$C^\infty(R_n)$ if the sequence $\varphi_j \to 0$ in $C_0^\infty(R_n)$. Then there exists one and only one distribution u such that $U\varphi = u * \varphi$, $\varphi \in C_0^\infty(R_n)$.

Proof. By hypothesis the linear form

$$C_0^\infty \ni \check{\varphi} \to (U\varphi)(0)$$

is a distribution u, hence $(U\varphi)(0) = u(\check{\varphi}) = (u * \varphi)(0)$. Replacing φ by $\tau_{-h}\varphi$ and using the fact that τ_{-h} commutes with U and the convolution operator, we obtain $(U\varphi)(h) = (u * \varphi)(h)$, which proves the theorem.

If u is a distribution with compact support, it is clear that $\varphi \to u * \varphi$ maps C_0^∞ continuously into C_0^∞. (By continuity we always mean sequential continuity here.) It is also clear that the definition of $u * \varphi$ given by (1.6.1) can then be extended to all $\varphi \in C^\infty$ and gives a continuous mapping of C^∞ into C^∞.

We can now define the convolution of two distributions u_1 and u_2, one of which has compact support. In fact, the mapping

$$C_0^\infty(R_n) \ni \varphi \to u_1 * (u_2 * \varphi) \in C^\infty(R_n)$$

is linear, translation invariant and continuous. Hence Theorem 1.6.4 shows that there is a unique distribution u such that

$$u_1 * (u_2 * \varphi) = u * \varphi, \quad \varphi \in C_0^\infty(R_n). \tag{1.6.4}$$

In order to maintain the associativity of the convolution product we have to make the following definition.

Definition 1.6.2. *The convolution of the distributions u_1 and u_2 in $\mathcal{D}'(R_n)$, one of which has compact support, is defined as the distribution u satisfying* (1.6.4), *and is denoted by $u_1 * u_2$.*

Note that it follows from Theorem 1.6.2 that this definition agrees with Definition 1.6.1 if u_2 is a test function. Similarly, if $u_1 \in \mathcal{E}'(R_n)$ and $u_2 \in C^\infty(R_n)$, a modification of Theorem 1.6.2 shows that the definition agrees with the previous one.

Example. For every $u \in \mathcal{D}'(R_n)$ we have $u * \delta = u$.

The convolution thus defined is obviously associative,

$$u_1 * (u_2 * u_3) = (u_1 * u_2) * u_3$$

if all u_j except one have compact support.

Theorem 1.6.5. *The convolution is commutative, that is, $u_1 * u_2 = u_2 * u_1$, if one of the distributions u_1 and u_2 has compact support. We have* $\operatorname{supp}(u_1 * u_2) \subset \operatorname{supp} u_1 + \operatorname{supp} u_2$.

Proof. First note that two distributions v_1 and v_2 are equal if $v_1 * (\varphi * \psi) = v_2 * (\varphi * \psi)$ for all $\varphi, \psi \in C_0^\infty$. For then we obtain using Theorem 1.6.2 that $v_1 * \varphi = v_2 * \varphi$ for all $\varphi \in C_0^\infty$ and hence that $v_1 = v_2$. Thus consider

$$(u_1 * u_2) * (\varphi * \psi) = u_1 * (u_2 * (\varphi * \psi)) = u_1 * ((u_2 * \varphi) * \psi)$$
$$= u_1 * (\psi * (u_2 * \varphi)) = (u_1 * \psi) * (u_2 * \varphi).$$

Here we have used the fact that convolution of functions is commutative, and also Theorem 1.6.2 and the similar result where u has compact support and one of the functions φ and ψ is in C^∞ only. In the same way one obtains

$$(u_2 * u_1) * (\varphi * \psi) = (u_2 * u_1) * (\psi * \varphi) = (u_2 * \varphi) * (u_1 * \psi) = (u_1 * \psi) * (u_2 * \varphi),$$

which proves the commutativity. To prove the last statement in the theorem, finally, we choose φ_ε as in Theorem 1.6.3 and note that since

$$(u_1 * u_2) * \varphi_\varepsilon = u_1 * (u_2 * \varphi_\varepsilon)$$

it follows from Theorem 1.6.1 that the support of $(u_1 * u_2) * \varphi_\varepsilon$ is contained in $\operatorname{supp} u_1 + \operatorname{supp} u_2 + \{x; |x| \leqq \varepsilon\}$. If we let $\varepsilon \to 0$, it now follows that $\operatorname{supp} u_1 + \operatorname{supp} u_2$ contains $\operatorname{supp}(u_1 * u_2)$. The proof is complete.

A differentiation can also be written as a convolution. In fact,

$$D^\alpha u = (D^\alpha \delta) * u \tag{1.6.5}$$

where δ is the Dirac measure at 0. To prove this we use (1.6.2) twice,

$$(D^\alpha u) * \varphi = u * (D^\alpha \varphi) = u * (D^\alpha \varphi) * \delta = u * (D^\alpha \delta) * \varphi, \quad \varphi \in C_0^\infty(R_n),$$

which implies (1.6.5). Hence we now obtain

$$D^\alpha(u_1 * u_2) = (D^\alpha u_1) * u_2 = u_1 * (D^\alpha u_2) \tag{1.6.6}$$

by using (1.6.5) and the associativity and commutativity of the convolution. It is clear that (1.6.6) contains (1.6.5) if we choose $u_1 = u$ and $u_2 = \delta$.

