

4. Suppose $b \in L^\infty(Y)$ and we set

$$T^\#_{b,G} = \int b(y)\, U(y)GU(y)^*\, dm(y). \tag{5.27}$$

Show that

$$\|T^\#_{b,G}\| \le C_0\|b\|_{L^\infty}\, \|G\|_{\mathrm{TR}}. \tag{5.28}$$

5. Let $Y = \mathbb{R}^{2n}$, with Lebesgue measure, $y = (q, p)$. Set $U(y) = e^{iq\cdot X}e^{ip\cdot D} = \tilde{\pi}(0, q, p)$, as in Exercises 1 and 2 of §1. Show that the identity (5.19) holds, for $f, g \in L^2(\mathbb{R}^n) = \mathcal{H}$, with $C_0 = (2\pi)^{-n}$. (*Hint*: Make use of the Plancherel theorem.)
6. Deduce that if $a(x, D)$ is a trace class operator,

$$\|(b * a)(x, D)\|_{\mathcal{L}(L^2)} \le C\|b\|_{L^\infty}\, \|a(x, D)\|_{\mathrm{TR}}. \tag{5.29}$$

(*Hint*: Look at Exercises 3–4 of §1.)

7. Suppose $p(x, \xi) \in S^0_{0,0}$. Set

$$a(x, \xi) = \psi(x)\psi(\xi), \quad b(x, \xi) = (1 - \Delta_x)^k(1 - \Delta_\xi)^k p(x, \xi), \tag{5.30}$$

where k is a positive integer, $\hat{\psi}(\xi) = \langle\xi\rangle^{-2k}$. Show that if k is chosen large enough, then $a(x, D)$ is trace class. Note that, for all $k \in \mathbb{Z}^+$, $b \in L^\infty(\mathbb{R}^{2n})$, provided $p \in S^0_{0,0}$. Show that

$$p(x, D) = (b * a)(x, D), \tag{5.31}$$

and deduce the $\rho = 0$ case of the Calderon-Vaillancourt estimate (5.19).

8. Sharpen the results of problems 3–4 above, showing that

$$\|T_{L,G}\|_{\mathcal{L}(\mathcal{H})} \le C_0^2\, \|L\|_{\mathcal{L}(L^2(Y))}\, \|G\|_{\mathrm{TR}}. \tag{5.32}$$

This is stronger than (5.26) in view of Proposition 5.1.

6. Gårding's inequality

In this section we establish a fundamental estimate, first obtained by L. Gårding in the case of differential operators.

Theorem 6.1. *Assume* $p(x, D) \in OPS^m_{\rho,\delta}$, $0 \le \delta < \rho \le 1$, *and*

$$\text{(6.1)} \qquad Re\ p(x,\xi) \ge C|\xi|^m, \quad \text{for } |\xi| \text{ large.}$$

Then, for any $s \in \mathbb{R}$, *there are* C_0, C_1 *such that, for* $u \in H^{m/2}(\mathbb{R}^n)$,

$$\text{(6.2)} \qquad Re\left(p(x,D)u, u\right) \ge C_0\|u\|^2_{H^{m/2}} - C_1\|u\|^2_{H^s}.$$

Proof. Replacing $p(x, D)$ by $\Lambda^{-m/2}p(x, D)\Lambda^{-m/2}$, we can suppose without loss of generality that $m = 0$. Then, as in the proof of Theorem 5.3, take

$$\text{(6.3)} \qquad A(x,\xi) = \left(\text{Re } p(x,\xi) - \frac{1}{2}C\right)^{1/2} \in S^0_{\rho,\delta},$$

so

$$\text{(6.4)} \qquad \begin{aligned} A(x,D)^*A(x,D) &= \text{ Re } p(x,D) - \frac{1}{2}C + r(x,D), \\ r(x,D) &\in OPS^{-(\rho-\delta)}_{\rho,\delta}. \end{aligned}$$

This gives

$$\text{(6.5)} \qquad \begin{aligned} \text{Re } (p(x,D)u, u) &= \|A(x,D)u\|^2_{L^2} + \frac{1}{2}C\|u\|^2_{L^2} + \left(r(x,D)u, u\right) \\ &\ge \frac{1}{2}C\|u\|^2_{L^2} - C_1\|u\|^2_{H^s} \end{aligned}$$

with $s = -(\rho - \delta)/2$, so (6.2) holds in this case. If $s < -(\rho-\delta)/2 = s_0$, use the simple estimate

$$\text{(6.6)} \qquad \|u\|^2_{H^s} \le \varepsilon\|u\|^2_{L^2} + C(\varepsilon)\|u\|^2_{H^{s_0}}$$

to obtain the desired result in this case.

