2

CONVEXITY AND FIXED POINT THEOREMS

2.1. Kakutani-Markov fixed point theorem

In the next two sections we describe fixed point theorems which in particular will yield results about the existence of invariant finite measures for certain group actions.

DEFINITION 2.1.1. *If a group G acts on a space X a point $x \in X$ is called a fixed point if $gx = x$ for all g.*

We recall that an action of the group $\mathbb{Z}$ on X is specified by giving a single invertible map $\varphi: X \to X$. In this case, there are some fixed point theorems known by topological methods. For example, the Brouwer fixed point theorem asserts that any continuous map $\varphi: X \to X$ has a fixed point if X is homeomorphic to the closed ball in $\mathbb{R}^n$. It is clear, however, that one may have homeomorphisms of other spaces with no fixed points. For example, a non-trivial rotation of the circle clearly has no fixed points. Any fixed point of course defines an invariant measure by taking the point mass at the fixed point. On the other hand, rotation on the circle, while having no fixed points, clearly leaves the arc length measure invariant. The Kakutani-Markov theorem, which is the main goal of this section, implies that every homeomorphism of a compact space has an invariant measure. There are two main ingredients. First is the compactness of $M(X)$ in the weak-*-topology (Corollary 1.1.29). The other is convexity.

DEFINITION 2.1.2. *If E is a vector space, a set $A \subset E$ is called convex if $x, y \in A$, $t \in [0,1]$ implies $tx + (1-t)y \in A$.*

EXAMPLE 2.1.3: (a) If E is a vector space and $\| \ \|$ is a seminorm on E, then open or closed $\| \ \|$-balls around any point x_0 (i.e. $\{x \mid \|x - x_0\| < r\}$ or $\{x \mid \|x - x_0\| \le r\}$) are convex.
(b) If X is a compact space, $M(X) \subset C(X)^*$ is convex.

(c) If $\lambda \in E^*$ (and $k = \mathbf{R}$), then for any $r \in \mathbf{R}$, $\{x \in E \mid \lambda(x) < r\}$ and $\{x \mid \lambda(x) \leq r\}$ are convex.
(d) If $\{A_\alpha\}$ are convex sets, so is $\bigcap A_\alpha$.
(e) If E is a TVS and $A \subset E$ is convex, so is $\overline{A}$.

DEFINITION 2.1.4. *(a) If $A \subset E$, we define the convex hull of A, denoted by* co(A)*, to be the unique smallest convex set containing A. This exists by 2.1.3 (d) and is equal to $\bigcap\{B \subset E \mid A \subset B$ and B is convex$\}$.*
(b) If E is a TVS, and $A \subset E$, we define the closed convex hull, denoted by $\overline{co}(A)$, to be the unique smallest closed convex set containing A. By 2.1.3 (d), (e), $\overline{co}(A) = \overline{co(A)} = \cap\{B \subset E \mid A \subset B$ and B is closed and convex$\}$.

THEOREM 2.1.5. (Kakutani-Markov) *Let E be a TVS whose topology is defined by a sufficient family of seminorms. Suppose G is an abelian group and $\pi: G \to Aut(E)$ is a representation. Let $A \subset E$ be a compact convex set that is G-invariant, i.e. $\pi(g)A \subset A$ for all $g \in G$. Then there is a G-fixed point in A.*

PROOF: For each $g \in G$ and $n \geq 0$, define $M_{n,g} \in B(E)$ by $M_{n,g} = \frac{1}{n}\sum_{i=0}^{n-1} \pi(g^i)$. Since A is convex and G-invariant, we have $M_{n,g}(A) \subset A$ for all n, g. Let G^* be the semigroup of operators generated $\{M_{n,g} \mid n \geq 0, g \in G\}$, (i.e. all finite compositions of such operators). Since G is abelian, G^* is commutative and we clearly have $T(A) \subset A$ for all $T \in G^*$. We claim $\bigcap_{T \in G^*} T(A) \neq \phi$, and that every element is a G-fixed point. To see the intersection is non-empty, since each $T(A)$ is compact (since A is compact and T is continuous), it suffices to see that for any finite set $T_1, \ldots, T_n \in G^*$, $\bigcap_{i=1}^n T_i(A) \neq \phi$. However, if we let $S = T_1 \circ \cdots \circ T_n \in G^*$, then $S(A) \subset T_1(T_2 \circ \cdots \circ T_n(A)) \subset T_1(A)$. Since G^* is commutative, we also have $S = T_2 \circ T_1 \circ \cdots \circ T_n$, and hence $S(A) \subset T_2(A)$. Similarly, $S(A) \subset T_i(A)$ for each i, showing that $\phi \neq S(A) \subset \bigcap T_i(A)$. Now suppose $y \in \bigcap_{T \in G^*} T(A)$. Then for each $n \geq 0$ and $g \in G$, there is some $x \in A$ such that $y = \frac{1}{n}(x + \cdots + \pi(g^{n-1})x)$. Then $\pi(g)y - y = (\pi(g^n)x - x)/n$. Let $\|\ \|$ be one of the seminorms defining the topology. Then for each n we have $\|\pi(g)y - y\| \leq 2B/n$ where $B = \sup\{\|a\| \mid a \in A\}$. (This exists since A is compact and $\|\ \|: E \to \mathbf{R}$ is continuous by definition.) Since this is true for all