The convolution $u_1 * u_2$ may also be defined in some cases where neither u_1 nor u_2 has compact support. For example, when u_1 and u_2 both have their supports in a half space $\{x; \langle x, N\rangle \geqq 0\}$ and the support of one of them is contained in the smaller cone $\{x; \langle x, N\rangle \geqq \varepsilon|x|\}$ for some $\varepsilon > 0$, we can define $u_1 * u_2$ by the formula

$$(u_1 * u_2) * \varphi = u_1 * (u_2 * \varphi), \quad \varphi \in C_0^\infty(R_n).$$

This definition will be used in section 5.6. We leave to the reader as an exercise to verify that the definition is meaningful and to extend the results proved above to this situation.

1.7. Fourier transforms of distributions. The Fourier transform $\hat{f}$ of a function $f \in L_1(R_n)$ is defined by

$$\hat{f}(\xi) = \int e^{-i\langle x, \xi\rangle} f(x)\, dx \tag{1.7.1}$$

where $\langle x, \xi\rangle = x_1\xi_1 + \cdots + x_n\xi_n$. If $\hat{f}$ also happens to be integrable, one can express f in terms of $\hat{f}$ by means of the Fourier inversion formula (Theorem 1.7.1 and Theorem 1.7.3),

$$f(x) = (2\pi)^{-n} \int e^{i\langle x, \xi\rangle} \hat{f}(\xi)\, d\xi. \tag{1.7.2}$$

Thus $\hat{f}(\xi)$ is the density of the frequency ξ in the harmonic decomposition of f.

To study the Fourier transform, and in particular prove (1.7.2), we first consider functions in a subset $\mathscr{S}$ of C^∞, containing C_0^∞, and then the dual space $\mathscr{S}'$ of $\mathscr{S}$, which is a subspace of $\mathscr{D}'(R_n)$ containing $\mathscr{E}'(R_n)$.

Definition 1.7.1. *By $\mathscr{S}$ or $\mathscr{S}(R_n)$ we denote the set of all functions $\varphi \in C^\infty(R_n)$ such that*

$$\sup_x |x^\beta D^\alpha \varphi(x)| < \infty \tag{1.7.3}$$

for all multi-indices α and β. The topology in $\mathscr{S}$ is defined by the semi-norms in the left-hand side of (1.7.3).[1]

It is clear that $C_0^\infty(R_n) \subset \mathscr{S}(R_n)$. Another example of a function in $\mathscr{S}$ is $\varphi(x) = e^{-a|x|^2}$, if $a > 0$. The importance of the class $\mathscr{S}$ is due to the following result.

Lemma 1.7.1. *The Fourier transformation $\varphi \to \hat{\varphi}$ maps $\mathscr{S}$ continuously into $\mathscr{S}$. The Fourier transform of $D_j\varphi$ is $\xi_j\hat{\varphi}(\xi)$ and the Fourier transform of $x_j\varphi$ is $-D_j\hat{\varphi}$.*

Proof. Differentiation of (1.7.1) under the integral sign gives

$$D^\alpha \hat{\varphi}(\xi) = \int e^{-i\langle x, \xi\rangle}(-x)^\alpha \varphi(x)\, dx\,,$$

and is legitimate since the integral obtained is uniformly convergent. Hence $\hat{\varphi} \in C^\infty$ and $D^\alpha\hat{\varphi}$ is the Fourier transform of $(-x)^\alpha\varphi$. Integrating by parts we also obtain

$$\xi^\beta D^\alpha \hat{\varphi}(\xi) = \int e^{-i\langle x, \xi\rangle} D^\beta((-x)^\alpha \varphi(x))\, dx\,. \tag{1.7.4}$$

These operations are legitimate since $\varphi \in \mathscr{S}$. From (1.7.4) and the fact that $D^\beta((-x)^\alpha\varphi(x)) \in \mathscr{S} \subset L_1$, it now follows that $\xi^\beta D^\alpha\hat{\varphi}(\xi)$ is bounded so that $\hat{\varphi} \in \mathscr{S}$. Also note that when $\alpha = 0$ we obtain that $\xi^\beta\hat{\varphi}$ is the Fourier transform of $D^\beta\varphi$. Now the right-hand side of (1.7.4) can be estimated by a constant times $\sup_x (1+|x|)^{n+1}|D^\beta((-x)^\alpha\varphi(x))|$, since $(1+|x|)^{-n-1}$ is integrable, and this proves the continuity of the Fourier transformation in $\mathscr{S}$.