This Gårding inequality has been improved to a sharp Gårding inequality, of the form

$$\text{(6.7)} \qquad \text{Re } \left(p(x,D)u, u\right) \ge -C\|u\|^2_{L^2} \quad \text{when Re } p(x,\xi) \ge 0,$$

first for scalar $p(x,\xi) \in S^1_{1,0}$, by Hörmander, then for matrix-valued symbols, with Re $p(x,\xi)$ standing for $(1/2)\left(p(x,\xi) + p(x,\xi)^*\right)$, by P. Lax and L. Nirenberg. Proofs and some implications can be found in Vol. 3 of [Ho5], and in [T1] and [Tre]. A very strong improvement due to C. Fefferman and D. Phong [FP] is that (6.7) holds for scalar $p(x,\xi) \in S^2_{1,0}$. See also [Ho5] and [F] for further discussion.

Exercises

1. Suppose $m > 0$ and $p(x, D) \in OPS^m_{1,0}$ has a symbol satisfying (6.1). Examine the solvability of

$$\frac{\partial u}{\partial t} = p(x,D)u,$$

for $u = u(t,x)$, $u(0,x) = f \in H^s(\mathbb{R}^n)$.
(*Hint*: Look ahead at §7 for some useful techniques. Solve

$$\frac{\partial u_\varepsilon}{\partial t} = J_\varepsilon p(x,D) J_\varepsilon u_\varepsilon$$

and estimate $(d/dt)\|\Lambda^s u_\varepsilon(t)\|^2_{L^2}$, making use of Gårding's inequality.)

7. Hyperbolic evolution equations

In this section we examine first-order systems of the form

$$\text{(7.1)} \qquad \frac{\partial u}{\partial t} = L(t,x,D_x)u + g(t,x), \quad u(0) = f.$$

We assume $L(t,x,\xi) \in S^1_{1,0}$, with smooth dependence on t, so

$$\text{(7.2)} \qquad |D^j_t D^\beta_x D^\alpha_\xi L(t,x;\xi)| \le C_{j\alpha\beta}\langle\xi\rangle^{1-|\alpha|}.$$

Here $L(t,x,\xi)$ is a $K \times K$ matrix-valued function, and we make the hypothesis of symmetric hyperbolicity:

$$\text{(7.3)} \qquad L(t,x,\xi)^* + L(t,x,\xi) \in S^0_{1,0}.$$

We suppose $f \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, $g \in C(\mathbb{R}, H^s(\mathbb{R}^n))$.

Our strategy will be to obtain a solution to (7.1) as a limit of solutions u_ε to

$$\text{(7.4)} \qquad \frac{\partial u_\varepsilon}{\partial t} = J_\varepsilon L J_\varepsilon u_\varepsilon + g, \quad u_\varepsilon(0) = f,$$

where

$$\text{(7.5)} \qquad J_\varepsilon = \varphi(\varepsilon D_x),$$

for some $\varphi(\xi) \in \mathcal{S}(\mathbb{R}^n)$, $\varphi(0) = 1$. The family of operators J_ε is called a *Friedrichs mollifier*. Note that, for any $\varepsilon > 0$, $J_\varepsilon \in OPS^{-\infty}$, while, for $\varepsilon \in (0,1]$, J_ε is bounded in $OPS^0_{1,0}$.

For any $\varepsilon > 0$, $J_\varepsilon L J_\varepsilon$ is a bounded linear operator on each H^s, and solvability of (7.4) is elementary. Our next task is to obtain estimates on u_ε, independent of $\varepsilon \in (0,1]$. Use the norm $\|u\|_{H^s} = \|\Lambda^s u\|_{L^2}$. We derive an estimate for

$$\text{(7.6)} \qquad \frac{d}{dt}\|\Lambda^s u_\varepsilon(t)\|^2_{L^2} = 2(\Lambda^s J_\varepsilon L J_\varepsilon u_\varepsilon) + 2(\Lambda^s g, \Lambda^s u_\varepsilon).$$

Write the first two terms on the right as

$$\text{(7.7)} \qquad 2(L\Lambda^s J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon) + 2([\Lambda^s, L]J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon).$$

By (7.3), $L + L^* = B(t,x,D) \in OPS^0_{1,0}$, so the first term in (7.7) is equal to

$$\text{(7.8)} \qquad \left(B(t,x,D)\Lambda^s J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon\right) \le C\|J_\varepsilon u_\varepsilon\|^2_{H^s}.$$

provided

$$\rho > \frac{1}{2}, \quad \delta = 1 - \rho. \tag{8.18}$$

One needs $\delta = 1 - \rho$ to ensure that $p(\mathcal{C}(t)(x,\xi)) \in S^m_{\rho,\delta}$, and one needs $\rho > \delta$ to ensure that the transport equations generate $q_j(t,x,\xi)$ of progressively lower order.

