

**Subject Areas:**

35B44 (primary); 35B40, 35L71,
74K15 (secondary).

Keywords:

semi-linear wave equation; finite-time
singularities; quenching; self-similar
solution; MEMS.

Author for correspondence:

Heiko Gimperlein

e-mail: heiko.gimperlein@uibk.ac.at

Quenching for a semi-linear
wave equation for MEMS

Heiko Gimperlein^{1,2}, Runan He³ and
Andrew A. Lacey⁴

¹ Leopold-Franzens-Universität Innsbruck, Engineering
Mathematics, Technikerstraße 13, 6020 Innsbruck,
Austria

² Department of Mathematical, Physical and Computer
Sciences, University of Parma, 43124, Parma, Italy

³ Institut für Mathematik, Martin-Luther-Universität
Halle-Wittenberg, 06120 Halle (Saale), Germany

⁴ Maxwell Institute for Mathematical Sciences and
Department of Mathematics, Heriot-Watt University,
Edinburgh, EH14 4AS, United Kingdom

We consider the formation of finite-time quenching singularities for solutions of semi-linear wave equations with negative power nonlinearities, as can model micro-electro-mechanical systems (MEMS). For radial initial data we obtain, formally, the existence of a sequence of quenching self-similar solutions. Also from formal asymptotic analysis, a solution to the PDE which is radially symmetric and increases strictly monotonically with distance from the origin quenches at the origin like an explicit spatially independent solution. The latter analysis and numerical experiments suggest a detailed conjecture for the singular behaviour.

1. Introduction

Micro-electro-mechanical systems, or MEMS, are micro-scale devices which transform electrical energy into mechanical energy and vice versa. MEMS combine the mechanics of deformable membranes, springs or levers with electric circuits, resistors, capacitors and inductors. They are crucial components of modern technology, from smart-phones and printer heads, to airbags, micro-valves and a wide variety of sensors.

An idealized electrostatically actuated MEMS capacitor device contains two conducting plates which, when the device is uncharged and at equilibrium, are close and parallel to each other.

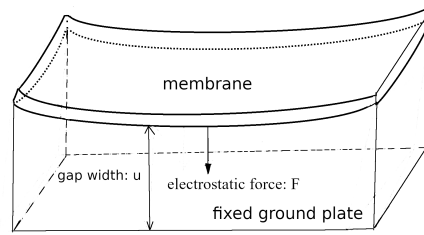


Figure 1. Idealized electrostatically actuated MEMS capacitor.

More generally, we suppose a fixed potential difference is applied; this potential difference acts across the plates, so that the MEMS device forms a capacitor. The upper of the plates is flexible but pinned around its edges. The other, lower, plate is taken to be rigid and flat. The flexible plate is here taken to behave as a membrane, and it is driven by the electrostatic force between the different electric potentials and so deforms towards to the rigid plate (see Fig. 1). The plate separation $u(x, t)$ can be modelled by the semi-linear equation (1.1), below, in space dimensions $n = 1, 2$, under the realistic assumption that: (i) the width of the gap, between the membrane and the bottom plate, is small compared to the device length; (ii) damping can be neglected. Such a device can suffer from an instability in which the plates touch when the voltages are increased beyond a certain critical value. This touchdown (also known as pull-in instability) restricts the range of stable operation of the device.

In the present paper we now consider the semi-linear wave equation in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$:

$$\frac{\partial^2 u}{\partial t^2} = \Delta u - \frac{1}{u^2}, \quad (1.1)$$

the case of $n = 1$ having been looked at in [1]. The solution u of (1.1) may cease to exist after a finite time T_{\max} . The main purpose of this paper is to study the local behaviour of u near $t = T_{\max}$.

The slightly more general model,

$$\frac{\partial^2 u}{\partial t^2} + \epsilon \frac{\partial u}{\partial t} - \Delta u = -\frac{\beta_F}{u^2}, \quad u|_{\partial\Omega} = 1, \quad (1.2)$$

with a first-derivative term $\frac{\partial u}{\partial t}$ modelling damping, is to date, better understood.

The pull-in voltage (the applied voltage is represented here by β_F) separates the stable operation regime for which the membrane can approach a steady state, from the “touchdown” regime for which the membrane collapses onto the rigid plate. There exists a critical value β_F^* such that touchdown takes place in finite time for large $\beta_F > \beta_F^*$, while there is at least one stationary solution of (1.2) for small $\beta_F < \beta_F^*$ [2], see also [3]. Mathematically, the collapse, with finite-time existence occurring through the touchdown, is seen to be through quenching, $\inf_{x \in \Omega} u \rightarrow 0$ as $t \rightarrow T_{\max}$. **A point where u tends to 0 is a quenching point; T_{\max} is the quenching time.**

Quenching in (1.2) has been thoroughly studied when the damping term dominates, i.e. $\epsilon \gg 1$ [4]. Then the semi-linear damped wave equation (1.2) can be reduced to a parabolic equation, for which general techniques based on the maximum principle have been developed. Specifically for the quenching profile we refer to [5], as well as [6–10].

This article studies the quenching profile when the contribution of the inertial terms dominates, i.e. $\epsilon \ll 1$. In this regime, (1.2) simplifies to the hyperbolic model

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = -\frac{\beta_F}{u^2}, \quad u|_{\partial\Omega} = 1. \quad (1.3)$$

Local and global existence results for (1.3) and the dynamical behaviour of solutions, depending on the parameter β_F , have been obtained in [11]. The analysis of quenching in semi-linear wave equations goes back to [12], who showed, in our language, the existence of a critical value β_F^* separating quenching from stationary behaviour in one dimension. While qualitative results are now well-understood for a variety of hyperbolic models [2,11,13–15], the behaviour close to quenching, i.e. the quenching profile, has not been studied previously for higher dimensions $n \geq 2$.

To understand the local behaviour of u near the quenching time, we consider the existence of self-similar, radial solutions to (1.1) in \mathbb{R}^n . We look for these self-similar solutions using new variables:

$$u(x, t) = (T_{\max} - t)^{\frac{2}{3}} v(\eta), \quad \eta = \frac{|x|}{T_{\max} - t}.$$

The form $u = (T_{\max} - t)^{\frac{2}{3}} v$ is indicated by the spatially uniform solution of (1.1). For v to indeed give a solution u at quenching, i.e. as $t \rightarrow T_{\max}$ so $\eta \rightarrow \infty$ for fixed $|x|$, v should behave as $\text{const.} \times \eta^{\frac{2}{3}}$ for large η . A key aim of the present paper is thus to investigate the existence of smooth v having the right growth condition at infinity, such that u satisfies (1.1) in \mathbb{R}^n .

In dimension $n = 1$, reference [11] shows that the trivial, constant $v_0 \equiv c^* = \left(\frac{9}{2}\right)^{\frac{1}{3}}$ is the only smooth global v . In this article we show that in dimensions $2 \leq n \leq 7$, however, non-trivial radial smooth quenching solutions exist in \mathbb{R}^n : There is a sequence of non-trivial smooth global solutions v_j with $v_j(0) \rightarrow 0^+$, which satisfy the growth condition at infinity.

A precise statement of our results is given in Theorem 2.1 in Section 2.

Results related to those of Section 2 have been proved for blow-up of solutions to semi-linear PDEs with nonlinearities u^p in [16] for wave equations and in [17] for heat equations. Although the formal results of the present paper are closely related to those of those two works, being somewhat weaker, the structure of the relevant self-similar equation (2.3) differs from that of the corresponding ODEs for papers [16] and [17]: a global solution is not guaranteed *a-priori* in our case. This means that key steps do not apply, and the Lyapunov function used in [16] might not exist here. Other related works on blow-up are referenced in [17] and [16].

The relevance of the global self-similar solutions v_j for quenching in (1.3) hinges on their stability. In Section 3 we go into this by showing formal and numerical evidence that in n dimensions the trivial self-similar solution $v_0 = c^*$ is stable, other than to a very particular class of (spatially uniform) perturbations which correspond to solutions quenching at different times. The generic local behaviour of a strictly increasing radial quenching solution u near the quenching time T_{\max} is given by $u(x, t) = (T_{\max} - t)^{\frac{2}{3}} V(\xi, t)$, with $\xi = \frac{r}{(T_{\max} - t)^{1/2}}$, $r = |x|$ being the distance to the quenching point, and

$$V(\xi, t) \sim C\xi^{\frac{4}{3}} \quad \text{as } \xi \rightarrow \infty \quad \text{and} \quad V(0, t) \rightarrow c^*$$

for $t \rightarrow T_{\max}$. Correspondingly,

$$u(x, T_{\max}) \sim Cr^{\frac{4}{3}} \quad \text{as } r \rightarrow 0 \quad \text{with} \quad u(0, t) \sim c^* (T_{\max} - t)^{\frac{2}{3}} \quad \text{as } t \rightarrow T_{\max},$$

provided u is a strictly increasing function of r near the quenching time. Here C is a constant which depends on n , and on the initial and boundary conditions for the semi-linear wave equation. We conjecture that for $2 \leq n \leq 7$, the smooth self-similar solutions v_j are unstable for $j \in \mathbb{N}$, as supported by the formal and numerical results in the paper [11] for the problem in one space dimension.

A precise statement is given in Conjecture 3.1 in Section 3.

We discuss the importance of our results in the concluding Section 4.

2. Global Existence of Self-Similar Solutions

To understand the local behaviour of the solution u to (1.1) near the quenching time, we consider the existence of self-similar, radial solutions to the equation in \mathbb{R}^n :

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r} \frac{\partial u}{\partial r} - \frac{1}{u^2}. \quad (2.1)$$

Here, $r = |x|$ and $u = u(r, t)$. We can look for self-similar solutions using new variables:

$$u(r, t) = (T_{\max} - t)^{\frac{2}{3}} v(\eta), \quad \eta = \frac{r}{T_{\max} - t}. \quad (2.2)$$

From (2.1), v then satisfies the self-similar equation

$$(1 - \eta^2) \frac{d^2 v}{d\eta^2} + \left(\frac{n-1}{\eta} - \frac{2}{3} \eta \right) \frac{dv}{d\eta} + \frac{2}{9} v - \frac{1}{v^2} = 0. \quad (2.3)$$

It can be noted, from the transformations (2.2), that for v to indeed give a solution u at quenching, i.e. as $t \rightarrow T_{\max}$ so $\eta \rightarrow \infty$ for fixed r , v should behave as $\text{const.} \times \eta^{\frac{2}{3}}$ for large η . A key aim of the present paper is thus to investigate the existence of smooth solutions of (2.3) having the right growth condition at infinity.