Theorem 1.7.1. *The Fourier inversion formula* (1.7.2) *is valid in $\mathscr{S}$.*

Proof. We have to compute the iterated integral

$$\int e^{i\langle x, \xi\rangle} d\xi \int \varphi(y)\, e^{-i\langle y, \xi\rangle} dy\,,$$

where $\varphi \in \mathscr{S}$. Since the double integral does not converge absolutely, the order of integration cannot be inverted. However, if we introduce a factor $\psi(\xi)$, where $\psi \in \mathscr{S}$, we obtain absolute convergence, and inverting the order of integration now gives

$$\begin{aligned}\int \hat{\varphi}(\xi)\, \psi(\xi)\, e^{i\langle x, \xi\rangle} d\xi &= \int \hat{\psi}(y - x)\, \varphi(y)\, dy \\ &= \int \hat{\psi}(y)\, \varphi(x+y)\, dy\,.\end{aligned} \tag{1.7.5}$$

[1] It is easy to see that $\mathscr{S}(R_n)$ is the largest subspace of $L_1(R_n)$ which is invariant under the differential operators D_j and under multiplication by x_j, $j = 1, \ldots, n$.

(So far we have only used that φ and $\psi \in L_1$.) Replacing $\psi(\xi)$ by $\psi(\varepsilon\xi)$, $\varepsilon > 0$, which has the Fourier transform $\varepsilon^{-n}\hat{\psi}(y/\varepsilon)$, we thus obtain

$$\int \hat{\varphi}(\xi)\, \psi(\varepsilon\xi)\, e^{i\langle x, \xi\rangle} d\xi = \int \hat{\psi}(y)\, \varphi(x + \varepsilon y)\, dy\,.$$

Since $\hat{\varphi}$ and $\hat{\psi}$ are in $\mathscr{S}$, hence integrable, and φ and ψ are bounded and continuous, we may pass to the limit under the integral signs when $\varepsilon \to 0$, and this gives

$$\psi(0) \int \hat{\varphi}(\xi)\, e^{i\langle x, \xi\rangle} d\xi = \varphi(x) \int \hat{\psi}\, dy\,.$$

To complete the proof we only need to note that if $\psi(x) = e^{-|x|^2/2}$ we have $\hat{\psi}(y) = (2\pi)^{n/2} e^{-|y|^2/2}$ and $\int \hat{\psi}\, dy = (2\pi)^n$.

Corollary 1.7.1. *The Fourier transformation is an isomorphism of $\mathscr{S}$ onto $\mathscr{S}$.*

It is now easy to prove the following fundamental properties of the Fourier transformation on $\mathscr{S}$.

Theorem 1.7.2. *If φ and ψ are in $\mathscr{S}$, we have*

$$\int \hat{\varphi}\psi\, dx = \int \varphi\hat{\psi}\, dx\,, \tag{1.7.6}$$

$$\int \varphi\bar{\psi}\, dx = (2\pi)^{-n} \int \hat{\varphi}\bar{\hat{\psi}}\, dx \quad \text{(Parseval's \textit{formula})}\,, \tag{1.7.7}$$

$$\widehat{(\varphi * \psi)} = \hat{\varphi}\hat{\psi}\,, \tag{1.7.8}$$

$$\widehat{(\varphi\psi)} = (2\pi)^{-n}\hat{\varphi} * \hat{\psi}\,. \tag{1.7.9}$$

Proof. (1.7.6) is a special case of (1.7.5) when $x = 0$. To prove (1.7.7) we set $(2\pi)^{-n}\bar{\hat{\psi}} = \chi$ and obtain, using the Fourier inversion formula

$$\overline{\hat{\chi}(\xi)} = (2\pi)^{-n} \int \hat{\psi}(x)\, e^{i\langle x, \xi\rangle} dx = \psi(\xi)\,.$$

Hence (1.7.7) follows if we apply (1.7.6) with ψ replaced by χ. Like (1.7.5), the formula (1.7.8) is completely elementary and may be verified by the reader. To obtain (1.7.9), finally, we note that the Fourier transform of $\widehat{(\varphi\psi)}$ is $(2\pi)^n\varphi(-x)\,\psi(-x)$ and that the Fourier transform of $\hat{\varphi} * \hat{\psi}$ is $(2\pi)^n\varphi(-x)\,(2\pi)^n\psi(-x)$ in view of Theorem 1.7.1 and (1.7.8). The proof is complete.

Definition 1.7.2. *A continuous linear form u on $\mathscr{S}$ is called a temperate distribution. The set of all temperate distributions is denoted by $\mathscr{S}'$.*

The restriction of a temperate distribution to $C_0^\infty(R_n)$ is obviously a distribution in $\mathscr{D}'(R_n)$. We can in fact *identify* $\mathscr{S}'$ with a subspace of $\mathscr{D}'(R_n)$ since the following lemma shows that a distribution $u \in \mathscr{S}'$ which vanishes on $C_0^\infty(R_n)$ must also vanish on $\mathscr{S}$.

Lemma 1.7.2. *C_0^∞ is dense in $\mathscr{S}$.*

Proof. Let $\varphi \in \mathscr{S}$ and take $\psi \in C_0^\infty$ such that $\psi(x) = 1$ when $|x| \leq 1$. Put $\varphi_\varepsilon(x) = \varphi(x)\,\psi(\varepsilon x)$. Then it is clear that $\varphi_\varepsilon \in C_0^\infty$, and since $\varphi_\varepsilon(x) - \varphi(x) = \varphi(x)\,(\psi(\varepsilon x) - 1) = 0$ if $|x| < 1/\varepsilon$, it is easy to see that $\varphi_\varepsilon \to \varphi$ in $\mathscr{S}$ when $\varepsilon \to 0$.