Exercises

1. Let $\chi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism that is a linear map outside some compact set. Define $\chi^* : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by $\chi^* f(x) = f(\chi(x))$. Show that

$$P \in OPS^m_{1,0} \Longrightarrow (\chi^*)^{-1} P \chi^* \in OPS^m_{1,0}. \tag{8.19}$$

(*Hint*: Reduce to the case where χ is homotopic to a linear map through diffeomorphisms, and show that the result in that case is a special case of Theorem 8.1, where $A(t,x,D)$ is a t-dependent family of real vector fields on $\mathbb{R}^n$.)

2. Let $a \in C_0^\infty(\mathbb{R}^n)$, $\varphi \in C^\infty(\mathbb{R}^n)$ be real-valued, and $\nabla\varphi \neq 0$ on supp a. If $P \in OPS^m$, show that

$$P\big(a\, e^{i\lambda\varphi}\big) = b(x,\lambda)\, e^{i\lambda\varphi(x)}, \tag{8.20}$$

where

$$b(x,\lambda) \sim \lambda^m\big[b_0^\pm(x) + b_1^\pm(x)\lambda^{-1} + \cdots\big], \quad \lambda \to \pm\infty. \tag{8.21}$$

(*Hint*: Using a partition of unity and Exercise 1, reduce to the case $\varphi(x) = x\cdot\xi$, for some $\xi \in \mathbb{R}^n \setminus 0$.)

3. If a and φ are as in Exercise 2 above and Γ_r is as in Exercise 2 of §7, show that, mod $O(\lambda^{-\infty})$,

$$\Gamma_r\big(a\, e^{i\lambda\varphi}\big) = \cos r\sqrt{-\Delta}\big(A_r(x,\lambda)e^{i\lambda\varphi}\big) + \frac{\sin r\sqrt{-\Delta}}{\sqrt{-\Delta}}\big(B_r(x,\lambda)e^{i\lambda\varphi}\big), \tag{8.22}$$

where

$$A_r(x,\lambda) \sim \lambda^{-1/2}\big[a_{0r}^\pm(x) + a_{1r}^\pm(x)\lambda^{-1} + \cdots\big],$$
$$B_r(x,\lambda) \sim \lambda^{1/2}\big[b_{0r}^\pm(x) + b_{1r}^\pm(x)\lambda^{-1} + \cdots\big],$$

as $\lambda \to \pm\infty$.

9. Microlocal regularity

We define the notion of wave front set of a distribution $u \in H^{-\infty}(\mathbb{R}^n) = \cup_s H^s(\mathbb{R}^n)$, which refines the notion of singular support. If $p(x,\xi) \in S^m$ has principal symbol $p_m(x,\xi)$, homogeneous in ξ, then the characteristic set of $P = p(x,D)$ is given by

$$\text{Char}\, P = \{(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0) : p_m(x,\xi) = 0\}. \tag{9.1}$$

If $p_m(x,\xi)$ is a $K \times K$ matrix, take the determinant. Equivalently, (x_0,ξ_0) is noncharacteristic for P, or P is elliptic at (x_0,ξ_0), if $|p(x,\xi)^{-1}| \le C|\xi|^{-m}$, for (x,ξ) in a small conic neighborhood of (x_0,ξ_0) and $|\xi|$ large. By definition, a conic set is invariant under the dilations $(x,\xi) \mapsto (x, r\xi)$, $r \in (0,\infty)$. The wave front set is defined by

$$\text{WF}(u) = \bigcap\{\text{Char}\, P : P \in OPS^0,\ Pu \in C^\infty\}. \tag{9.2}$$

Clearly, WF(u) is a closed conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$.

Proposition 9.1. *If π is the projection $(x,\xi) \mapsto x$, then*

$$\pi(WF(u)) = \textit{sing supp}\ u.$$

Proof. If $x_0 \notin$ sing supp u, there is a $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi = 1$ near x_0, such that $\varphi u \in C_0^\infty(\mathbb{R}^n)$. Clearly, $(x_0,\xi) \notin$ Char φ for any $\xi \neq 0$, so $\pi(WF(u)) \subset$ sing supp u.