Our analysis in this section will be based on a shooting method: imposing initial conditions at $\eta = 0$,

$$v(0) = c > 0, \quad \frac{dv}{d\eta}(0) = 0, \quad (2.4)$$

we determine those c for which v extends smoothly across $\eta = 1$ and has the physically relevant behaviour $v \sim \text{const.} \times \eta^{2/3}$ as $\eta \rightarrow \infty$.

Note that (2.3) admits the trivial, constant solution $v_0 \equiv c^* = \left(\frac{9}{2}\right)^{\frac{1}{3}}$. In dimension $n = 1$, reference [11] shows that this is the only smooth global solution of (2.3). In dimensions $2 \leq n \leq 7$, however, non-trivial global smooth solutions exist:

Theorem 2.1 (Global Existence of Self-Similar Solutions, Formal Theorem). *a) There exists a discrete set of initial conditions $c = c_j < c^*$ in (2.4), with $c_j \rightarrow 0^+$ as $j \rightarrow \infty$, such that the self-similar equation (2.3) has a non-trivial smooth global solution v_j for $2 \leq n \leq 7$.*

b) The solutions $v_j(\eta) \sim a\eta^{\frac{2}{3}}$ for $c_j^{\frac{3}{2}} \ll \eta \ll 1$, where $a = \left[\frac{2}{3} \left(n - \frac{4}{3}\right)\right]^{-\frac{1}{3}}$, and $v_j(\eta) \sim b_j \eta^{\frac{2}{3}}$ for $\eta \rightarrow \infty$, where b_j is a constant depending upon n and j .

Theorem 2.1 is proved formally in this section below.

Formal asymptotics also indicate that the smooth solutions $v_j(\eta)$ are monotonically increasing with respect to η for large enough j .

Note that (2.3) presents challenges through the degeneracy of the coefficient of the highest-order derivative at $\eta = 1$, the light cone, and the possibility of limited existence of solution, as v could vanish at some finite value of η .

We briefly remark that, for a steady state, equation (2.1) reduces to a radial ODE (2.5) for which the qualitative behaviour has been studied extensively:

$$\frac{d}{d\eta} \left(\eta^{n-1} \frac{dv}{d\eta} \right) + \lambda \eta^{n-1} \mathcal{F}(v) = 0, \quad (2.5)$$

here writing $v = u$ and $\eta = r$, with nonlinearity $\mathcal{F}(v) = -1/v^2$, similar to an equation of Emden-Fowler type. For $n > 2$ equation (2.5) has been considered in the classical work [18], establishing the local behaviour of a singular solution. For existence, nonexistence and multiplicity of solutions we refer to the seminal paper by Gelfand [19].

(a) Asymptotic Behaviours of Self-Similar Solutions

To prove Theorem 2.1 formally we must understand the asymptotic behaviours of the self-similar solutions, as indicated by the following Proposition 2.1, Lemma 2.1 and Lemma 2.2.

Asymptotic Behaviour of Self-Similar Solution Near 0

Proposition 2.1. For $0 < c \ll (\frac{9}{2})^{\frac{1}{3}}$, the asymptotic behaviour of the solution to the self-similar equation (2.3) with the initial values (2.4) for $c^{\frac{3}{2}} \ll \eta \ll 1$ is of the form, when $2 \leq n \leq 7$,

$$v \sim a\eta^{\frac{2}{3}} + A_1 c^{\frac{3}{4}n - \frac{1}{2}} \eta^{1 - \frac{n}{2}} \cos\left(b \ln(\eta c^{-\frac{3}{2}}) + A_2\right) + \dots; \quad (2.6)$$

and when $n \geq 8$,

$$v \sim a\eta^{\frac{2}{3}} + c^{\frac{3}{4}n - \frac{1}{2}} \eta^{1 - \frac{n}{2}} \left(A_3 c^{-\frac{3}{2}b} \eta^b + A_4 c^{\frac{3}{2}b} \eta^{-b} \right) + \dots \quad (2.7)$$

Here, $a = [\frac{2}{3}(n - \frac{4}{3})]^{-\frac{1}{3}}$, $b = \left| (\frac{n}{2} - \frac{7}{3})^2 - \frac{8}{3} \right|^{1/2} \geq 0$, and $\{A_j\}_{j=1}^4$ are suitable constants dependent on n , but not c .

Asymptotic Behaviour of Self-Similar Solution Near 1

Lemma 2.1. Any positive solution v of the equation (2.3) admits a decomposition into a smooth and a singular part, $v = v_{reg} + v_{sing, \pm}$, at $\eta = 1^{\pm}$. We have

$$v_{sing, \pm}(\eta) = \sum_{\alpha \in E} c_{\alpha}^{\pm} |\eta - 1|^{\alpha},$$

where c_{α}^{\pm} are constants and E is specified below (2.26). The smallest exponent $\alpha \in E$ is given by $\alpha = \frac{n}{2} + \frac{1}{6}$.

Asymptotic Behaviour of Self-Similar Solution Near Infinity

Lemma 2.2. A solution v of the equation (2.3) existing for large η is either asymptotically constant $v(\eta) \rightarrow c^*$ for $\eta \rightarrow \infty$, or it has an asymptotic form

$$v(\eta) \sim \sum_{\alpha \in E} b_{\alpha} \eta^{\alpha} = b_{\frac{2}{3}} \eta^{\frac{2}{3}} + \dots, b_{\frac{2}{3}} > 0, \quad (2.8)$$

for η near ∞ . The dots denote lower-order terms.

Proof of Proposition 2.1

For small $0 < c \ll (\frac{9}{2})^{\frac{1}{3}}$, we set

$$v = cV, \quad \eta = c^{\frac{3}{2}} \bar{r}, \quad V = V(\bar{r}), \quad v = v(\eta).$$

From (2.3), we obtain

$$\frac{d^2 V}{d\bar{r}^2} + \frac{n-1}{\bar{r}} \frac{dV}{d\bar{r}} - \frac{1}{V^2} = c^3 \left(\bar{r}^2 \frac{d^2 V}{d\bar{r}^2} + \frac{2}{3} \bar{r} \frac{dV}{d\bar{r}} - \frac{2}{9} V \right), \quad V(0) = 1, \quad \frac{dV}{d\bar{r}}(0) = 0. \quad (2.9)$$

Decomposing V as

$$V = V_0 + c^3 V_1, \quad (2.10)$$

the leading-order term V_0 is a solution of the initial-value problem

$$\frac{d^2 V_0}{d\bar{r}^2} + \frac{n-1}{\bar{r}} \frac{dV_0}{d\bar{r}} - \frac{1}{V_0^2} = 0, \quad V_0(0) = 1, \quad \frac{dV_0}{d\bar{r}}(0) = 0. \quad (2.11)$$

A solution to an ODE like that of (2.11) has been extensively studied in terms of phase plane analysis in the paper [18] of Joseph and Lundgren.

Rescale the variables V_0 and \bar{r} by

$$V_0 = \bar{r}^{\frac{2}{3}} Z_0, \quad \bar{r} = e^{\bar{s}}, \quad \bar{s} = \ln \bar{r}. \quad (2.12)$$

Note that the initial conditions in (2.11) give

$$Z_0 \sim e^{-\frac{2}{3}\bar{s}} \quad \text{and} \quad \frac{dZ_0}{d\bar{s}} \sim -\frac{2}{3}e^{-\frac{2}{3}\bar{s}} \quad \text{as } \bar{s} \rightarrow \infty. \quad (2.13)$$

The differential equation of (2.11) in $\bar{r} > 0$ transforms into the following equation in $\bar{s} \in \mathbb{R}$:

$$\frac{d^2 Z_0}{d\bar{s}^2} + \left(n - \frac{2}{3}\right) \frac{dZ_0}{d\bar{s}} + \frac{2}{3} \left(n - \frac{4}{3}\right) Z_0 = \frac{1}{Z_0^2}. \quad (2.14)$$

We rewrite (2.14) as a system

$$\frac{d}{d\bar{s}} \begin{pmatrix} Z_0 \\ Z_0' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{2}{3} \left(\frac{4}{3} - n\right) & \frac{2}{3} - n \end{pmatrix} \begin{pmatrix} Z_0 \\ Z_0' \end{pmatrix} + \begin{pmatrix} 0 \\ Z_0^{-2} \end{pmatrix}. \quad (2.15)$$

An equilibrium point of the system (2.15) is given by the equation

$$\frac{d}{d\bar{s}} \begin{pmatrix} Z_0 \\ Z_0' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$(Z_0, Z_0') = (a, 0)$ is a possible solution, and an equilibrium point of the system (2.15) in the phase plane, provided $a = \left[\frac{2}{3} \left(n - \frac{4}{3}\right)\right]^{-\frac{1}{3}}$. Correspondingly, $a\eta^{\frac{2}{3}}$ is a possible solution of (2.14).

It is easily seen that in fact $(a, 0)$ is the only equilibrium of (2.15), that the solution of (2.14) satisfying condition (2.13) has $Z_0 > 0$ for all \bar{s} , and that (2.14) has the Lyapunov function $\frac{1}{2} \left(\frac{dZ_0}{d\bar{s}}\right)^2 + \frac{1}{3} \left(n - \frac{4}{3}\right) Z_0^2 + \frac{1}{Z_0}$. It is thus clear that $(Z_0, Z_0') \rightarrow (a, 0)$ as $\bar{s} \rightarrow \infty$.

We linearize (2.15) around the stable point $(Z_0, Z_0') = (a, 0)$ and obtain

$$\frac{d}{d\bar{s}} \begin{pmatrix} Z_0 \\ Z_0' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{2}{3} \left(\frac{4}{3} - n\right) - \frac{2}{a^3} & \frac{2}{3} - n \end{pmatrix} \begin{pmatrix} Z_0 \\ Z_0' \end{pmatrix} := \mathcal{A} \begin{pmatrix} Z_0 \\ Z_0' \end{pmatrix}. \quad (2.16)$$

The characteristic equation of the matrix \mathcal{A} is given by

$$\lambda \left(\lambda - \frac{2}{3} + n\right) - \frac{2}{3} \left(\frac{4}{3} - n\right) + \frac{2}{a^3} = \lambda \left(\lambda - \frac{2}{3} + n\right) + 2 \left(n - \frac{4}{3}\right) = 0. \quad (2.17)$$

Its roots give us the following for the eigenvalues of \mathcal{A}

$$\lambda_{\pm} = \frac{1}{3} - \frac{n}{2} \mp \sqrt{\Delta}, \quad \Delta = \left(\frac{n}{2} - \frac{7}{3}\right)^2 - \frac{8}{3}.$$

If $2 \leq n \leq 7$, then $\Delta < 0$, λ_+ and λ_- are complex eigenvalues with negative parts. Thus $(Z_0, Z_0') = (a, 0)$ is a stable spiral point.

