

Functional formalism for classical and quantum algebraic field theory

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1. Motivation

Formalisms of QFT

1) Canonical Quantization

Problem: UV divergences, Haag's thm.: No fixed Hilbert space
 rep. for interacting theories.

2.) Axiomatic QFT Wightman

Can derive e.g. PCT thm., antiparticles have same mass as particles
 spike-shar. thm.

problem: multiply fields \Rightarrow multiply distributions

↑
 interactions are too singular!

Haag-Kastler approach: local C^* -algebras, more powerful, but
 still can't treat interactions.

3.) Path-integral approach (Feynman)

can't define measure in "most" cases \rightarrow not rigorous.

4.) Constructive QFT: (Glimm-Jaffe, ...)

restricted - can't define realistic 4-dim theories

5.) Perturbative renormalized QFT

more rigorously Bogoliubov; — BPHZ-method
 Epstein-Glaser

gives formal power series, inserting numbers makes it nonrigor.
 radius of convergence = 0.

Open problem: Inclusion of gravity.

Simplified situation: QFT on curved spacetime.

Fields: ✓

Particles: no good notion

Vacuum: — "

Locality: ✓

(alg) Quantization: ✓

Hilbert space of States: no distinguished one

positivity of energy doesn't make sense (no translation invariance)
Specular condition

2. Functors, functionals, wavefront sets

(M, g) globally hyperbolic, (co)orientation w/f. (e.g. 4-dim contractible)
fix time-orientation → not nearly complete
→ Modifies $\mathbb{R} \times \sum_{t,x} \mathcal{S} \sum_{t,x} \text{Piem. w/f.}$

$\mathcal{E}(M)$ space of smooth fns = "configurations of scalar field"

$\mathcal{D}(M) = \mathcal{E}(M) \cap \{\text{compact support}\}$

$\mathcal{F}(M) = \text{maps: } \mathcal{E}(M) \rightarrow \mathbb{C}$ "observables"

$N \xrightarrow[X]{\text{embedding}}$ (of spacetimes N, M) preserving metric, time orientation
and causal relations (if γ causal curve connecting two points
 $x, y \in X(N) \Rightarrow \gamma \subseteq X(M)$)
such X are called admissible

Glob Hyp = category of spacetimes w/ admissible embeddings as morphisms

$\mathcal{E}: \text{Glob Hyp} \rightarrow \text{Top Vect}$ contravariant

$$(\mathcal{E}, X) \circ \phi = \psi \circ X$$

$\mathcal{D}: \text{Glob Hyp} \rightarrow \text{Top Vect}$ covariant

$$(\mathcal{D}, X) f = f \circ X^{-1}$$

$\mathcal{F}: \text{Glob Hyp} \rightarrow \text{Top Vect}$ invariant

Example: $\mathcal{F}(M) \ni F : F(\phi) = \int_{\text{supp } M(x)} \phi(x)^4 f(x) , f \in \mathcal{D}(M)$

$$(FG)(\phi) := F(\phi)G(\phi)$$

F is called smooth if F is ∞ -times diff.

i.e. $\langle F^{(n)}[\phi], \psi^{(n)} \rangle := \frac{d^n}{dx^n} F(\phi + \lambda \psi)|_{\lambda=0}$ exists for $\forall \phi, \psi$

and $(\phi, \psi) \mapsto \langle F^{(n)}[\phi], \psi^{(n)} \rangle$ is continuous

$$= \int dx, \int dx n! g(x_i) - g(x_n) \frac{\delta^n F(\phi)}{\delta \phi(x_i) \dots \delta \phi(x_n)}$$

F is called regular if it is smooth and the fractional derivs are smooth fcts.

F is called local if it is smooth and $\frac{\delta^2 F}{\delta \phi(x) \delta \phi(y)} = 0$ if $x \neq y$.