Conversely, if $x_0 \notin \pi(WF(u))$, then for any $\xi \neq 0$ there is a $Q \in OPS^0$ such that $(x_0,\xi) \notin$ Char Q and $Qu \in C^\infty$. Thus we can construct finitely many $Q_j \in OPS^0$ such that $Q_j u \in C^\infty$ and each (x_0,ξ) (with $|\xi| = 1$) is noncharacteristic for some Q_j. Let $Q = \sum Q_j^* Q_j \in OPS^0$. Then Q is elliptic near x_0 and $Qu \in C^\infty$, so u is C^∞ near x_0.

We define the associated notion of ES(P) for a pseudodifferential operator. Let U be an open conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$. We say that $p(x,\xi) \in S^m_{\rho,\delta}$ has order $-\infty$ on U if for each closed conic set V of U we have estimates, for each N,

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \le C_{\alpha\beta N V}\langle\xi\rangle^{-N}, \quad (x,\xi) \in V. \tag{9.3}$$

If $P = p(x,D) \in OPS^m_{\rho,\delta}$, we define the essential support of P (and of $p(x,\xi)$) to be the smallest closed conic set on the complement of which $p(x,\xi)$ has order $-\infty$. We denote this set by ES(P).

From the symbol calculus of §3, it follows easily that

$$\text{ES}(P_1P_2) \subset \text{ES}(P_1) \cap \text{ES}(P_2) \tag{9.4}$$

provided $P_j \in OPS^{m_j}_{\rho_j,\delta_j}$ and $\rho_1 > \delta_2$. To relate WF(Pu) to WF(u) and ES(P), we begin with the following.

Lemma 9.2. *Let $u \in H^{-\infty}(\mathbb{R}^n)$, and suppose that U is a conic open set satisfying*

$$WF(u) \cap U = \emptyset.$$

If $P \in OPS^m_{\rho,\delta}$, $\rho > 0$, $\delta < 1$, and $ES(P) \subset U$, then $Pu \in C^\infty$.

Proof. Taking $P_0 \in OPS^0$ with symbol identically 1 on a conic neighborhood of ES(P), so $P = PP_0$ mod $OPS^{-\infty}$, it suffices to conclude that $P_0 u \in C^\infty$, so we can specialize the hypothesis to $P \in OPS^0$.

By hypothesis, we can find $Q_j \in OPS^0$ such that $Q_j u \in C^\infty$ and each $(x,\xi) \in ES(P)$ is noncharacteristic for some Q_j, and if $Q = \sum Q_j^* Q_j$, then $Qu \in C^\infty$ and $\text{Char}\, Q \cap ES(P) = \emptyset$. We claim there exists an operator $A \in OPS^0$ such that $AQ = P \bmod OPS^{-\infty}$. Indeed, let $\tilde{Q}$ be an elliptic operator whose symbol equals that of Q on a conic neighborhood of $ES(P)$, and let $\tilde{Q}^{-1}$ denote a parametrix for $\tilde{Q}$. Now simply set set $A = P\tilde{Q}^{-1}$. Consequently, (mod C^∞) $Pu = AQu \in C^\infty$, so the lemma is proved.

We are ready for the basic result on the preservation of wave front sets by a pseudodifferential operator.

Proposition 9.3. *If $u \in H^{-\infty}$ and $P \in OPS^m_{\rho,\delta}$, with $\rho > 0$, $\delta < 1$, then*

$$WF(Pu) \subset WF(u) \cap ES(P). \tag{9.5}$$

Proof. First we show $WF(Pu) \subset ES(P)$. Indeed, if $(x_0,\xi_0) \notin ES(P)$, choose $Q = q(x,D) \in OPS^0$ such that $q(x,\xi) = 1$ on a conic neighborhood of (x_0,ξ_0) and $ES(Q) \cap ES(P) = \emptyset$. Thus $QP \in OPS^{-\infty}$, so $QPu \in C^\infty$. Hence $(x_0,\xi_0) \notin WF(Pu)$.

In order to show that $WF(Pu) \subset WF(u)$, let Γ be any conic neighborhood of $WF(u)$, and write $P = P_1 + P_2$, $P_j \in OPS^m_{\rho,\delta}$, with $ES(P_1) \subset \Gamma$ and $ES(P_2) \cap WF(u) = \emptyset$. By Lemma 9.2, $P_2 u \in C^\infty$. Thus $WF(u) = WF(P_1 u) \subset \Gamma$, which shows $WF(Pu) \subset WF(u)$.