If $n \geq 8$, then $\Delta \geq 0$, λ_+ and λ_- are real and negative eigenvalues, hence $(Z_0, Z_0') = (a, 0)$ is a stable node.

The asymptotic form of a solution Z_0 to the equation (2.14) near the steady state is given by $Z_0 \sim a + \tilde{Z}_0$ for $\bar{s} \rightarrow \infty$, where $\tilde{Z}_0 = \tilde{Z}_0(\bar{s})$ satisfies

$$\frac{d^2 \tilde{Z}_0}{d\bar{s}^2} + \left(n - \frac{2}{3}\right) \frac{d\tilde{Z}_0}{d\bar{s}} + 2 \left(n - \frac{4}{3}\right) \tilde{Z}_0 = 0. \quad (2.18)$$

Solutions of (2.18) are of the form

$$\tilde{Z}_0(\bar{s}) = A e^{-l_+ \bar{s}} + B e^{-l_- \bar{s}}, \quad (2.19)$$

where A and B are arbitrary constants and l_+ and l_- are the roots of

$$l_{\pm}^2 - \left(n - \frac{2}{3}\right) l_{\pm} + 2 \left(n - \frac{4}{3}\right) = 0.$$

Then $l_{\pm} = -\lambda_{\pm}$,

$$l_{\pm} = \frac{n}{2} - \frac{1}{3} \pm \sqrt{\Delta} = \frac{n}{2} - \frac{1}{3} \pm bi, \quad \text{when } \frac{4}{3} < n < 8,$$

$$l_{\pm} = \frac{n}{2} - \frac{1}{3} \pm \sqrt{\Delta} = \frac{n}{2} - \frac{1}{3} \pm b, \quad \text{when } n \geq 8.$$

Here, $b = \sqrt{|\Delta|} \geq 0$ and $\Delta = \left(\frac{n}{2} - \frac{7}{3}\right)^2 - \frac{8}{3}$. Hence, we obtain the asymptotic forms of Z_0

$$Z_0(\bar{s}) \sim a + A_1 e^{\left(\frac{1}{3} - \frac{n}{2}\right)\bar{s}} \cos(b\bar{s} + A_2), \quad \text{when } 2 \leq n \leq 7, \quad (2.20a)$$

$$Z_0(\bar{s}) \sim a + A_3 e^{\left(b + \frac{1}{3} - \frac{n}{2}\right)\bar{s}} + A_4 e^{\left(-b + \frac{1}{3} - \frac{n}{2}\right)\bar{s}}, \quad \text{when } n \geq 8, \quad (2.20b)$$

for $\bar{s} \rightarrow \infty$. Here, constants $A_i, i = 1, 2, 3, 4$, are independent of c since both differential equation (2.14) and initial conditions (2.13) involve only the dimension n .

Because $V \sim (\bar{r})^{\frac{2}{3}} Z_0$, and $\bar{r} = e^{\bar{s}}$, then, for $\bar{r} \gg 1$,

$$V \sim a(\bar{r})^{\frac{2}{3}} + A_1(\bar{r})^{1 - \frac{n}{2}} \cos(b \ln \bar{r} + A_2), \quad \text{when } 2 \leq n \leq 7, \quad (2.21)$$

$$V \sim a(\bar{r})^{\frac{2}{3}} + (\bar{r})^{1 - \frac{n}{2}} + \left(A_3(\bar{r})^b + A_4(\bar{r})^{-b}\right), \quad \text{when } n \geq 8. \quad (2.22)$$

Since $cV = v$ and $\eta = c^{\frac{3}{2}}\bar{r}$, we obtain the asymptotic form of v , when $2 \leq n \leq 7$,

$$v \sim a\eta^{\frac{2}{3}} + A_1 c^{\frac{3}{4}n - \frac{1}{2}} \eta^{1 - \frac{n}{2}} \cos\left(b \ln(\eta c^{-\frac{3}{2}}) + A_2\right), \quad (2.23)$$

and when $n \geq 8$,

$$v \sim a\eta^{\frac{2}{3}} + c^{\frac{3}{4}n - \frac{1}{2}} \eta^{1 - \frac{n}{2}} \left(A_3 c^{-\frac{3}{2}b} \eta^b + A_4 c^{\frac{3}{2}b} \eta^{-b}\right). \quad (2.24)$$

This concludes our discussion of the asymptotics in Proposition 2.1.

Proof of Lemma 2.1

Without loss of generality, we consider the case $\eta > 1$. Let v be a positive solution of the equation (2.3), and $v(1) = c_0^+ > 0$.

We set $\hat{\eta} = \eta - 1$, $\hat{v}(\hat{\eta}) = v(\eta)$. Then \hat{v} satisfies

$$-\hat{\eta}(\hat{\eta} + 2) \frac{d^2 \hat{v}}{d\hat{\eta}^2} + \left(\frac{n-1}{\hat{\eta}+1} - \frac{2}{3}(\hat{\eta} + 1)\right) \frac{d\hat{v}}{d\hat{\eta}} + \frac{2}{9}\hat{v} - \frac{1}{\hat{\eta}^2}, \quad \hat{v}(0) = C_1. \quad (2.25)$$

The asymptotic expansion of the solution to (2.25) takes the form

$$\hat{v}(\hat{\eta}) \sim \sum_{\alpha \in E \cup \mathbb{N}_0} c_\alpha^+ \hat{\eta}^\alpha = \hat{v}_{reg,+}(\hat{\eta}) + \hat{v}_{sing,+}(\hat{\eta}), \quad (2.26)$$

where $\hat{v}_{sing,+}(\hat{\eta}) \sim \sum_{\alpha \in E} c_\alpha^+ \hat{\eta}^\alpha$. Here, $E \subset \mathbb{C}$ with $\min\{\text{Re } \alpha : \alpha \in E\} > 0$, and $c_0^+ > 0$ as above.

To determine the smallest $\underline{\alpha} \in E$, we substitute (2.26) into (2.25) and obtain

$$\begin{aligned} & -\hat{\eta}(\hat{\eta} + 2) \left(2\hat{c}_2^+ + \dots + \underline{\alpha}(\underline{\alpha} - 1)\hat{c}_\alpha^+ \hat{\eta}^{\underline{\alpha}-2} + [\underline{\alpha} + 1]\underline{\alpha}\hat{c}_{\underline{\alpha}+1}^+ \hat{\eta}^{\underline{\alpha}-1} + \dots\right) \\ & + \left[(n-1)(1 - \hat{\eta} + \hat{\eta}^2 + \dots) - \frac{2}{3}(\hat{\eta} + 1)\right] \left[\hat{c}_1^+ + 2\hat{c}_2^+ \hat{\eta} + \dots + \underline{\alpha}\hat{c}_\alpha^+ \hat{\eta}^{\underline{\alpha}-1} + (\underline{\alpha} + 1)\hat{c}_{\underline{\alpha}+1}^+ \hat{\eta}^\alpha + \dots\right] \\ & + \frac{2}{9} \left(\hat{c}_0^+ + \hat{c}_1^+ \hat{\eta} + \hat{c}_2^+ \hat{\eta}^2 + \dots + \hat{c}_\alpha^+ \hat{\eta}^\alpha + \hat{c}_{\underline{\alpha}+1}^+ \hat{\eta}^{\underline{\alpha}+1} + \dots\right) \\ & - (\hat{c}_0^+)^{-2} + \frac{2\hat{c}_1^+}{(\hat{c}_0^+)^3} \hat{\eta} + \frac{2\hat{c}_2^+}{(\hat{c}_0^+)^3} \hat{\eta}^2 + \dots + \frac{2\hat{c}_\alpha^+}{(\hat{c}_0^+)^3} \hat{\eta}^\alpha + \frac{2\hat{c}_{\underline{\alpha}+1}^+}{(\hat{c}_0^+)^3} \hat{\eta}^{\underline{\alpha}+1} + \dots = 0. \end{aligned}$$

Dots indicate higher-order terms.

In order to find the smallest non-integer exponent $\underline{\alpha}$ in the set E , we balance the terms of $\hat{\eta}^{\underline{\alpha}-1}$ and get $-2\underline{\alpha}(\underline{\alpha} - 1) + \left(n - \frac{5}{3}\right)\underline{\alpha} = 0$, or $\underline{\alpha} = \frac{n}{2} + \frac{1}{6}$. Hence,

$$\hat{v}(\hat{\eta}) \sim \hat{v}_{reg,+}(\hat{\eta}) + \hat{c}_\alpha^+ \hat{\eta}^{\frac{n}{2} + \frac{1}{6}} + \dots$$

After possibly modifying the splitting $\hat{v} = \hat{v}_{reg,-} + \hat{v}_{sing,-}$, we may assume that $\hat{v}_{reg,+} = \hat{v}_{reg,-}$. This concludes the proof of Lemma 2.1.

Proof of Lemma 2.2

The asymptotic expansion of the solution to (2.3) takes the form $v(\eta) \sim \sum_{\alpha \in E} b_\alpha \eta^\alpha$ for some $E \subset \mathbb{C}$, and $b_{\alpha_0} \neq 0$ for some α_0 of maximal real part. α_0 is determined by substituting the expansion into (2.3):

$$b_{\alpha_0}(-\alpha_0(\alpha_0 - 1) - \frac{2}{3}\alpha_0 + \frac{2}{9})\eta^{\alpha_0} - b_{\alpha_0}^{-2}\eta^{-2\alpha_0} + \dots = 0, \quad (2.27)$$

where dots indicate lower order terms. Depending on the sign of $\text{Re } \alpha_0$, we obtain:

When $\text{Re } \alpha_0 > 0$, the nonlinear term $b_{\alpha_0}^{-2}\eta^{-2\alpha_0}$ is of lower order in (2.27), and $\alpha_0 = \frac{2}{3}$ is the unique positive root of $-\alpha_0(\alpha_0 - 1) - \frac{2}{3}\alpha_0 + \frac{2}{9} = 0$.

For $\text{Re } \alpha_0 = 0$, (2.27) leads to $\alpha_0 = 0$ and $b_0 = c^*$. Finally, for $\text{Re } \alpha_0 < 0$, the nonlinear term is the leading term in (2.27) and leads to the contradiction $b_{\alpha_0}^{-2} = 0$.

We conclude that either $\alpha_0 = \frac{2}{3}$ or $\alpha_0 = 0$, and therefore the assertion of Lemma 2.2.

(b) Formal Proof of Global Existence of Self-Similar Solutions

We now prove Theorem 2.1 formally.