It is obvious that $\mathcal{E}' \subset \mathcal{S}'$. Other examples of elements in $\mathcal{S}'$ are measures $d\mu$ such that for some m

$$\int (1+|x|)^{-m} |d\mu(x)| < \infty .$$

In particular, we thus have $L_p(R_n) \subset \mathcal{S}'$ for every p. It is also clear that $\mathcal{S}'$ is closed under differentiation and under multiplication by functions in $\mathcal{S}$.

Definition 1.7.3. *If $u \in \mathcal{S}'$, the Fourier transform $\hat{u}$ is defined by*

$$\hat{u}(\varphi) = u(\hat{\varphi}), \quad \varphi \in \mathcal{S} . \tag{1.7.10}$$

It follows from Lemma 1.7.1 that $\hat{u} \in \mathcal{S}'$, and from (1.7.6) it follows that the definition agrees with (1.7.1) if $u \in L_1(R_n)$. (Note that the proof of (1.7.6) is valid for all φ and $\psi \in L_1$.)

Fourier's inversion formula proved in Theorem 1.7.1 may be written $\hat{\hat{\varphi}} = (2\pi)^n \check{\varphi}$, where $\check{\varphi}(x) = \varphi(-x)$. If we define $\check{u}(\varphi) = u(\check{\varphi})$, when u is a distribution, we therefore obtain

Theorem 1.7.3. *Fourier's inversion formula is valid for every $u \in \mathcal{S}'$, that is, $\hat{\hat{u}} = (2\pi)^n \check{u}$. The Fourier transformation maps $\mathcal{S}'$ onto $\mathcal{S}'$.*

Proof. By definition we have $\hat{\hat{u}}(\varphi) = u(\hat{\hat{\varphi}}) = (2\pi)^n u(\check{\varphi}) = (2\pi)^n \check{u}(\varphi)$, $\varphi \in \mathcal{S}$, which proves the theorem.

We shall always use the weak topology in $\mathcal{S}'$. It is then obvious that the Fourier transformation is continuous in $\mathcal{S}'$, and from Theorem 1.7.3 it thus follows that the Fourier transformation is an isomorphism of $\mathcal{S}'$ onto $\mathcal{S}'$.

Theorem 1.7.4. *If $u \in L_2(R_n)$, the Fourier transform $\hat{u}$ is also in $L_2(R_n)$ and Parseval's formula is valid,*

$$\int |\hat{u}|^2 dx = (2\pi)^n \int |u|^2 dx . \tag{1.7.11}$$

Proof. From (1.7.7) and (1.7.10) it follows that

$$|\hat{u}(\varphi)| = |u(\hat{\varphi})| \leq \|u\|_2 \|\hat{\varphi}\|_2 = (2\pi)^{n/2} \|u\|_2 \|\varphi\|_2, \; \varphi \in C_0^\infty .$$

Since the form $\hat{u}$ is continuous for the L_2 norm it has a density $\in L_2$, which we also denote by $\hat{u}$, and $\|\hat{u}\|_2 \leq (2\pi)^{n/2} \|u\|_2$. Iterating this inequality we obtain

$$(2\pi)^n \|u\|_2 = \|\hat{\hat{u}}\|_2 \leq (2\pi)^{n/2} \|\hat{u}\|_2 \leq (2\pi)^n \|u\|_2 ,$$

which implies (1.7.11).

Theorem 1.7.5. *The Fourier transform of a distribution $u \in \mathcal{E}'$ is the function (cf. (1.7.1))*

$$\hat{u}(\xi) = u_x(e^{-i\langle x, \xi \rangle}) .$$

The right-hand side is also defined for every complex vector $\xi \in C_n$ and is an entire analytic function of ξ, called the Fourier-Laplace transform of u.

Proof. The theorem is trivial if u is a function. To prove it in general, we choose $\varphi \in C_0^\infty$ satisfying (1.2.1) and set $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon)$. In view of Theorem 1.6.3 we then have $u * \varphi_\varepsilon \to u$ in the weak topology of $\mathcal{E}'$, hence in $\mathcal{S}'$. This proves that the Fourier transform of $u * \varphi_\varepsilon$ converges to $\hat{u}$ in $\mathcal{S}'$ when $\varepsilon \to 0$. On the other hand, the Fourier-Laplace transform of the function $u * \varphi_\varepsilon$ is the analytic function

$$(u * \varphi_\varepsilon)(e^{-i\langle x, \xi \rangle}) = u(\check{\varphi}_\varepsilon * e^{-i\langle x, \xi \rangle}) = \hat{\varphi}(\varepsilon \xi)\, u(e^{-i\langle x, \xi \rangle}) .$$

Since $\hat{\varphi}(\varepsilon\xi) \to \hat{\varphi}(0) = 1$ uniformly on every compact subset of C_n when $\varepsilon \to 0$, it now follows that $u(e^{-i\langle x, \xi \rangle})$ is an entire function of ξ and that its restriction to real arguments is the Fourier transform of u.