Note: local + regular \Leftrightarrow linear

$$\begin{aligned} \text{supp } F &= \{x \in M : \text{Unbddl U of } x \exists \phi \in \mathcal{E}(M), f \in \mathcal{D}(M), \text{supp } f \subset \\ &\quad \text{s.t. } F(\phi + f) = F(\phi)\} \\ &= \bigcup_{\phi} \text{supp } F^{(n)}(\phi) \end{aligned}$$

Wave front set of a distribution: $t \in \mathcal{D}'(\mathbb{R}^n), x \in \mathbb{R}^n, f \in \mathcal{D}(\mathbb{R}^n), f(x) \neq 0$.

$$(ft)^{\wedge}(k) := \langle ft, e^{ik \cdot} \rangle \text{ smooth fct of } k.$$

If $(ft)^{\wedge}$ strongly decays $\Rightarrow t$ is smooth in a nbhd of x .

(x, k) regular point of t if \exists conical nbhd in which $(tf)^{\wedge}$ decays fast.

$$\text{WF}(t) = \{ \text{points which are not regular} \} \subseteq \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$$

Example: $\text{WF } \delta : \langle \delta f, e^{ik \cdot} \rangle = f(0) \Rightarrow \text{WF } \delta = \{0\} \times (\mathbb{R}^n \setminus \{0\})$

$$\lim_{\epsilon \rightarrow 0^+} \overline{\frac{1}{x+i\epsilon}} : \left\langle \frac{1}{x+i\epsilon} f(x), e^{ik \cdot} \right\rangle \xrightarrow{\epsilon \rightarrow 0^+} \int_k^{\infty} dk' f(k')$$

$$\Rightarrow \text{WF } (\cdot) = \{0\} \times \mathbb{R}_+$$

strongly decaying for $k \rightarrow 0$
not for $k \rightarrow \infty$ if $f(0) \neq 0$.

\mathcal{F} microcausal if \mathcal{F} smooth and $WF(F^{(n)}[\phi]) \subseteq \Xi_n$

$$\Xi_n = \{ (x_1, \dots, x_n; \ell_1, \dots, \ell_n) \in T^* M^n : (\ell_1, \dots, \ell_n) \notin V_f^n \cup V_b^n \}$$

↑
forward lightcone ↘
backward lightcone

3.) Lagrangians, Poissons bracket, Poisson algebra of classical field theory

$L: D \rightarrow \mathcal{F}$ natural transformation
 $D(M) \xrightarrow{\sim} \mathcal{F}(M)$

i.e. ($M \xrightarrow{\cong} N$ admiss. embedding) the following diagram commutes:

$$\begin{array}{ccc} D(M) & \xrightarrow{\cong} & D(N) \\ L_M \downarrow & \curvearrowright & \downarrow L_N \\ \mathcal{F}(M) & \xrightarrow{\cong} & \mathcal{F}(N) \end{array}$$

L_N, L_M should be linear (or depend locally on f)

Example: $L_M(f) = \int d\text{vol}(x) f(x) \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) - \frac{1}{2} m^2 \phi(x)^2 \right]$

All natural trasf. are local functionals, covariant under space-time symmetries.
smoothness has to be assumed.

We say that L, L' are equivalent ($L \sim L'$) if
 $\text{supp } (L_M - L'_M)(f) \subseteq \text{supp } df$

action: $S = [L]$.

$$S': \mathcal{E} \rightarrow D', \quad \langle S'_M(\phi), f \rangle = \langle L_M(h^{(1)}), f \rangle \quad \text{where}$$

$h \equiv 1$ on $\text{supp } f$.

field eqn: $S'_M(\phi) = 0$.

linearized field eqn:
 $S''_M[\phi]: \mathcal{E}(M) \rightarrow D'(M)$

$$\langle S''_M[\phi]\psi, f \rangle = \langle L_M(h^{(2)})[\phi], \psi \otimes f \rangle \quad h \equiv 1 \text{ on } \text{supp } f.$$

Example: $\mathcal{L} = \int dx \left\{ \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \right\}$

$$S'(\phi)(x) = \square \phi(x) + V'(\phi(x))$$

$$(S''[\phi]\psi)(x) = -D\psi(x) - V''(\phi(x))\psi(x)$$

Assume: $S_M''[\phi]$ is normally hyperbolic diff. op., i.e.