A smooth solution of (2.3) is given by

$$v(\eta) \sim a\eta^{\frac{2}{3}} + v_l(\eta). \quad (2.28)$$

Here, a is given in Proposition 2.1 and v_l is a general solution to the linearized equation (2.29)

$$(1 - \eta^2) \frac{d^2 v_l}{d\eta^2} + \left(\frac{n-1}{\eta} - \frac{2}{3}\eta\right) \frac{dv_l}{d\eta} + \left(\frac{2}{9} + \frac{2}{v_l^3}\right) v_l = 0, \quad (2.29)$$

which is obtained by linearizing (2.3) around $v_s = a\eta^{\frac{2}{3}}$. Then v_l can be found by first making a change of variable, as earlier, $s = \ln \eta$, so that (2.29) becomes, using the definition of v_s ,

$$\left(1 - e^{2s}\right) \frac{d^2 v_l}{ds^2} + \left((n-2) + \frac{1}{3}e^{2s}\right) \frac{dv_l}{ds} + \left(\frac{4}{3}\left(n - \frac{4}{3}\right) + \frac{2}{9}e^{2s}\right) v_l = 0, \quad (2.30)$$

with v_l regular at $s = 0$ ($\eta = 1$). We are interested in the asymptotic behaviour of v_l for $s \rightarrow -\infty$ ($\eta \rightarrow 0$), in particular for cases of $2 \leq n \leq 7$.

Note that the characteristic equation

$$\lambda^2 + (n-2)\lambda + \frac{4}{3}\left(n - \frac{4}{3}\right) = 0.$$

associated with the limiting form of (2.30),

$$\frac{d^2 v_l}{ds^2} + \frac{1}{3}e^{2s} \frac{dv_l}{ds} + \frac{4}{3}\left(n - \frac{4}{3}\right) v_l = 0,$$

got by taking e^{2s} is small, i.e. s large and negative, has roots $(1 - \frac{n}{2}) \pm ib$, with $b = \sqrt{\frac{n^2}{4} - \frac{7}{3} + \frac{25}{9}}$. Then repeated use of variation of parameters to solve (2.30), writing

$$v_l = v_1 e^{(1 - \frac{n}{2})s} \cos bs + v_2 e^{(1 - \frac{n}{2})s} \sin bs,$$

gives for $2 \leq n \leq 7$, first that v_1 and v_2 are bounded for $s \rightarrow -\infty$, and then that v_1 and v_2 both have limits as $s \rightarrow -\infty$.

It follows that

$$v_l \sim B_1 e^{(1 - \frac{n}{2})s} \cos(bs + B_2) \text{ for } s \rightarrow -\infty$$

and then that

$$v \sim a\eta^{\frac{2}{3}} + B_1 \eta^{1 - \frac{n}{2}} \cos(b \ln \eta + B_2) \text{ for } \eta \rightarrow 0, \quad (2.31)$$

where B_2 is a constant determined by n .¹

¹It should be noted that the ODE (2.29) can be solved in terms of hypergeometric functions, and the limiting behaviour (2.31) got from this explicit solution. For example, in the case of $n=3$, we have $v_l = C_l \eta^{1 - \frac{n}{2}} \text{Im}\{\beta_2 e^{ib \ln \eta} {}_2F_1\left(\left[\frac{2}{3} - \frac{n}{4} + \frac{ib}{2}, \frac{1}{6} - \frac{n}{4} + \frac{ib}{2}\right], [1 + ib], \eta^2\right)\}$, b as above with $n=3$, $\beta_2 = \left[6\Gamma\left(1 - \frac{\sqrt{71}i}{6}\right) \pi \sqrt{3}(-2)^{\frac{2}{3}}\right] / \left[5\Gamma\left(\frac{2}{3}\right) \Gamma\left(-\frac{1}{12} - \frac{\sqrt{71}i}{12}\right) \Gamma\left(-\frac{7}{12} - \frac{\sqrt{71}i}{12}\right)\right]$ and C_l is a real constant.

Matching (2.31) with (2.6),

$$v \sim a\eta^{\frac{2}{3}} + A_1 c^{\frac{3}{4}n - \frac{1}{2}} \eta^{1 - \frac{n}{2}} \cos\left(b \ln(\eta c^{-\frac{3}{2}}) + A_2\right) = a\eta^{\frac{2}{3}} + A_1 c^{\frac{3}{4}n - \frac{1}{2}} \eta^{1 - \frac{n}{2}} \cos\left(b \ln \eta - \frac{3}{2} b \ln c + A_2\right),$$

gives

$$B_l = A_1 c^{\frac{3}{4}n - \frac{1}{2}}, \quad B_2 + 2\pi k = -\frac{3}{2} b \ln c + A_2, \quad (2.32)$$

and

$$-\ln c = \frac{2}{3b}(B_2 - A_2 + 2\pi k) \quad (2.33)$$

where k is a large positive integer.

This results in a sequence of solutions c_k with $c_k \geq 0$. The discrete set of such values c_k for c is what we look for.

As we consider the smooth solution v existing beyond $\eta = 1$, the integral formulation (A 23) in the proof of Lemma A.3 leads to $\beta = 0$.

Moreover, v is monotonically increasing for all $\eta > 0$, as we shall now demonstrate.

First, from the equation (2.3), it is easy to see v is monotonically increasing for η near 0.

According to the formal argument above, with the asymptotic form of smooth v for the discrete set of small values $c_k \ll 1$ of c , v remains close to $v^* = a\eta^{\frac{2}{3}}$ in bounded intervals such as $0 \leq \eta \leq \eta^* + \epsilon$, where η^* is where v^* crosses c^* : $a(\eta^*)^{\frac{2}{3}} = c^*$, $\eta^* > 1$ for $2 \leq n \leq 7$, $\epsilon > 0$ is small.

Next note that, from the equation (2.3), v has no local:

- (a) maximum if $v < c^*$ and $0 < \eta < 1$; (b) minimum if $v < c^*$ and $\eta > 1$;
- (c) minimum if $v > c^*$ and $0 < \eta < 1$; (d) maximum if $v > c^*$ and $\eta > 1$.

Now, for c_k small enough (sufficiently large k), since v is close to v^* , it too will cross c for the first time at some point $\eta^{**} > 1$: $v < c^*$ for $0 \leq \eta < \eta^{**}$, $v = c^*$ at $\eta = \eta^{**}$.

In the interval $0 \leq \eta \leq 1$, $v < c^*$, $0 = v(0) < v(1)$, and by (a) above, there is no local maximum: Here v is monotonic increasing.

In the interval $1 \leq \eta < \eta^{**}$, $v < c^*$, $v(1) < v(\eta^{**})$, and by (b) above, there is no local minimum: Here v is monotonic increasing.

Since η^{**} is the first point at which c^* is crossed, $v'(\eta^{**}) \geq 0$. By uniqueness of (regular) solutions of (2.3), $v'(\eta^{**}) \neq 0$, giving $v'(\eta^{**}) > 0$ and $v > c^*$ in a right neighbourhood of η^{**} . From (d) it follows that v remains increasing for $\eta \geq \eta^{**}$.

Hence, we conclude v is increasing with respect to $\eta > 0$ and $v > 0$ for all $\eta > 0$, as a result, the smooth solution v exists globally and then, by Lemma 2.2, $v \sim b \frac{2}{3} \eta^{\frac{2}{3}}$ for $\eta \rightarrow \infty$, with the value of $\sim b \frac{2}{3}$ fixed by that of c_k .

For $n \geq 8$, we don't know whether there is a smooth solution of the equation (2.3), according to the above analysis.

This concludes the formal proof of Theorem 2.1.

Note that it might, with some effort, be possible to make some of the arguments in the above proof rigorous by adapting arguments of [17] to obtain error bounds for neglected terms in the formal asymptotic series. However, it should be noted that even if such work were to be successful, the results will be expected to be weaker than those of [17] through the already noted property of the key ODE (2.3): a global solution of the ODE cannot be expected for all initial values c ; it is then not clear that the index k will be identified with the number of times that solution v_k crosses v^* .

3. Approximate Calculations of the Quenching Profile

Having looked for self-similar solutions, which might serve to give the local form of quenching for solutions of (1.1) in the previous section, we now look at approximate solutions local to a quenching point. The relevance of the global self-similar solutions v_j in Theorem 2.1 in (1.3) hinges on their stability.

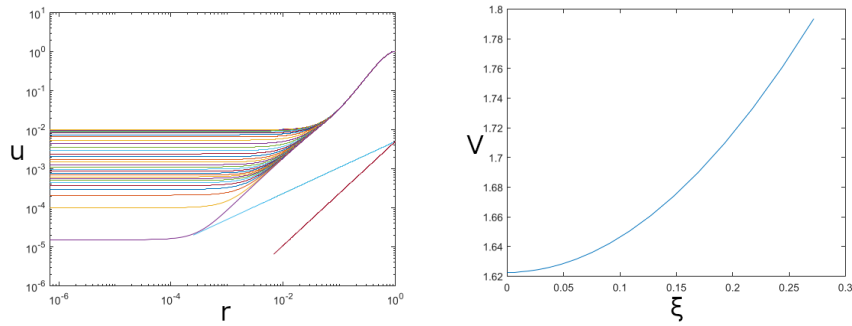


Figure 2. (A) Solution u for $a = 1$, (B) rescaled solution V for small ξ .

Should u be only approximately self-similar, so that v varies with a transformed time variable $\tau = -\ln(T_{\max} - t)$ as well as η , it is given by the PDE (3.1) instead of the ODE (2.3):

$$\frac{\partial^2 v}{\partial \tau^2} + 2\eta \frac{\partial^2 v}{\partial \eta \partial \tau} - \frac{1}{3} \frac{\partial v}{\partial \tau} = (1 - \eta^2) \frac{\partial^2 v}{\partial \eta^2} + \left(\frac{n-1}{\eta} - \frac{2}{3}\eta \right) \frac{\partial v}{\partial \eta} + \frac{2}{9}v - \frac{1}{v^2}; \quad (3.1)$$

note that the new variable satisfies $\tau \rightarrow \infty$ as $t \rightarrow T_{\max}$. To discuss stability of a true self-similar solution v_j , we interpret it as an equilibrium of the PDE (3.1) for $v(\eta, \tau)$.