Theorem 1.7.6. *If $u_1 \in \mathcal{E}'$ and $u_2 \in \mathcal{S}'$, it follows that $u_1 * u_2 \in \mathcal{S}'$, and the Fourier transform of $u_1 * u_2$ is $\hat{u}_1 \hat{u}_2$. (The product is defined since $\hat{u}_1 \in C^\infty$ in virtue of Theorem 1.7.5.)*

Proof. If $\varphi \in C_0^\infty$, we have by definition

$$(u_1 * u_2)(\varphi) = u_1 * (u_2 * \check{\varphi})(0) = \check{u}_1(u_2 * \check{\varphi}) .$$

Hence there exists a compact set $K \subset R_n$ and constants C, k such that

$$|(u_1 * u_2)(\varphi)| \leq C \sum_{|\alpha| \leq k} \sup_K |D^\alpha (u_2 * \check{\varphi})| = C \sum_{|\alpha| \leq k} \sup_K |(D^\alpha u_2) * \check{\varphi})| .$$

Since $D^\alpha u_2 \in \mathcal{S}'$, the right-hand side can be estimated by semi-norms of φ in $\mathcal{S}$. The linear form $u_1 * u_2$ on C_0^∞ can thus be extended by continuity to a continuous linear form on $\mathcal{S}$, for C_0^∞ is dense in $\mathcal{S}$. Hence $u_1 * u_2 \in \mathcal{S}'$.

To prove the last result we first assume that $u_1 \in C_0^\infty$. If $\psi \in \mathcal{S}$ and $\hat{\psi} \in C_0^\infty$, we then obtain

$$(u_1 * u_2)(\hat{\psi}) = u_2(\check{u}_1 * \hat{\psi}) = \hat{u}_2(\hat{u}_1 \psi) = (\hat{u}_1 \hat{u}_2)(\psi) .$$

Here we have used the fact, which follows from (1.7.5), that the Fourier transform of $\hat{u}_1 \psi$ is $\check{u}_1 * \hat{\psi}$. Since C_0^∞ is dense in $\mathcal{S}$, this proves that the Fourier transform of $u_1 * u_2$ is $\hat{u}_1 \hat{u}_2$ if $u_1 \in C_0^\infty$ and $u_2 \in \mathcal{S}'$. If $u_1 \in \mathcal{E}'$ and $u_2 \in \mathcal{S}'$, we choose a function $\varphi \in C_0^\infty$ and apply the result already proved to the identity $(\varphi * u_1) * u_2 = \varphi * (u_1 * u_2)$, which gives

$$\hat{\varphi}\, \widehat{(u_1 * u_2)} = \widehat{(\varphi * u_1)}\, \hat{u}_2 = \hat{\varphi}\, \hat{u}_1 \hat{u}_2 .$$

Since we can choose φ so that $\hat{\varphi} \neq 0$ at any given point, this implies that the Fourier transform of $u_1 * u_2$ is $\hat{u}_1 \hat{u}_2$. The proof is complete.

We shall now study the Fourier transform of a distribution with compact support more closely.

Theorem 1.7.7. (Paley-Wiener) *An entire analytic function $U(\zeta)$ is the Fourier-Laplace transform of a distribution with support in the ball $B_A = \{x; |x| \leq A\}$ if and only if for some constants C and N we have*

$$|U(\zeta)| \leq C(1 + |\zeta|)^N e^{A|\mathrm{Im}\,\zeta|} . \tag{1.7.12}$$

U is the Fourier-Laplace transform of a function u in $C_0^\infty(B_A)$ if and only if for every integer N there exists a constant C_N such that

$$|U(\zeta)| \leq C_N(1 + |\zeta|)^{-N} e^{A|\mathrm{Im}\,\zeta|} . \tag{1.7.13}$$

Proof. a) To prove the *necessity* of (1.7.12) we use the fact, proved in section 1.5, that if $u \in \mathscr{E}'$ there exist constants C and N such that

$$|u(\varphi)| \leqq C \sum_{|\alpha| \leqq N} \sup |D^\alpha \varphi|, \quad \varphi \in C_0^\infty .$$

Let $\psi \in C^\infty(R_1)$ be equal to 1 on $(-\infty, \frac{1}{2})$ and 0 on $(1, \infty)$. Then the function

$$\varphi_\zeta(x) = e^{-i\langle x, \zeta\rangle} \psi(|\zeta|(|x| - A))$$

is in C_0^∞ and coincides with $e^{-i\langle x,\zeta\rangle}$ in a neighborhood of B_A. Hence

$$|\hat{u}(\zeta)| = |u(\varphi_\zeta)| \leqq C \sum_{|\alpha| \leqq N} \sup |D^\alpha \varphi_\zeta| . \tag{1.7.14}$$

Since $|x| \leqq A + |\zeta|^{-1}$ in the support of φ_ζ, it follows that $|e^{-i\langle x,\zeta\rangle}| \leqq$ $\leqq e^{A|\mathrm{Im}\,\zeta|+1}$ there. Hence (1.7.12) follows from (1.7.14).