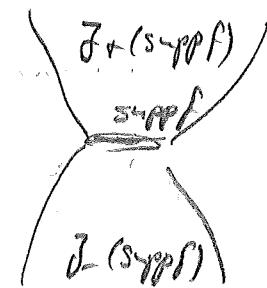
i.e. $S_M''(\phi) = \square + a^r \partial_\mu + b$ $\forall \phi \in M$
 smooth fcts of ϕ

more generally: only need unique advanced and retarded Green's fcts $G_{R,A}$

$$S_M''[\phi] \circ G_{R,A}[\phi] = G_{A,R}[\phi] = S_M''[\phi] = i\partial D(H)$$

uniquely fixed by $\text{supp } G_{R,A} \subseteq J^\pm(\text{supp } f)$

$$G[\phi] = G_R[\phi] - G_A[\phi] \quad \text{commutator fct.}$$



Poisson bracket $\{F_1, F_2\} := \langle F_1^{(1)}, G F_2^{(1)} \rangle$

for suitable F_1, F_2

Poisson bracket! antisymmetric ✓

Leibniz rule ✓

Jacobi identity not so trivial

crucial: 3rd derivative symmetric in indices.

\mathcal{J}_S ideal generated by $S'(\phi)$: $\ni \langle S'(\phi), X(\phi) \rangle$
 (wrt pointwise product)

Prop: \mathcal{J}_S is Poisson ideal ($\{\mathcal{J}_S, \cdot\} \subseteq \mathcal{J}_S$)

$(\mathcal{F}(M), \{ \cdot, \cdot \}) / \mathcal{J}_S$ Poisson algebra of classical field theory

4. Deformation quantization of free field theory and the algebra of Wick polynomials

S action s.t. S'' independent of ϕ

$$F_1 * F_2 = m \circ e^{i\frac{\hbar}{2} P_{1,2}} (F_1 \otimes F_2) , \quad P_{1,2} = \left\langle G, \frac{\delta^2}{\delta \phi_1 \delta \phi_2} \right\rangle$$

↑
pointwise multiplication
formal power series
in \hbar

$$= \int dx dy G(x,y) \frac{\delta(F_1)}{\delta \phi_1(x)} \frac{\delta(F_2)}{\delta \phi_2(y)}$$

F_1, F_2 regular (i.e., $F_1^{(n)}[\phi], F_2^{(n)}[\phi] \in \mathcal{D}(M)$)

Example: (i) $F_{1,2}[\phi] = \int d\text{vol}(x) f_{1,2}(x) \phi(x) , f_i \in \mathcal{D}(M)$

$$(F_1 * F_2)(\phi) = F_1(\phi) F_2(\phi) + \frac{i\hbar}{2} \langle f_1, G f_2 \rangle$$

$$[F_1, F_2]_* = i\hbar \{F_1, F_2\}$$

Linear functionals generate algebra of canonical commutation relations (modulo ideal of field equation)

$$(ii) F_3[\phi] = e^{iF_{1,2}[\phi]}$$

$$F_{3,4}^{[n]}[\phi] = i^n f_{1,2}^{\otimes n} F_{3,4}[\phi]$$

$$(F_3 \otimes F_4)(\phi) = e^{-i\frac{\hbar}{2} \langle f_1, G f_2 \rangle} e^{i(F_1[\phi] + F_2[\phi])}$$

\Rightarrow generates Weyl algebra of canonical commutation relations (modulo ideal of field eqn)

Neither algebra of CCR nor Weyl algebra behaves well for nonlinear local functionals.

Question: Can the $*$ -product be defined for nonlinear local functionals?

$$F_{1,2}[\phi] = \int d\text{vol}(x) f_{1,2}(x) \phi(x)^2$$

$$F_{1,2}^{(1)}[\phi] = 2 f_{1,2} \phi$$

$$F_{1,2}^{(2)}[\phi](x,y) = 2 f_{1,2}(x) \delta(x-y)$$

2nd order int: $(F_1 * F_2)^{(2)} = \left(\frac{i\pi}{2}\right)^2 \cdot 4 \int d\text{vol}(x) f_1(x) f_2(y) G(x,y)^2$

$$\text{WF}(G) = \{ (x, x', k, k') \in T^*(M \times M) , (k, k') \neq 0, \exists \text{ "null"-geodesic } x \sim x', \\ k \text{ lightlike, parallel to geodesic: } \dot{\gamma} \parallel k \\ k' \text{ lightlike, Par transport of } k \text{ along } \gamma = -k' \}$$