In this section go into this by showing formal and numerical evidence for the following conjecture in n dimensions:

Conjecture 3.1. *a) The trivial self-similar solution $v_0 = c^*$ to (2.3) is stable as a solution of the PDE (3.1), excepting for (spatially uniform) perturbations which grow as $\exp \tau$, which correspond to solutions quenching at different times.*

b) The generic local behaviour of a strictly increasing radial quenching solution to (2.1) near the quenching time T_{\max} is given by $u(r, t) = (T_{\max} - t)^{\frac{2}{3}} V(\xi, t)$, with $\xi = \frac{r}{(T_{\max} - t)^{1/2}}$ and

$$V(\xi, t) \sim C\xi^{\frac{4}{3}} \quad \text{as } \xi \rightarrow \infty \quad \text{and} \quad V(0, t) \rightarrow c^* \quad (3.2)$$

for $t \rightarrow T_{\max}$. Correspondingly,

$$u(r, T_{\max}) \sim Cr^{\frac{4}{3}} \quad \text{as } r \rightarrow 0 \quad \text{with} \quad u(0, t) \sim c^*(T_{\max} - t)^{\frac{2}{3}} \quad \text{as } t \rightarrow T_{\max}, \quad (3.3)$$

provided u is a strictly increasing function of r near the quenching time. Here C is a constant which depends on n , and on the initial and boundary conditions for the semi-linear wave equation (2.1).

c) For $2 \leq n \leq 7$, the smooth self-similar solutions v_j to (2.3) are unstable for $j \in \mathbb{N}$ as solutions of the PDE (3.1).

The behaviour in c) is supported by the formal and numerical results in the paper [11] for the problem in one space dimension.

We start in Subsec. (a) by looking at numerical solutions of the PDE (2.1) for dimension $n = 2$. In the Subsecs. (b) - (d) we consider a formal asymptotic solution of the PDE in any dimension n .

(a) Numerical Solution of the PDE

We illustrate the behaviour of radial solutions near the quenching time with numerical experiments. We consider the semi-linear wave equation (1.1) in the unit ball $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ with homogeneous Neumann boundary conditions $\partial_\nu u = 0$ on $\partial\Omega$. The initial condition is of

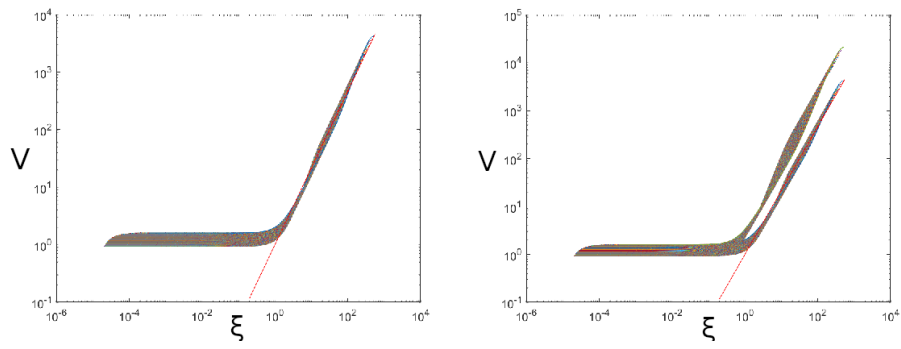


Figure 3. Rescaled solution V for (A) $a = 1$, (B) $a = 1$ and $a = 5$.

the form

$$u(x, 0) = \epsilon + a \sin\left(\frac{\pi|x|}{2}\right)^2, \quad \frac{\partial u}{\partial t}(x, 0) = 0 \quad (x \in \Omega),$$

where $\epsilon = 10^{-2}$ and $a \in \{1, 5\}$. As we are looking at the local behaviour near the quenching point, the following observations are independent of the type of boundary condition and the precise boundary data. They seem to reflect the behaviour of the solution not only in this special case, but for generic initial conditions in the numerically feasible dimensions $n = 2, 3, 4, 5$. The results refine and complement those obtained in [11] for dimension $n = 1$.

We discretise the weak formulation of (1.1) by piecewise linear finite elements in $r \in (0, 1)$ on an algebraically graded mesh with nodes $r_j = \left(\frac{j}{N}\right)^2, 0 \leq j \leq N, N$ fixed. Note that integration in polar coordinates involves a Jacobian r^{n-1} , with resulting numerical difficulties for $n \geq 6$. An implicit Euler method with variable time step is used for the time discretisation. Note that the numerical method becomes ill-conditioned with increasing dimension n .

For $a = 1$, Figure 2(A) shows u as a function of the radius $r = |x|$ at several times t close to T_{\max} . Plotting both u and r in a logarithmic scale, a solution $u \sim \tilde{C}r^\alpha$ corresponds to a straight line of slope α . The straight red line corresponds to $\alpha = \frac{4}{3}$, in agreement with the behaviour of u at all times t . The straight blue line depicts the behaviour expected for non-trivial self-similar solutions, where $\alpha = \frac{2}{3}$.

Figure 3(A) depicts $V = (T_{\max} - t)^{-2/3}u(r, t)$ as a function of the rescaled radius $\xi = \frac{r}{(T_{\max} - t)^{1/2}}$, again in logarithmic scales. As $t \rightarrow T_{\max}$, V approaches a limiting profile, indicating an asymptotically self-similar evolution. The straight red line agrees with $V(\xi, t) \sim C\xi^{4/3}$ as $t \rightarrow T_{\max}$. Note the scaling of the variable ξ in agreement with Conjecture 3.1, rather than (2.2).

Figure 3(B) compares the rescaled solutions $V = (T_{\max} - t)^{-2/3}u(r, t)$ for initial data corresponding to $a = 1$, respectively $a = 5$. In both cases $V(\xi, t) \sim C\xi^{4/3}$ as $t \rightarrow T_{\max}$, but the constant C depends on the initial condition.

Figure 2(B) plots $V = (T_{\max} - t)^{-2/3}u(r, t)$ near $r = 0$, when $a = 1$ and t is the last time step before T_{\max} . For $t \rightarrow T_{\max}$ and $\xi \rightarrow 0$, V is independent of t , and the limiting value $V(0, t) \simeq 1.62$ is compatible with $c^* = \left(\frac{9}{2}\right)^{\frac{1}{3}} \approx 1.65$.

(b) Formal Asymptotics: Inner Solution

“Inner solution” refers to an asymptotic form of a quasi-self-similar solution v satisfying (3.1) for η not large. We assume that v has an asymptotic form

$$v \sim c^* + \underline{V}, \quad \text{with} \quad \lim_{\tau \rightarrow \infty} \underline{V}(\eta, \tau) = 0.$$

We look for an asymptotic form of \underline{V} , $\underline{V} \sim e^{-\lambda\tau} \varphi_\lambda(\eta)$, where φ_λ satisfies

$$(1 - \eta^2) \frac{d^2 \varphi}{d\eta^2} + \left[2 \left(\lambda - \frac{1}{3} \right) \eta + \frac{n-1}{\eta} \right] \frac{d\varphi_\lambda}{d\eta} + \left(\frac{2}{3} - \frac{\lambda}{3} - \lambda^2 \right) \varphi_\lambda = 0. \quad (3.4)$$

We seek an even solution of (3.4) as a power-series given by

$$\varphi_\lambda(\eta) = \sum_{k=0}^{\infty} a_k \eta^{2k}. \quad (3.5)$$

Plugging (3.5) into (3.4), after some manipulations and ratio test, we see that the radius of convergence of the power series for $\varphi_\lambda(\eta)$ is 1, consistent with φ_λ having a singularity at $\eta = 1$, the solution (3.5) is not smooth unless it terminates. The series (3.5) terminates, with φ_λ being a polynomial, and hence smooth for all η , for $\lambda = \lambda_k^\pm$ with $\lambda_k^- = 2k - 1$ for $k = 0, 1, 2, \dots$ and $\lambda_k^+ = 2k + \frac{2}{3}$ for $k = 0, 1, 2, \dots$

The case $k = 0$ and $\lambda = -1$ corresponds to a shift in T_{\max} , which we ignore.

For $k = 0$ and $\lambda = \frac{2}{3}$, \underline{V} is a slower shrinking solution, uniform in η , and the asymptotic form of \underline{V} is given by

$$\underline{V} \sim b e^{-\frac{2}{3}\tau}, \quad \text{where } b \text{ is an arbitrary constant.}$$

For $k = 1$ and $\lambda = 1$, the asymptotic form of \underline{V} is quadratic in η ,

$$\underline{V} \sim a e^{-\tau} (\eta^2 + 3n), \quad \text{where } a \text{ is an arbitrary constant.}$$

We consider these two terms, as they are the slowest decaying in time, and therefore dominant. Other contributions decay more rapidly and we neglect them. As a result,

$$v \sim c^* + b e^{-\frac{2}{3}\tau} + a e^{-\tau} (\eta^2 + 3n) + \dots \quad (3.6)$$

Returning to earlier variables (2.2), for $t \rightarrow T_{\max}$, $\eta \rightarrow \infty$, we obtain

$$\begin{aligned} u &\sim c^* (T_{\max} - t)^{\frac{2}{3}} + b (T_{\max} - t)^{\frac{4}{3}} + a (T_{\max} - t)^{\frac{5}{3}} (\eta^2 + 3n) + \dots \\ &\sim (T_{\max} - t)^{\frac{2}{3}} \left(c^* + b (T_{\max} - t)^{\frac{2}{3}} + a (T_{\max} - t) \eta^2 + \dots \right) \\ &\sim (T_{\max} - t)^{\frac{2}{3}} \left(c^* + b (T_{\max} - t)^{\frac{2}{3}} + a (T_{\max} - t)^{-1} r^2 + \dots \right), \end{aligned} \quad (3.7)$$

for $t \rightarrow T_{\max}$, $r \rightarrow 0$. Here terms c^* and $a (T_{\max} - t)^{-1} r^2$ are in the same size for $\eta \sim O \left[(T_{\max} - t)^{-\frac{1}{2}} \right]$, $\eta \rightarrow \infty$ i.e. $r \sim O \left[(T_{\max} - t)^{\frac{1}{2}} \right]$, $r \rightarrow 0$.

This indicates that there is significant behaviour of the solution v of (3.1) and u for large η of the size $(T_{\max} - t)^{-\frac{1}{2}}$.

(c) Formal Asymptotics: Outer Solution

We want an approximation for a solution of equation (3.1) valid for large η .

For large $\eta \gg 1$, terms $\frac{\partial^2 v}{\partial \eta^2}$ and $\frac{n-1}{\eta} \frac{\partial v}{\partial \eta}$ in (3.1) are relatively small, i.e. we might neglect the Laplacian term Δu in the original equation (1.1) to get

$$\frac{d^2 u}{dt^2} = -\frac{1}{u^2} \quad (3.8)$$

which has the exact solution given by (see (6.7) in [11])

$$T_{\max} - t + A = \frac{u^{\frac{1}{2}}}{B} (2 + Bu)^{\frac{1}{2}} - 2B^{-\frac{3}{2}} \ln \left[\left(\frac{Bu}{2} \right)^{\frac{1}{2}} + \left(1 + \frac{Bu}{2} \right)^{\frac{1}{2}} \right], \quad (3.9)$$

where $A = A(r) > 0$, $B = B(r) > 0$ with $\lim_{r \rightarrow 0} A(r) = \lim_{r \rightarrow 0} B(r) = 0$.