The necessity of (1.7.13) follows from (1.7.4) with $\varphi = u$ and $\alpha = 0$.

b) The *sufficiency* of (1.7.13) is proved as follows. Let

$$u(x) = (2\pi)^{-n} \int U(\xi)\, e^{i\langle x,\xi\rangle} d\xi . \tag{1.7.15}$$

Then we have $\hat{u} = U$ and the proof of Lemma 1.7.1 shows that $u \in C^\infty$. To prove that $\mathrm{supp}\, u \subset B_A$ we note that (1.7.13) permits us to shift the integration in (1.7.15) into the complex domain, which gives

$$u(x) = (2\pi)^{-n} \int U(\xi + i\eta)\, e^{i\langle x, \xi + i\eta\rangle} d\xi \tag{1.7.15'}$$

for an arbitrary fixed vector η. Estimating the integral by means of (1.7.13) with $N = n + 1$, we obtain

$$|u(x)| \leqq C_N e^{A|\eta| - \langle x,\eta\rangle} (2\pi)^{-n} \int d\xi/(1 + |\xi|)^{n+1}, \tag{1.7.16}$$

and the integral is convergent. If we choose $\eta = tx$ and let $t \to +\infty$, it now follows that $u(x) = 0$ if $|x| > A$.

To prove the sufficiency of (1.7.12), we first note that $U \in \mathscr{S}'$, hence $U = \hat{u}$ for some $u \in \mathscr{S}'$. Let $\varphi \in C_0^\infty$ satisfy (1.2.1) and set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Then the Fourier transform of $u_\varepsilon = u * \varphi_\varepsilon$ is equal to $\hat{\varphi}_\varepsilon \hat{u}$, and this function satisfies estimates of the form (1.7.13) with A replaced by $A + \varepsilon$. Hence it follows that $\mathrm{supp}\,(u * \varphi_\varepsilon) \subset B_{A+\varepsilon}$, and when $\varepsilon \to 0$ this implies that $\mathrm{supp}\, u \subset B_A$. The proof is complete.

We shall now prove a result analogous to Theorem 1.7.7 for the singular support. (See Definition 1.3.3.)

Theorem 1.7.8. *Let* $u \in \mathscr{E}'(R_n)$. *In order that* $\mathrm{sing\,supp}\, u \subset B_A = \{x; |x| \leqq A\}$, *it is necessary and sufficient that there be a constant* N *and a sequence of constants* C_m, $m = 1, 2, \ldots$ *such that*

$$|\hat{u}(\zeta)| \leqq C_m (1 + |\zeta|)^N e^{A|\mathrm{Im}\,\zeta|} \text{ if } |\mathrm{Im}\,\zeta| \leqq m \log(|\zeta| + 1);\ m = 1, 2, \ldots \tag{1.7.17}$$

Proof. a) To show that (1.7.17) is *necessary* we note that we can write $u = u_1 + u_2$ where $\mathrm{supp}\, u_1 \subset B_{A+1/m}$ and $u_2 \in C_0^\infty(R_n)$. If $N - 1$ is the order of u, which is equal to the order of u_1, it follows from the proof of Theorem 1.7.7 that

$$|\hat{u}_1(\zeta)| \leqq C_m (1 + |\zeta|)^{N-1} e^{(A+1/m)|\mathrm{Im}\,\zeta|}$$

for every ζ, hence

$$|\hat{u}_1(\zeta)| \leqq C_m (1 + |\zeta|)^N e^{A|\mathrm{Im}\,\zeta|} \quad \text{if} \quad |\mathrm{Im}\,\zeta| \leqq m \log(1 + |\zeta|) .$$

It also follows from (1.7.13) that $\hat{u}_2(\zeta) \to 0$ when $\zeta \to \infty$ in this set. Hence (1.7.17) follows with a larger C_m.

b) The proof of the *sufficiency* of (1.7.17) is analogous to that of Theorem 1.7.7. Since the assumptions and statements are invariant under orthogonal transformations, it is sufficient to prove that (1.7.17) implies that u is in C^∞ when $x_n > A$. Now we have if $\psi \in C_0^\infty(R_n)$

$$u(\psi) = (2\pi)^{-n} \int \hat{u}(\xi)\, \hat{\psi}(-\xi)\, d\xi .$$

Since $\hat{\psi}$ satisfies an estimate of the form (1.7.13), the product $\hat{u}(\zeta)\, \hat{\psi}(-\zeta)$ tends to 0 faster than any power of $(1 + |\zeta|)^{-1}$ if $|\zeta| \to \infty$ while $|\mathrm{Im}\,\zeta|/\log(|\zeta| + 1)$ remains bounded. Hence it is permissible to deform the integration contours to obtain the formula

$$u(\psi) = (2\pi)^{-n} \int d\xi' \int \hat{u}(\xi', \zeta_n)\, \hat{\psi}(-\xi', -\zeta_n)\, d\zeta_n \tag{1.7.18}$$

where $\xi' = (\xi_1, \ldots, \xi_{n-1}) \in R_{n-1}$ and the integration with respect to ζ_n is made over the contour defined by

$$\mathrm{Im}\,\zeta_n = m \log((|\xi'|^2 + |\mathrm{Re}\,\zeta_n|^2)^{\frac{1}{2}} + 1) .$$