n, $\|\pi(g)y - y\| = 0$, and since this is true for all seminorms in a sufficient family, $\pi(g)y = y$ for any $g \in G$.

COROLLARY 2.1.6. *Let G be an abelian group acting continuously on a compact metric space X. Then there is a G- invariant probability measure on X.*

PROOF: $M(X) \subset C(X)^*$ is compact, convex with the weak-*-topology. By 1.3.13, 1.3.14 we have a representation of G on $C(X)^*$ leaving $M(X)$ invariant. Thus, Theorem 2.1.5 implies the result.

EXAMPLE 2.1.7: It is not true that any group acting on a compact metric space has an invariant measure. For example, if we let $\varphi_1 : [0,1] \to [0,1]$ be $\varphi_1(x) = x^2$, the only invariant probability measures are supported on $\{0,1\}$. (See exercise 2.9.) Thus, if we identify φ_1 with a homeomorphism of S^1 by identifying 0 and 1, we obtain a homeomorphism whose only invariant measure is supported at a given point x_0. Let φ_2 be any homeomorphism of S^1 moving x_0, e.g. a rotation. Then the (non-abelian) group generated by φ_1, φ_2 has no invariant measure on S^1.

2.2. Haar measure for compact groups

Let G be a topological group. Then G acts on itself by left (or right) translation. I.e. for $g \in G$, we define the action of g on G to be $g \cdot h = gh$, where gh is simply multiplication. Under the assumption that G is locally compact, the following theorem asserts that there is always an essentially unique invariant measure. This result is fundamental for many aspects of the study of such groups.

THEOREM 2.2.1. (Haar) *Let G be a locally compact (second countable) group. Then:*

(i) *There is a measure μ which is invariant under left translations and is finite on compact subsets.*

(ii) *μ is unique up to positive scalar multiple.*

(iii) *The measure class of μ is the unique invariant measure class. More precisely if ν is a measure such that $g_*\nu \sim \nu$ (i.e. they have the same null sets), then $\nu \sim \mu$.*

(iv) *$\mu(G) < \infty$ if and only if G is compact.*

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A FUNCTIONAL ANALYSIS PROOF OF THE EXISTENCE OF HAAR MEASURE ON LOCALLY COMPACT ABELIAN GROUPS

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ABSTRACT. A simple proof of the existence of Haar measure on locally compact abelian groups is given. The proof uses the Markov-Kakutani fixed-point theorem.

It is very well known that every locally compact group has a Haar measure and that the Haar measure is unique up to a positive multiplicative constant. Several different proofs have been given, all of them somewhat difficult. (See [N] for two proofs as well as references to others). In most of these proofs, the existence and uniqueness of Haar measure are established separately. For compact groups, a simple proof of the existence and uniqueness of Haar measure was given by von Neumann [vN], and his proof can be made even simpler by using the Kakutani fixed-point theorem (see [R2]). For locally compact *abelian* groups, uniqueness of Haar measure is easily established (see [R1, p. 2]). The purpose of this short note is to present a simple proof of the *existence* of Haar measure for these groups. The proof will make use of the Markov-Kakutani fixed-point theorem, which we recall below. It is known that this fixed-point theorem can be used to prove that every locally compact abelian group has an invariant mean (see [P, p. 113]). For compact groups an invariant mean and a Haar measure are the same thing, but for noncompact groups this is obviously not the case.