One says that a pseudodifferential operator of type (ρ,δ), with $\rho > 0$ and $\delta < 1$, is *microlocal*. As a corollary, we have the following sharper form of local regularity for elliptic operators, called *microlocal regularity*.

Corollary 9.4. *If $P \in OPS^m_{\rho,\delta}$ is elliptic, $0 \le \delta < \rho \le 1$, then*

$$WF(Pu) = WF(u). \tag{9.6}$$

Proof. We have seen that $WF(Pu) \subset WF(u)$. On the other hand, if $E \in OPS^{-m}_{\rho,\delta}$ is a parametrix for P, we see that $WF(u) = WF(EPu) \subset WF(Pu)$. In fact, by an argument close to the proof of Lemma 9.2, we have for general P that

$$WF(u) \subset WF(Pu) \cup \text{Char}\, P. \tag{9.7}$$

We next discuss how the solution operator e^{itA} to a scalar hyperbolic equation $\partial u/\partial t = iA(x,D)u$ propagates the wave front set. We assume $A(x,\xi) \in S^1_{cl}$, with real principal symbol. Suppose $WF(u) = \Sigma$. Then there is a countable family of operators $p_j(x,D) \in OPS^0$, each of whose complete symbols vanishes in a neighborhood of Σ, but such that

$$\Sigma = \bigcap_j \{(x,\xi) : p_j(x,\xi) = 0\}. \tag{9.8}$$

We know that $p_j(x,D)u \in C^\infty$ for each j. Using Egorov's theorem, we want to construct a family of pseudodifferential operators $q_j(x,D) \in OPS^0$ such that $q_j(x,D)e^{itA}u \in C^\infty$, this family being rich enough to describe the wave front set of $e^{itA}u$.

Indeed, let $q_j(x,D) = e^{itA} p_j(x,D) e^{-itA}$. Egorov's theorem implies that $q_j(x,D) \in OPS^0$ (modulo a smoothing operator) and gives the principal symbol of $q_j(x,D)$. Since $p_j(x,D)u \in C^\infty$, we have $e^{itA}p_j(x,D)u \in C^\infty$, which in turn implies $q_j(x,D)e^{itA}u \in C^\infty$. From this it follows that $WF(e^{itA}u)$ is contained in the intersection of the characteristics of the $q_j(x,D)$, which is precisely $C(t)\Sigma$, the image of Σ under the canonical transformation $C(t)$, generated by H_{A_1}. In other words,

$$WF(e^{itA}u) \subset C(t)WF(u).$$

However, our argument is reversible; $u = e^{-itA}(e^{itA}u)$. Consequently, we have the following result:

Proposition 9.5. *If $A = A(x,D) \in OPS^1$ is scalar with real principal symbol, then, for $u \in H^{-\infty}$,*

$$WF(e^{itA}u) = C(t)WF(u). \tag{9.9}$$

The same argument works for the solution operator $S(t,0)$ to a time-dependent, scalar, hyperbolic equation.

Exercises

1. If $a \in C_0^\infty(\mathbb{R}^n)$, $\varphi \in C^\infty(\mathbb{R}^n)$ is real-valued, $\nabla\varphi \ne 0$ on supp a, as in Exercise 2 of §8, and $P = p(x,D) \in OPS^m$, so
$$P\big(a\, e^{i\lambda\varphi}\big) = b(x,\lambda)e^{i\lambda\varphi(x)},$$
as in (8.20), show that, mod $O(|\lambda|^{-\infty})$, $b(x,\lambda)$ depends only on the behavior of $p(x,\xi)$ on an arbitrarily small conic neighborhood of
$$C_\varphi = \big\{\big(x, \lambda d\varphi(x)\big) : x \in \text{supp}\, a, \lambda \ne 0\big\}.$$
If C_φ^+ is the subset of C_φ on which $\lambda > 0$, show that the asymptotic behavior of $b(x,\lambda)$ as $\lambda \to +\infty$ depends only on the behavior of $p(x,\xi)$ on an arbitrarily small conic neighborhood of C_φ^+.
2. If Γ_r is as in (8.22), show that, given $r > 0$,
$$(\cos r\sqrt{-\Delta})(a\, e^{i\lambda\varphi}) = \Gamma_r Q_r(a\, e^{i\lambda\varphi}), \quad \text{mod } O(\lambda^{-\infty}),\ \lambda > 0, \tag{9.10}$$
for some $Q_r \in OPS^{1/2}$. Consequently, analyze the behavior of the left side of (9.10), as $\lambda \to +\infty$, in terms of the behavior of Γ_r analyzed in §7 of Chapter 6.