Let Γ_m be the surface described by (ξ', ζ_n) in C_n. The inverse Fourier transform

$$(2\pi)^{-n} \int_{\Gamma_m} e^{i\langle x,\zeta\rangle} \hat{u}(\zeta)\, d\zeta , \tag{1.7.19}$$

where $d\zeta = d\xi' d\zeta_n$, is absolutely convergent if $m(x_n - A) > N + n$. In fact, it follows from (1.7.17) that

$$|e^{i\langle x,\zeta\rangle} \hat{u}(\zeta)| \leqq C_m (1 + |\zeta|)^N (1 + |\mathrm{Re}\,\zeta|)^{m(A - x_n)} \quad \text{if} \quad \zeta \in \Gamma_m ,$$

and it is clear that the quotients $|\zeta|/|\mathrm{Re}\,\zeta|$ and $|d\zeta_n|/|d\,\mathrm{Re}\,\zeta_n|$ are bounded on Γ_m. Taking ψ with support in the set where $m(x_n - A) > N + n$ and introducing the definition of $\hat{\psi}$ in (1.7.18), we may thus invert the order of integration and obtain that u is there equal to the function (1.7.19),

$$u(x) = (2\pi)^{-n} \int_{\Gamma_m} e^{i\langle x,\zeta\rangle} \hat{u}(\zeta)\, d\zeta . \tag{1.7.20}$$

This integral remains absolutely convergent after j differentiations with respect to x if $m(x_n - A) > N + n + j$. Hence $u \in C^j$ in the set defined by this inequality and since m may be chosen arbitrarily large, this proves that $u \in C^\infty$ when $x_n > A$. The proof is complete.

We shall finally discuss briefly the definition of a partial Fourier transform for distributions in a half space

$$R_n^+ = \{x; x \in R_n, x_n > 0\},$$

the closure of which will be denoted by $\bar{R}_n^+$. Set

$$\mathring{\mathscr{S}}(\bar{R}_n^+) = \{\varphi; \varphi \in \mathscr{S}(R_n), \operatorname{supp}\varphi \subset \bar{R}_n^+\}.$$

This is a closed subspace of $\mathscr{S}(R_n)$ and we give it the induced topology. From the proof of Lemma 1.7.2 it follows that $C_0^\infty(R_n^+)$ is a dense subset of $\mathring{\mathscr{S}}(\bar{R}_n^+)$. In fact, if $\varphi \in \mathring{\mathscr{S}}(\bar{R}_n^+)$, the function $\varphi_\varepsilon(x) = \varphi(x', x_n - \varepsilon)$ where $x' = (x_1, \ldots, x_{n-1})$, vanishes in a neighborhood of the boundary of $\bar{R}_n^+$ and converges to φ in $\mathring{\mathscr{S}}(\bar{R}_n^+)$ when $\varepsilon \searrow 0$. Using the proof of Lemma 1.7.2 we can approximate φ_ε in $\mathring{\mathscr{S}}(\bar{R}_n^+)$ by functions in $C_0^\infty(R_n^+)$, which proves the assertion. Hence a continuous linear form on $\mathring{\mathscr{S}}(\bar{R}_n^+)$ is uniquely determined by its restriction to $C_0^\infty(R_n^+)$ and can thus be identified with an element in $\mathscr{D}'(R_n^+)$.

Definition 1.7.4. *A continuous linear form u on $\mathring{\mathscr{S}}(\bar{R}_n^+)$ is called a temperate distribution in R_n^+; the set of such forms is a subset of $\mathscr{D}'(R_n^+)$ which is denoted by $\mathscr{S}'(\bar{R}_n^+)$.*

When $\varphi \in \mathring{\mathscr{S}}(\bar{R}_n^+)$, the partial Fourier transform is defined by

$$\hat{\varphi}_n(\xi', x_n) = \int e^{-i\langle x', \xi'\rangle} \varphi(x', x_n)\, dx', \tag{1.7.21}$$

where $x' = (x_1, \ldots, x_{n-1})$, $\xi' = (\xi_1, \ldots, \xi_{n-1})$ and $\langle x', \xi'\rangle = x_1\xi_1 + \ldots + x_{n-1}\xi_{n-1}$. An obvious modification of Lemma 1.7.1, which may be left to the reader, shows that the mapping $\varphi \to \hat{\varphi}_n$ is a continuous mapping of $\mathring{\mathscr{S}}(\bar{R}_n^+)$ into $\mathring{\mathscr{S}}(\bar{R}_n^+)$. From Theorem 1.7.1 it follows that the Fourier inversion formula is valid,

$$\varphi(x', x_n) = (2\pi)^{1-n} \int e^{i\langle x', \xi'\rangle} \hat{\varphi}_n(\xi', x_n)\, d\xi', \quad \varphi \in \mathring{\mathscr{S}}(\bar{R}_n^+). \tag{1.7.22}$$