Theorem (Markov-Kakutani). *Let K be a nonempty compact convex subset of a (Hausdorff) topological vector space. Let $\mathscr{F}$ be a commuting family of continuous affine mappings of K into itself. Then there exists a point $p \in K$ such that $Tp = p$ for all $T \in \mathscr{F}$.*

A proof can be found in [C, pp. 155–156]. (There the theorem is stated only for locally convex spaces, but local convexity is not needed in the proof.)

We will also need two lemmas.

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Lemma 1. *Suppose G is a topological group and N is a neighborhood of the identity in G that is symmetric (i.e., $N^{-1} = N$). Then there exists a subset S of G such that for each g in G the set $gN \cdot N$ contains at* least *one element of S and the set gN contains at* most *one element of S.*

Proof. Let $\mathscr{G}$ be the collection of all subsets T of G such that

$$p^{-1}q \notin N \cdot N \quad \text{for all } p, q \in T.$$

By applying Zorn's lemma, we see that $\mathscr{G}$ has a maximal element S. Now if $g \in G$, then there is some s in S such that $g^{-1}s \in N \cdot N$, for otherwise the set $S \cup \{g\}$ would be a member of $\mathscr{G}$ strictly containing S. Moreover, if there were two distinct points s_1, s_2 in S such that both $g^{-1}s_1$ and $g^{-1}s_2$ were in N, then we would have $s_1^{-1}s_2 = s_1^{-1}gg^{-1}s_2 \in N^{-1} \cdot N = N \cdot N$, a contradiction. Thus, there is at most one s in S such that $g^{-1}s \in N$. □

Lemma 2. *Let X be a vector space, and let X^* denote the space of* all *linear functionals on X with the weak*-topology (i.e., the weak topology induced by X). If K is a closed subset of X^* such that for each $x \in X$ the set $\{\Lambda x : \Lambda \in K\}$ is bounded, then K is compact.*

The proof of this lemma is very similar to the proof of the Banach-Alaoglu theorem and is essentially contained in [DS, pp. 423–424]. A more succinct statement of the conclusion is that every closed bounded set in X^* is compact.

Proof of the existence of Haar measure on locally compact abelian groups. Let G be a locally compact abelian group. Let $C_c(G)$ denote the space of compactly supported continuous functions on G, and let $C_c(G)^*$ denote the space of *all* linear functionals on $C_c(G)$ with the weak*-topology (i.e., the weak topology induced by $C_c(G)$). If $f \in C_c(G)$ and $a \in G$, then f_a (the translate of f by a) is defined by $f_a(x) = f(a+x)$. For each a in G, define $T_a : C_c(G)^* \to C_c(G)^*$ by the equation

$$(T_a\Lambda)(f) = \Lambda(f_a) \qquad (\Lambda \in C_c(G)^*,\ f \in C_c(G)).$$

Then each T_a is a continuous linear operator. To establish the existence of Haar measure on G we must simply show that there is a nonzero positive linear functional on $C_c(G)$ that is fixed by every T_a.

Fix a symmetric neighborhood N of the identity in G with compact closure. Let K be the set of positive linear functionals Λ on $C_c(G)$ that satisfy the following two conditions:

(i) $\Lambda(f) \leq 1$ whenever f is a nonnegative function in $C_c(G)$ that is bounded above by 1 and whose support is contained in $a + N$ for some $a \in G$, and

(ii) $\Lambda(f) \geq 1$ whenever f is a nonnegative function in $C_c(G)$ that is equal to 1 on $a + N + N$ for some $a \in G$.

Then K is clearly closed and convex in $C_c(G)^*$. Moreover, by a partition of unity argument every nonnegative function in $C_c(G)$ can be written as a finite sum of nonnegative continuous functions each of which has support in $a + N$ for some $a \in G$. It follows that condition (i) in the definition of K implies that for each function f in $C_c(G)$, the set $\{\Lambda(f) : \Lambda \in K\}$ is bounded. Therefore by Lemma 2, K is compact.

Let S be as in Lemma 1, and note that the functional that consists of a point mass at each point of S (i.e., the functional $f \mapsto \sum_{s \in S} f(s)$) is in K. Thus K is nonempty.

It is clear from the definition of K that each of the operators T_a maps K into itself. Hence, since the operators T_a ($a \in G$) form a commuting family, the Markov-Kakutani fixed-point theorem shows that they have a common fixed-point in K. Since all the elements of K are nonzero positive linear functionals on $C_c(G)$, the proof is complete. □

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