Taking $u \rightarrow 0$ in (3.9) and doing the appropriate eliminations give

$$T_{\max} - t + A(r) \sim \frac{\sqrt{2}}{3} u^{\frac{3}{2}} + c_o B(r) u^{\frac{5}{2}} + \dots \quad (3.10)$$

Here and the following parts, c_o denotes a number, no physical dependence. This implies

$$u \sim \left[\frac{3}{\sqrt{2}} (T_{\max} - t) + \dots \right]^{\frac{2}{3}}.$$

In more detail

$$\begin{aligned} u &\sim \left[\frac{3}{\sqrt{2}} (T_{\max} - t) + A(r) + c_o B(r) (T_{\max} - t)^{\frac{5}{3}} + \dots \right]^{\frac{2}{3}} \\ &\sim c^* (T_{\max} - t)^{\frac{2}{3}} \left[1 + c_o A(r) (T_{\max} - t)^{-1} + c_o B(r) (T_{\max} - t)^{\frac{2}{3}} + \dots \right]^{\frac{2}{3}} \\ &\sim (T_{\max} - t)^{\frac{2}{3}} \left[c^* + c_o A(r) (T_{\max} - t)^{-1} + c_o B(r) (T_{\max} - t)^{\frac{2}{3}} + \dots \right]. \end{aligned} \quad (3.11)$$

Another way for solving the equation (3.8) can be seen in [11].

(d) Formal Asymptotics: Matching

Recall the approximation for the inner region is of the form (3.7) and the the approximation for the outer region has the form (3.11). In an intermediate region, the inner solution (3.7) and the approximation (3.11) to outer solution should be the same, which implies

$$B(r) \sim c_o b \quad \text{for } r \rightarrow 0, \quad A(r) \sim c_o a r^2 \quad \text{for } r \rightarrow 0.$$

(e) Quenching Profile

Taking $t \rightarrow T_{\max}$ in (3.9) gives

$$A = \frac{u^{\frac{1}{2}}}{B} (2 + Bu)^{\frac{1}{2}} - 2B^{-\frac{3}{2}} \ln \left[\left(\frac{Bu}{2} \right)^{\frac{1}{2}} + \left(1 + \frac{Bu}{2} \right)^{\frac{1}{2}} \right].$$

Then for $r \rightarrow 0$ and $u \rightarrow 0$, we obtain

$$c_o a r^2 \sim \frac{\sqrt{2}}{3} u^{\frac{3}{2}}$$

as (3.10). Thus, the local profile of the solution at quenching time $t = T_{\max}$ is

$$u \sim c_o a^{\frac{2}{3}} r^{\frac{4}{3}}.$$

The coefficient a is determined by the initial and boundary conditions of the equation (1.1).

This gives us an $r^{\frac{4}{3}}$ dependence of the solution profile near the quenching point $r = 0$, which agrees with the numerical results in Section (a).

4. Discussion

Motivated by touch-down singularities in MEMS capacitors, we have, in this article, studied the (radial) profile of solutions to the semi-linear wave equation (1.1) near a quenching time. The results are related to classical works on self-similar solutions for the stationary problem [18,19]. They have led to a detailed conjecture for quenching behaviour in (1.1), with a different scaling from that in recent rigorous work on blow-up [20,21].

In Section 2 we found that there were non-trivial smooth self-similar solutions which could, potentially, give the local profile of quenching solutions to the semi-linear wave equation (1.1) in dimension n from 2 to 7, the dimension $n = 2$ being physically relevant for touchdown in MEMS devices. Quenching governed by such a self-similar solution would produce a local quenching profile $u(r, T_{\max}) \sim C_{nm} r^{\frac{2}{3}}$ for $r \rightarrow 0$, with the positive constant C_{nm} fixed by dimension n and

index m specifying the particular similarity solution, equivalent to a specific value of the shooting parameter $c = v(0)$ in Section 2.

However, in Section 3 we found that both numerical solutions and formal asymptotics indicate that these non-trivial self-similar solutions ($C_{nm} > 0$) are unstable, while the trivial, constant solution is stable: generically, the solution to (3.1) satisfies $v \rightarrow c^* = \left(\frac{9}{2}\right)^{\frac{1}{3}}$ as $\tau \rightarrow \infty$ for fixed r . This then means that, at least for radially symmetric cases, as quenching is approached, i.e. $t \rightarrow T_{\max}$, $u(0, t) \sim c^* t^{\frac{2}{3}}$, and that at quenching time, $t = T_{\max}$, u has a flatter local profile, $u(r, T_{\max}) \sim Cr^{\frac{4}{3}}$ for $r \rightarrow 0$, with the positive constant C fixed by the imposed initial and boundary conditions. These results lend support to the detailed behaviour of the solution proposed in Conjecture 3.1 at the quenching time. Such less sharp local behaviour near a quenching point gives, at the quenching time, a nonlinear term in (1.1) of size $r^{-\frac{8}{3}}$, whose integral is infinite for $n = 1, 2$. This could suggest that, for these physically important cases, any continuation of the solution beyond quenching time might require significant modification of the model.

A. Local Existence of Self-Similar Solutions

In this Appendix we prove the local solvability of the initial value problem given by (2.3), (2.4) in any dimension n .

Proposition A.1. (A) Equation (2.3) with the initial condition (2.4) admits a solution v with maximal existence interval $[0, \eta_{\max})$.

(B) If $\eta_{\max} < \infty$, then

$$\lim_{\eta \rightarrow \eta_{\max}^-} v(\eta) = 0.$$

(C) If $\eta_{\max} \leq 1$, the solution v is unique.

(D) If $\eta_{\max} > 1$, there exists a 1-parameter family of solutions $\{v_\beta\}_{\beta \in \mathbb{R}}$ to (2.3) with initial condition (2.4). η_{\max} depends on both $v_\beta(0)$ and β . v_β is independent of β in the interval $[0, 1]$, and it admits a decomposition $v_\beta = v_{\beta, \text{reg}} + v_{\beta, \text{sing}}$ into a smooth and a singular part at $\eta = 1^+$. We have

$$v_{\beta, \text{sing}}(\eta) = \sum_{\alpha \in E} c_\alpha (\eta - 1)^\alpha,$$

whose coefficients c_α are determined by β , and the smallest exponent $\alpha \in E$ is given by $\alpha = \frac{n}{2} + \frac{1}{6}$.

(a) Integral Formulation of Self-Similar Equation

For the proof of Proposition A.1, we show that there exists a unique local solution to (2.3), with initial conditions imposed at $\eta_0 \geq 0$ given by

$$v(\eta_0) = c > 0, \quad \frac{dv}{d\eta}(\eta_0) = d \tag{A 1}$$

when $\eta_0 \neq 1$ (see Section (d) for details when $\eta_0 = 1$). The Picard-Lindelöf Theorem implies the existence of a unique local solution in an interval $[\eta_0, \eta_0 + \delta)$ away from $\eta = 0$ and 1, where $\delta > 0$. To prove Proposition A.1 it is then sufficient to consider local existence and uniqueness for η_0 near the singular points $\eta = 0, 1$. The expansion in Proposition A.1(D) follows from Lemma 2.1.

We reformulate (2.3), (A 1) as an integral equation, which is obtained from the linearisation (2.3) around c^* , given by

$$(1 - \eta^2) \frac{d^2 \bar{v}}{d\eta^2} + \left(\frac{n-1}{\eta} - \frac{2}{3} \eta \right) \frac{d\bar{v}}{d\eta} + \frac{2}{3} \bar{v} = 0. \tag{A 2}$$

For $n > 2$ the following hypergeometric functions P, \bar{P} form a basis of solutions to the linearized equation (A 2):

$$P(\eta) = \eta^{2-n} {}_2F_1 \left(\left[\frac{4}{3} - \frac{n}{2}, \frac{1}{2} - \frac{n}{2} \right], \left[2 - \frac{n}{2} \right], \eta^2 \right), \quad \bar{P}(\eta) = {}_2F_1 \left(\left[-\frac{1}{2}, \frac{1}{3} \right], \left[\frac{n}{2} \right], \eta^2 \right).$$

P is smooth near $\eta = 1$, with $P(1) = 1$, while \bar{P} is smooth near $\eta = 0$ and $\bar{P}(0) = 1$. In fact, P is a rational function for odd n and, specifically, $P(\eta) = \frac{\eta^2+3}{4\eta}$ when $n = 3$. A basis for $n = 1$ is given by 1 and η , while for $n = 2$ a basis is given by

$${}_2F_1\left(\left[-\frac{1}{2}, \frac{1}{3}\right], \left[-\frac{1}{6}\right], -\eta^2 + 1\right), (\eta^2 - 1)^{7/6} {}_2F_1\left(\left[\frac{2}{3}, \frac{3}{2}\right], \left[\frac{13}{6}\right], -\eta^2 + 1\right).$$

Given a solution Q of the linearized equation (A 2), with $Q(\eta_0) > 0$, we set

$$v(\eta) = c^* + w(\eta, z), \quad w(\eta, z) = Q(\eta)z, \quad z = z(\eta). \quad (\text{A } 3)$$

The self-similar equation (2.3) then becomes

$$\frac{d^2 z}{d\eta^2} + \left(\frac{2}{Q(\eta)} \frac{dQ(\eta)}{d\eta} + \frac{3(n-1) - 2\eta^2}{3\eta(1-\eta^2)} \right) \frac{dz}{d\eta} = \frac{f(w(\eta, z))}{(1-\eta^2)Q(\eta)}, \quad (\text{A } 4)$$

where

$$f(w(\eta, z)) = \frac{1}{[c^* + w(\eta, z)]^2} - \frac{1}{(c^*)^2} + \frac{2}{(c^*)^3} w(\eta, z).$$

The initial condition (A 1) becomes

$$z(\eta_0) = \frac{c - c^*}{Q(\eta_0)} =: z_0, \quad \frac{dz}{d\eta}(\eta_0) = \frac{d}{Q(\eta_0)} - \frac{c - c^*}{Q^2(\eta_0)} \frac{dQ}{d\eta}(\eta_0) =: z_0^*. \quad (\text{A } 5)$$

With the substitution $y = \frac{dz}{d\eta}$, one readily derives the explicit solution to (A 4), (A 5):

$$\begin{aligned} z(\eta) = z_0 + \alpha_1 \int_{\eta_0}^{\min\{\eta, 1\}} \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{Q^2(\zeta)\zeta^{n-1}} d\zeta + \beta \int_1^{\max\{\eta, 1\}} \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{Q^2(\zeta)\zeta^{n-1}} d\zeta \\ - \int_{\eta_0}^{\eta} \left[\int_{\xi}^{\eta} \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{Q^2(\zeta)\zeta^{n-1}} d\zeta \right] \left[\frac{Q(\xi)\xi^{n-1}}{(\xi^2 - 1)^{\frac{1}{6} + \frac{n}{2}}} \right] f(w(\xi, z)) d\xi, \end{aligned} \quad (\text{A } 6)$$

where α_1 is determined by the initial condition at η_0 ,

$$\alpha_1 = z_0^* Q^2(\eta_0) (\eta_0^2 - 1)^{\frac{5}{6} - \frac{n}{2}} \eta_0^{n-1}, \quad (\text{A } 7)$$

$\beta \in \mathbb{R}$ is an arbitrary constant, and H denotes the Heaviside function. Note that the existence of a unique solution to the differential equation (A 4), (A 5) is equivalent to the existence of a unique solution to the integral equation (A 6), (A 7).