The partial Fourier transformation can now be defined when $u \in \mathscr{S}'(\bar{R}_n^+)$ by means of the formula

$$\hat{u}_n(\varphi) = u(\hat{\varphi}_n), \quad \varphi \in \mathring{\mathscr{S}}(\bar{R}_n^+). \tag{1.7.23}$$

This definition coincides with (1.7.21) if (1.7.21) is absolutely convergent and it is clear that the Fourier inversion formula (1.7.22) can be extended to $\mathscr{S}'(\bar{R}_n^+)$. It is immediately verified that the partial Fourier transform of $D_j u$ is $\xi_j \hat{u}_n$, if $j < n$, and $D_n \hat{u}_n$, if $j = n$.

The restriction to R_n^+ of a distribution in $\mathscr{S}'(R_n)$ is obviously in $\mathscr{S}'(\bar{R}_n^+)$ and conversely it follows from the Hahn-Banach theorem that every element in $\mathscr{S}'(\bar{R}_n^+)$ can be extended to an element in $\mathscr{S}'(R_n)$. By analogy it is thus natural to denote by $\mathscr{S}(\bar{R}_n^+)$ the set of all restrictions to R_n^+ of elements in $\mathscr{S}(R_n)$ and to denote by $\mathring{\mathscr{S}}'(\bar{R}_n^+)$ the set of distributions in $\mathscr{S}'(R_n)$ with support in $\bar{R}_n^+$. Again the spaces $\mathring{\mathscr{S}}'(\bar{R}_n^+)$ and $\mathscr{S}(\bar{R}_n^+)$ are dual. Since we shall not use this fact, the proof may be left to the reader. However, we shall need the obvious fact that the partial Fourier transformation (with respect to $x_1, \ldots, x_{n-1}$) is an isomorphism of $\mathscr{S}(\bar{R}_n^+)$ on itself.

1.8. Distributions on a manifold. In Chapter X we also have to consider some spaces of distributions on manifolds, in particular on manifolds which bound open sets in R_n. In this section we shall give the basic definitions which this requires.

Definition 1.8.1. (Cf. DE RHAM [1], STEENROD [1].) *An n-dimensional manifold is a topological space in which each point has a neighborhood homeomorphic to some open set in R_n. A C^∞ structure on a manifold Ω is a family $\mathscr{F}$ of homeomorphisms $\varkappa$, called coordinate systems, of open sets $\Omega_\varkappa \subset \Omega$ on open sets $\tilde{\Omega}_\varkappa \subset R_n$ such that*

i) *If $\varkappa$ and $\varkappa' \in \mathscr{F}$, then the mapping*

$$\varkappa'\varkappa^{-1}: \varkappa(\Omega_\varkappa \cap \Omega_{\varkappa'}) \to \varkappa'(\Omega_\varkappa \cap \Omega_{\varkappa'}) \tag{1.8.1}$$

(between open sets in R_n) is infinitely differentiable. (This is then true of the inverse mapping also.)

ii) $$\bigcup_{\varkappa \in \mathscr{F}} \Omega_\varkappa = \Omega.$$

iii) *If $\varkappa_0$ is a homeomorphism of an open set $\Omega_0 \subset \Omega$ on an open set in R_n and the mapping*

$$\varkappa\varkappa_0^{-1}: \varkappa_0(\Omega_0 \cap \Omega_\varkappa) \to \varkappa(\Omega_0 \cap \Omega_\varkappa)$$

as well as its inverse is infinitely differentiable for every $\varkappa \in \mathscr{F}$, it follows that $\varkappa_0 \in \mathscr{F}$.

A manifold with a C^∞ structure is called a C^∞ manifold. The sets $\Omega_\varkappa$ are called coordinate patches and the cartesian coordinates of $\varkappa x$, $x \in \Omega_\varkappa$, are called local coordinates in $\Omega_\varkappa$.

The condition iii) in Definition 1.8.1 is in a way superfluous. For if $\mathscr{F}$ satisfies i) and ii), we can extend $\mathscr{F}$ in one and only one way to a family $\mathscr{F}'$ satisfying i), ii) and iii). In fact, the only such family $\mathscr{F}'$ is the set of all homeomorphisms $\varkappa'$ of open subsets $\Omega_{\varkappa'}$ of Ω on open subsets of R_n such that the mapping (1.8.1) and its inverse are infinitely differentiable for every $\varkappa \in \mathscr{F}$. The simple verification is left to the reader; clearly every extension of $\mathscr{F}$ satisfying i) is contained in this family $\mathscr{F}'$, and that $\mathscr{F}'$ satisfies i) and ii) and contains $\mathscr{F}$ follows from the fact that $\mathscr{F}$ satisfies i) and ii). A C^∞ structure can thus be defined by an arbitrary family $\mathscr{F}$ satisfying i) and ii) only, but if the condition iii) is dropped there are many families defining the same structure. Such a family is called a *complete* set of C^∞ coordinate systems and two such sets are called equivalent if they define the same C^∞ structure.