Product of distributions t, s is well-def. if $\text{WF}(t) \oplus \text{WF}(s) \cap \text{f}^{-1} \{ 0 \} = \emptyset$,

$\text{WF}(G)$ decomposes into future/past components (i.e. $k \in \overline{V_{\pm}}$)

$$\Rightarrow G = G_+ - G_- , \text{WF}(G_+) = \text{future component} \\ \text{WF}(G_-) = \text{past component}$$

New $*$ -product $*_t$: Replace G by $2G_+$

$*_t$ is equivalent to $*$ on regular fcts:

$$G_+ - \frac{G_+^t}{||} = G_-$$

$$iG_+ = iH + \frac{iG}{2}, H \text{ symmetric}$$

$$\Gamma_H = \langle H, \frac{\delta^2}{\delta t^2} \rangle$$

$$e^{\pm \frac{i}{2} \Gamma_H} [e^{-\frac{i}{2} \Gamma_H} F_1 *_t e^{-\frac{i}{2} \Gamma_H} F_2] = F_1 *_+ F_2$$

This is what is meant
by "equivalent".

$e^{-\frac{i}{2} \Gamma_H}$ corresponds to Wick ordering.

$$e^{-\frac{t}{2} \int dxdy f(x,y) \phi(x)\phi(y)} \\ = \int dxdy f(x,y) (\phi(x)\phi(y) - t H(x,y))$$

→ Can define differentiable fcts. of ϕ .

G_+ can be chosen to be a distribution of positive type ($\langle f, G_+ f \rangle \geq 0 \forall f$)

~ $\omega: F \mapsto F(\phi)$ coherent state, $\phi=0$: vacuum.
i.e. $\omega(F^*F) \geq 0$

5. Time-ordered products and interacting fields

$$F_1 \cdot_T F_2 = \begin{cases} F_1 * F_2 & \text{if } \text{supp } F_1 \cap \mathcal{J}_-(\text{supp } F_2) \\ F_2 * F_1 & \text{if } \dots \mathcal{J}_+ \dots \end{cases}$$

$$\int dx \phi(x) f_1(x) + \int dy \phi(y) f_2(y)$$

$$= \int dxdy f_1(x) f_2(y) \phi(x)\phi(y) + i\hbar \int dxdy G_F(x,y) f_1(x) f_2(y)$$

$$G_F(x,y) = \begin{cases} \mathcal{D}(x,y) G_+(x,y), & x \notin \mathcal{J}_-(y) \\ \mathcal{D}(x,y) G_+(y,x) & x \in \mathcal{J}_+(y) \end{cases}$$

$$G_F = \mathcal{D} G_+ + (-\mathcal{D}) G_-, \quad \mathcal{D}(x,y) = \begin{cases} 1 & t(x) \geq t(y) \\ 0 & t(y) > t(x) \end{cases}$$

$$F_1 \cdot_T F_2 =_{\text{mo}} e^{i P_F} (F_1 \otimes F_2), \quad P_F = \langle G_F, \frac{\delta^2}{\delta \phi_1 \delta \phi_2} \rangle$$

What is G_F^2 ? $WF G_F \supset WF \delta \Rightarrow G_F^2$ ill-defined,
well-def. on $M \times M \setminus \text{diag.}$

Thm: Let $t_0 \in D'(\mathbb{R}^n \setminus \{0\})$, $\text{sd}(t_0) = \inf \{d \in \mathbb{R} : \lambda^d t_0(\lambda \cdot) \rightarrow 0\}$

- If $\text{sd}(t_0) = \infty \Rightarrow \exists!$ extension
- If $\text{sd}(t_0) < n \Rightarrow \exists! \text{ extension}$
- If $\text{sd}(t_0) < \infty \Rightarrow \exists \text{ extension, not unique}$

Action for the interacting field:

$S + V$ interaction

$$R_V F = \left(\exp_{+} \frac{i}{\hbar} V \right)^{*_{+}^{-1}} *_{+} \left(\exp_{+} \frac{i}{\hbar} V + F \right)$$

Then

$R_V \langle S' + V', f \rangle = \langle S', f \rangle$, i.e. it maps the interacting field eqn to free field eqn.