(b) Local Existence of Unique Self-Similar Solution for η_0 near 1^-

We consider $\eta_0 = 1 - \epsilon$ for a sufficiently small $\epsilon \in (0, \epsilon_0(n))$, $n > 2$. The function $Q = P$ is smooth near $\eta = 1$ and satisfies $P(1) = 1$. After possibly shrinking $\epsilon_0(n)$,

$$\frac{1}{2} \leq P(\eta) \leq 2, \quad \text{for all } 1 - \epsilon_0(n) \leq \eta \leq 1. \quad (\text{A } 8)$$

For the integral formulation given by (A 6), we prove the local existence of a solution, as stated in the following lemma:

Lemma A.1. *Let $\eta_0 = 1 - \epsilon$, $\epsilon \in (0, \epsilon_0(n))$. There exists a $\delta > 0$, depending only on the initial values (A 5) and on the dimension n , such that the integral equation (A 6) has a unique solution in the interval $\mathcal{I}_\delta = [\eta_0, \min\{\eta_0 + \delta, 1\}]$.*

Proof. For given $\delta > 0$, to be fixed later, we set $\mathcal{I}_\delta = [\eta_0, \min\{\eta_0 + \delta, 1\}]$ and

$$B_\delta = \{z \in C(\mathcal{I}_\delta; \mathbb{R}) : z(\eta_0) = z_0, \sigma_1 \leq c^* + w(\eta, z) \leq \sigma_2\}.$$

Here, σ_1 and σ_2 are positive constants depending on the initial values (A 5).

Consider the nonlinear operator Γ on B_δ given by (A 6),

$$\Gamma z(\eta) = z_0 + \alpha_1 \int_{\eta_0}^{\eta} \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{P^2(\zeta)\zeta^{n-1}} d\zeta - \int_{\eta_0}^{\eta} \left[\int_{\xi}^{\eta} \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{P^2(\zeta)\zeta^{n-1}} d\zeta \right] \frac{P(\xi)\xi^{n-1}}{(\xi^2 - 1)^{\frac{1}{6} + \frac{n}{2}}} f(w(\xi, z)) d\xi.$$

Contraction. We first prove that Γ is a contractive map on B_{δ_1} , for $\delta_1 > 0$ sufficiently small. For $z_1, z_2 \in B_{\delta_1}$, let $w_1 = w(\xi, z_1)$, $w_2 = w(\xi, z_2)$. Then

$$\begin{aligned} |f(w_1) - f(w_2)| &= \left| \frac{1}{(c^* + w_1)^2} - \frac{1}{(c^* + w_2)^2} + \frac{2}{(c^*)^3} (w_1 - w_2) \right| \\ &\leq \left| \frac{1}{(c^* + w_1)^2(c^* + w_2)} + \frac{1}{(c^* + w_1)(c^* + w_2)^2} + \frac{2}{(c^*)^3} \right| |w_1 - w_2| \leq \left| \frac{2}{(\sigma_1)^3} + \frac{2}{(c^*)^3} \right| |w_1 - w_2|. \end{aligned} \quad (\text{A } 9)$$

Hence,

$$\begin{aligned} |\Gamma z_1(\eta) - \Gamma z_2(\eta)| &= \left| \int_{\eta_0}^{\eta} \left[\int_{\xi}^{\eta} \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{P^2(\zeta)\zeta^{n-1}} d\zeta \right] \left[\frac{P(\xi)\xi^{n-1}}{(\xi^2 - 1)^{\frac{1}{6} + \frac{n}{2}}} [f(w_1) - f(w_2)] d\xi \right] \right| \\ &\leq \left| \frac{2}{(\sigma_1)^3} + \frac{2}{(c^*)^3} \right| \|w_1 - w_2\|_{C(\mathcal{I}_\delta; \mathbb{R})} \int_{\eta_0}^{\eta} \left| \int_{\xi}^{\eta} \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{P^2(\zeta)\zeta^{n-1}} d\zeta \right| \frac{P^2(\xi)\xi^{n-1}}{(\xi^2 - 1)^{\frac{1}{6} + \frac{n}{2}}} d\xi. \end{aligned} \quad (\text{A } 10)$$

Since $\eta_0 \leq \xi \leq \zeta \leq \eta \leq 1$, we have $\frac{\xi^{n-1}}{\zeta^{n-1}} \leq 1$. Together with (A 8), we conclude

$$\begin{aligned} \int_{\eta_0}^{\eta} \left| \int_{\xi}^{\eta} \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{P^2(\zeta)\zeta^{n-1}} d\zeta \right| \frac{P^2(\xi)\xi^{n-1}}{(\xi^2 - 1)^{\frac{1}{6} + \frac{n}{2}}} d\xi &= \int_{\eta_0}^{\eta} \left| \int_{\xi}^{\eta} \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{P^2(\zeta)\zeta^{n-1}} d\zeta \right| \frac{P^2(\xi)\xi^{n-1}}{(\xi^2 - 1)^{\frac{1}{6} + \frac{n}{2}}} d\xi \\ &\leq 16 \int_{\eta_0}^{\eta} \int_{\xi}^{\eta} \left| \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{(\xi^2 - 1)^{\frac{1}{6} + \frac{n}{2}}} \right| d\zeta d\xi \leq C_n |\eta - \eta_0| \end{aligned} \quad (\text{A } 11)$$

where C_n depends on n only. For $\delta_1 < \frac{1}{4C_n} \left| \frac{2}{(\sigma_1)^3} + \frac{2}{(c^*)^3} \right|^{-1}$ one concludes

$$|\Gamma z_1(\eta) - \Gamma z_2(\eta)| \leq \frac{1}{4} \|w_1 - w_2\|_{C(\mathcal{I}_\delta; \mathbb{R})} \leq \frac{1}{2} \|z_1 - z_2\|_{C(\mathcal{I}_\delta; \mathbb{R})} \quad \text{for all } \eta \in \mathcal{I}_{\delta_1}. \quad (\text{A } 12)$$

Preservation. To see that $\Gamma : B_{\delta_2} \rightarrow B_{\delta_2}$ for sufficiently small δ_2 , first note that $\Gamma z(\eta_0) = z_0$. It suffices to show that there exists $\delta_2 > 0$, such that for all $\eta \in \mathcal{I}_{\delta_2}$

$$\sigma_1 \leq c^* + w(\eta, \Gamma z(\eta)) \leq \sigma_2. \quad (\text{A } 13)$$

Here,

$$\begin{aligned} w(\eta, \Gamma z(\eta)) &= P(\eta)z_0 + \alpha_1 P(\eta) \int_{\eta_0}^{\eta} \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{P^2(\zeta)\zeta^{n-1}} d\zeta \\ &\quad - P(\eta) \int_{\eta_0}^{\eta} \left[\int_{\xi}^{\eta} \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{P^2(\zeta)\zeta^{n-1}} d\zeta \right] \left[P(\xi) (\xi^2 - 1)^{-\frac{1}{6} - \frac{n}{2}} \xi^{n-1} \right] f(w(\xi, z)) d\xi. \end{aligned}$$

Let $\rho = |\alpha_1|$. Using (A 8) we note that

$$\left| c^* + P(\eta)z_0 + \alpha_1 P(\eta) \int_{\eta_0}^{\eta} \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{P^2(\zeta)\zeta^{n-1}} d\zeta \right| \leq \frac{2}{3} \sigma_2 \quad (\text{A } 14)$$

for $\sigma_2 = \frac{3}{2} \left(c^* + 2|z_0| + \frac{8(1-\eta_0)\rho(2|1-\eta_0|)^{\frac{n}{2} - \frac{5}{6}}}{\eta_0^{n-1}} \right)$. Because the integrand is bounded,

$$\left| P(\eta) \int_{\eta_0}^{\eta} \left[\int_{\xi}^{\eta} \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{P^2(\zeta)\zeta^{n-1}} d\zeta \right] \left[P(\xi) (\xi^2 - 1)^{-\frac{1}{6} - \frac{n}{2}} \xi^{n-1} \right] f(w) d\xi \right| \leq C |\eta - \eta_0| \quad (\text{A } 15)$$

for a constant C only depending on σ_1 , σ_2 and n . If δ_2 is sufficiently small, we conclude from (A 14) and (A 15) that for $\eta \in \mathcal{I}_{\delta_2}$

$$w(\eta, \Gamma z(\eta)) \leq \sigma_2. \quad (\text{A } 16)$$

Similarly, for $\sigma_1 > 0$ with $P(\eta_0)z_0 \geq -c^* + 2\sigma_1$, continuity of P implies

$$c^* + P(\eta)z_0 \geq \frac{2}{3}\sigma_1 \quad (\text{A } 17)$$

for $\eta \in [\eta_0, \eta_0 + \tilde{\delta}]$, $\tilde{\delta}$ depending on σ_1 . For all η in this interval, the boundedness of the integrand implies

$$\left| \alpha_1 P(\eta) \int_{\eta_0}^{\eta} \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{P^2(\zeta)\zeta^{n-1}} d\zeta \right| \leq C|\eta - \eta_0|, \quad (\text{A } 18)$$

for a $C > 0$ depending only on α_1 and n . Hence, for sufficiently small δ_3 we deduce in combination with (A 15) that for $\eta \in \mathcal{I}_{\delta_3}$

$$c^* + w(\eta, \Gamma z(\eta)) \geq \sigma_1. \quad (\text{A } 19)$$

We conclude that $\Gamma : B_\delta \rightarrow B_\delta$ is a contraction for $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. According to the Banach fixed point theorem, Γ has a unique fixed point, and thus the integral equation (A 6) has a unique solution in \mathcal{I}_δ . This concludes the proof of Lemma A.1. \square

(c) Local Existence of Unique Self-Similar Solution for $\eta_0 = 0$

We describe how to adapt the arguments in Section (b) to the singular point $\eta_0 = 0$. In this case we use (A 6) with $Q = \bar{P}$, which is smooth near $\eta = 0$ and satisfies $\bar{P}(0) = 1$. From $\frac{d\bar{P}}{d\eta}(0) = 0$, the initial values (2.4) for v and the transformation (A 3) imply

$$z(0) = c - c^*, \quad \frac{dz}{d\eta}(0) = 0. \quad (\text{A } 20)$$

In particular, $\alpha_1 = 0$. The result is

Lemma A.2. *There exists a $\delta > 0$, depending only on the initial value (A 20) and on n , such that the integral equation (A 6) has a unique solution in $\mathcal{I}_\delta = [0, \delta]$.*

Proof. For given $\delta > 0$, to be fixed later, we denote $\mathcal{I}_\delta = [0, \delta]$ and

$$B_\delta = \{z \in C(\mathcal{I}_\delta; \mathbb{R}) : z(0) = z_0, \sigma_1 \leq c^* + w(\eta, z) \leq \sigma_2\}. \quad (\text{A } 21)$$

Here, σ_1 and σ_2 are positive constants depending on the initial values (A 20).

Consider the nonlinear operator Γ on B_δ given by (A 6) with $\alpha_1 = 0$,

$$\Gamma z(\eta) = z_0 - \int_0^\eta \left[\int_\xi^\eta \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{\bar{P}^2(\zeta)\zeta^{n-1}} d\zeta \right] \left[\bar{P}(\xi) \left(\xi^2 - 1 \right)^{-\frac{1}{6} - \frac{n}{2}} \xi^{n-1} \right] f(w(\xi, z)) d\xi. \quad (\text{A } 22)$$

The proof of Lemma A.2 closely follows the proof of Lemma A.1. We describe the necessary adaptations.

Contraction. It is easy to see that $\frac{1}{2} \leq \bar{P} \leq 2$ implies an estimate of the double integral in Γ corresponding to (A 11). Since (A 9) and (A 10) hold verbatim as in the proof of Lemma A.1, there again exists $\delta_1 > 0$, depending on the initial value (A 20) and n , such that the contraction estimate (A 12) holds for all $\eta \in \mathcal{I}_{\delta_1}$.

Preservation. The upper and lower bounds for \bar{P} again imply an estimate of the double integral in Γ corresponding to (A 15). Since $\alpha_1 = 0$, estimate (A 14) is replaced by the observation $|c^* + \bar{P}(\eta)z_0| \leq c^* + 2|z_0| =: \frac{2}{3}\sigma_2$. Together, these two estimates show as in Lemma A.1 that there exists a $\delta_2 > 0$, depending on the initial value (A 20) and the dimension n , such that (A 15) holds. Then

the upper bound (A 16) again holds for all $\eta \in \mathcal{I}_{\delta_2}$. The lower bound (A 19) follows verbatim for $\eta \in \mathcal{I}_{\delta_3}$ from (A 15) and the observation $c^* + \bar{P}(\eta)z_0 \geq 2\sigma_1$. The proof of Lemma A.2 concludes by setting $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. □

(d) Local Existence of a Family of Self-Similar Solutions for $\eta_0 = 1$

When $\eta_0 = 1$, the initial condition $\frac{dv}{d\eta}(1)$ is determined by $v(1)$ from the differential equation (2.3), and the integral formulation (A 6) contains a free parameter $\beta \in \mathbb{R}$. As in Section (b), we consider (A 6) with $Q = P$. Analogous to Lemma A.1, we obtain the local existence of a unique solution $v = v(\eta; [v(1), \beta])$ to (A 6) for given β :

Lemma A.3. *For any $\beta \in \mathbb{R}$ there exists a $\delta > 0$, depending on the initial value $v(1)$, on β and on n , such that the integral equation (A 6) has a unique solution in $\mathcal{I}_\delta = [1, 1 + \delta]$.*

Proof. For given $\delta > 0$, to be fixed later, we denote $\mathcal{I}_\delta = [1, \min\{1 + \delta, 2\}]$ and

$$B_\delta = \{z \in C(\mathcal{I}_\delta; \mathbb{R}) : z(1) = z_0, \sigma_1 \leq c^* + w(\eta, z) \leq \sigma_2\}.$$

Here, σ_1 and σ_2 are positive constants depending on the initial values z_1 and β . Consider the nonlinear operator Γ on B_δ given by (A 6),

$$\Gamma z(\eta) = z_0 + \beta \int_1^\eta \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{P^2(\zeta)\zeta^{n-1}} d\zeta - \int_1^\eta \int_\xi^\eta \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{P^2(\zeta)\zeta^{n-1}} d\zeta \frac{P(\xi)\xi^{n-1}}{(\xi^2 - 1)^{\frac{n}{6} + \frac{n}{2}}} f(w(\xi, z)) d\xi. \tag{A 23}$$

Note that the term involving α_1 in (A 6) vanishes when $\eta_0 = 1$. The proof of Lemma A.3 closely follows the proof of Lemma A.1, with a new term involving β . We here describe the necessary adaptations.

Contraction. The double integral in the definition of Γ is identical to the proof of Lemma A.1. The contraction argument is therefore unchanged, using $\frac{1}{2} \leq P \leq 2$.

Preservation. As in the contraction argument, all estimates for the double integral in Γ follow by using $\frac{1}{2} \leq P \leq 2$. For the term involving β , the inequality (A 14) is valid with $\alpha_1 := \beta$ and $\rho := |\beta|$ in the definition of σ_2 . As in the proof of Lemma A.1, these two estimates show the upper bound (A 16) for all $\eta \in \mathcal{I}_{\delta_2}$. The lower bound (A 19) follows verbatim for $\eta \in \mathcal{I}_{\delta_3}$, from (A 15) and the observations $c^* + P(\eta)z_0 \geq 2\sigma_1$, respectively

$$\left| \beta P(\eta) \int_1^\eta \frac{(\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}}}{P^2(\zeta)\zeta^{n-1}} d\zeta \right| \leq 8|\beta| \int_1^\eta \left| (\zeta^2 - 1)^{\frac{n}{2} - \frac{5}{6}} \right| d\zeta \leq C|\eta - 1|.$$

The proof of Lemma A.3 concludes by setting $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, which depends on z_0, β and the dimension n . □

References

1. Kavallaris N, Lacey A, Nikolopoulos C, Tzanetis D. 2011 A hyperbolic non-local problem modelling MEMS technology. *The Rocky Mountain Journal of Mathematics* pp. 505–534.
2. Flores G. 2014 Dynamics of a damped wave equation arising from MEMS. *SIAM Journal on Applied Mathematics* **74**, 1025–1035.
3. Guo JS, Huang BC. 2014 Hyperbolic quenching problem with damping in the Micro-Electro-Mechanical system device. *Discrete and Continuous Dynamical Systems-B* **19**, 419–434.
4. Laurençot P, Walker C. 2016 Some singular equations modeling MEMS. *Bulletin of the American Mathematical Society* **54**, 437–479.
5. Guo JS. 1990 On the quenching behavior of the solution of a semilinear parabolic equation. *Journal of Mathematical Analysis and Applications* **151**, 58–79.

6. Boni TK. 2000 On quenching of solutions for some semilinear parabolic equations of second order. *Bulletin of the Belgian Mathematical Society-Simon Stevin* **7**, 73–95.
7. Drosinou O, Kavallaris NI, Nikolopoulos CV. 2020 Impacts of noise on quenching of some models arising in MEMS technology. *arXiv preprint arXiv:2012.10922*.
8. Ghoussoub N, Guo Y. 2008 Estimates for the quenching time of a parabolic equation modeling electrostatic MEMS. *Methods and Applications of Analysis* **15**, 361–376.
9. Guo JS, Kavallaris NI. 2012 On a nonlocal parabolic problem arising in electrostatic MEMS control. *Discrete and Continuous Dynamical Systems-A* **32**, 1723–1746.
10. Kavallaris NI, Miyasita T, Suzuki T. 2008 Touchdown and related problems in electrostatic MEMS device equation. *Nonlinear Differential Equations and Applications NoDEA* **15**, 363–386.
11. Kavallaris NI, Lacey AA, Nikolopoulos CV, Tzanetis DE. 2015 On the quenching behaviour of a semilinear wave equation modelling MEMS technology. *Discrete and Continuous Dynamical Systems-A* **35**, 1009–1037.
12. Chang PH, Levine HA. 1981 The quenching of solutions of semilinear hyperbolic equations. *SIAM Journal on Mathematical Analysis* **12**, 893–903.
13. Levine H, Smiley M. 1984 Abstract wave equations with a singular nonlinear forcing term. *Journal of Mathematical Analysis and Applications* **103**, 409–427.
14. Liang C, Zhang K. 2014 Global solution of the initial boundary value problem to a hyperbolic nonlocal MEMS equation. *Computers and Mathematics with Applications* **67**, 549–554.
15. Smith RA. 1989 On a hyperbolic quenching problem in several dimensions. *SIAM Journal on Mathematical Analysis* **20**, 1081–1094.
16. Dai W, Duyckaerts T. 2021 Self-similar solutions of focusing semi-linear wave equations in R^N . *Journal of Evolution Equations* **21**, 4703–4750.
17. Collot C, Raphaël P, Szeftel J. 2019 On the stability of type I blow up for the energy super critical heat equation. *Memoirs of the American Mathematical Society* **1255**, v+97 pp.
18. Joseph DD, Lundgren TS. 1973 Quasilinear Dirichlet problems driven by positive sources. *Archive for Rational Mechanics and Analysis* **49**, 241–269.
19. Gelfand IM. 1959 Some problems in the theory of quasi-linear equations. *Uspekhi Matematicheskikh Nauk* **14**, 87–158.
20. Donninger R, Schörkhuber B. 2012 Stable self-similar blow up for energy subcritical wave equations. *Dynamics of Partial Differential Equations* **9**, 63–87.
21. Merle F, Raphael P, Rodnianski I, Szeftel J. 2022 On blow up for the energy super critical defocusing non linear Schrödinger equations. *Inventiones Mathematicae* **227**, 247–413.