ON SOBOLEV NORMS FOR LIE GROUP REPRESENTATIONS

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Abstract. We define Sobolev norms of arbitrary real order for a Banach representation (π, E) of a Lie group, with regard to a single differential operator $D = d\pi(R^2 + \Delta)$. Here, Δ is a Laplace element in the universal enveloping algebra, and $R > 0$ depends explicitly on the growth rate of the representation. In particular, we obtain a spectral gap for D on the space of smooth vectors of E. The main tool is a novel factorization of the delta distribution on a Lie group.

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1. Introduction

Let G be a Lie group and (π, E) be a Banach representation of G, that is, a morphism of groups $\pi : G \to \mathrm{GL}(E)$ such that the orbit maps

$$\gamma_v : G \to E, \quad g \mapsto \pi(g)v,$$

are continuous for all $v \in E$.

We say that a vector v is k-times differentiable if $\gamma_v \in C^k(G, E)$ and write $E^k \subset E$ for the corresponding subspace. The smooth vectors are then defined by $E^\infty = \bigcap_{k=0}^\infty E^k$.

The space E^k carries a natural Banach structure: if p is a defining norm for the Banach structure on E, then a k-th Sobolev norm of p on E^k is defined as follows:

$$p_k(v) := \Big(\sum_{m_1+\ldots+m_n \leq k} p(d\pi(X_1^{m_1} \cdot \ldots \cdot X_n^{m_n})v)^2 \Big)^{\frac{1}{2}} \quad (v \in E^k)\,. \tag{1.1}$$

Here $X_1, \ldots, X_n$ is a fixed basis for the Lie algebra $\mathfrak{g}$ of G, and $d\pi : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(E^\infty)$ is, as usual, the derived representation for the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. Then E^k, endowed with the norm p_k, is a Banach space and defines a Banach representation of G. Furthermore, E^∞ carries a natural Fréchet structure, induced by the Sobolev norms $(p_k)_{k\in\mathbb{N}_0}$. The corresponding G-action on E^∞ is smooth and of moderate growth, i.e. an SF-representation in the terminology of [2].

In case $(\pi, \mathcal{H})$ is a unitary representation on a Hilbert space $\mathcal{H}$, there is an efficient way to define the Fréchet structure on $\mathcal{H}^\infty$ via a Laplace element

$$\Delta = -\sum_{j=1}^n X_j^2 \tag{1.2}$$

in $\mathcal{U}(\mathfrak{g})$. More specifically, one defines the $2k$-th Laplace Sobolev norm in this case by

$$^{\Delta}p_{2k}(v) := p(d\pi((\mathbf{1} + \Delta)^k)v) \qquad (v \in E^{2k})\,. \tag{1.3}$$

The unitarity of the action then implies that the standard Sobolev norm p_{2k} is equivalent to $^{\Delta}p_{2k}$.

For a general Banach representation (π, E) we still have $E^\infty = \bigcap_{k=0}^\infty \mathrm{dom}(d\pi(\Delta^k))$, but it is no longer true that $^{\Delta}p_{2k}$, as defined in (1.3), is equivalent to p_{2k}: it typically fails that p_{2k} is dominated by $^{\Delta}p_{2k}$, for example if $-1 \in \mathrm{spec}(d\pi(\Delta))$ or if elliptic regularity fails as in Remark 4.2 below.

In the following we use Δ for the expression (2.1), a first-order modification of Δ as defined in (1.2), in order to make Δ selfadjoint on $L^2(G)$. In case G is unimodular, we remark that the two notions (2.1) and (1.2) coincide.

One of the main results of this note is that every Banach representation (π, E) admits a constant $R = R(E) > 0$ such that the operator $d\pi(R^2 + \Delta) : E^\infty \to E^\infty$ is invertible, see Corollary 3.3. The constant R is closely related to the growth rate of the representation, i.e. the growth of the weight $w_\pi(g) = \|\pi(g)\|$.

More precisely, for the Laplace Sobolev norms defined as

$$^{\Delta}p_{2k}(v) := p(d\pi((R^2 + \Delta)^k)v) \qquad (v \in E^{2k})\,, \tag{1.4}$$

we show that the families $(p_{2k})_k$ and $\left({}^\Delta p_{2k}\right)_k$ are equivalent in the following sense: Let m be the smallest even integer greater or equal to $1 + \dim G$. Then there exist constants $c_k, C_k > 0$ such that

$$c_k \cdot {}^\Delta p_{2k}(v) \leq p_{2k}(v) \leq C_k \cdot {}^\Delta p_{2k+m}(v) \qquad (v \in E^\infty)\,.$$

The above mentioned results follow from a novel factorization of the delta distribution $\delta_{\mathbf{1}}$ on G, see Proposition 2.4 in the main text for the more technical statement. This in turn is a consequence of the functional calculus for $\sqrt{\Delta}$, developed in [3], and previously applied to representation theory in [7] to derive factorization results for analytic vectors. The functional calculus allows us to define Laplace Sobolev norms for any order $s \in \mathbb{R}$ by

$${}^\Delta p_s(v) := p(d\pi((R^2 + \Delta)^{\frac{s}{2}})v) \qquad (v \in E^\infty)\,.$$

On the other hand [2] provided another definition of Sobolev norms for any order $s \in \mathbb{R}$; they were denoted Sp_s and termed induced Sobolev norms there. The norms Sp_s were based on a noncanonical localization to a neighborhood of $\mathbf{1} \in G$, identified with the unit ball in $\mathbb{R}^n$, and used the s-Sobolev norm on $\mathbb{R}^n$. We show that the two notions ${}^\Delta p_s$ and Sp_s are equivalent up to constant shift in the parameter s, see Proposition 4.3. The more geometrically defined norms ${}^\Delta p_s$ may therefore replace the norms Sp_s in [2].

Our motivation for this note stems from harmonic analysis on homogeneous spaces, see for example [1] and [4]. Here one encounters naturally the dual representation of some E^k and in this context it is often quite cumbersome to estimate the dual norm of p_k, caused by the many terms in the definition (1.1). On the other hand the dual norm of ${}^\Delta p_s$, as defined by one operator $d\pi((R^2 + \Delta)^{\frac{s}{2}})$, is easy to control and simplifies a variety of technical issues.

2. Some geometric analysis on Lie groups

Let G be a Lie group of dimension n and $\mathbf{g}$ a left invariant Riemannian metric on G. The Riemannian measure dg is a left invariant Haar measure on G. We denote by $d(g,h)$ the distance function associated to $\mathbf{g}$ (i.e. the infimum of the lengths of all paths connecting group elements g and h), by $B_r(g) = \{x \in G \mid d(x,g) < r\}$ the ball of radius r centered at g, and we set

$$d(g) := d(g, \mathbf{1}) \qquad (g \in G)\,.$$

Here are two key properties of $d(g)$, which will be relevant later, see [5]:

Lemma 2.1. *If $w : G \to \mathbb{R}_+$ is locally bounded and submultiplicative (i.e. $w(gh) \leq w(g)w(h)$), then there exist $c_1, C_1 > 0$ such that*

$$w(g) \leq C_1 e^{c_1 d(g)} \qquad (g \in G)\;.$$

Lemma 2.2. *There exists $c_G > 0$ such that for all $C > c_G$, $\int_G e^{-Cd(g)}\, dg < \infty$.*

Convolution in this article is always left convolution, i.e. for integrable functions $\varphi, \psi \in L^1(G)$ we define $\varphi * \psi \in L^1(G)$ by

$$\varphi * \psi(g) = \int_G \varphi(x)\psi(x^{-1}g)\, dx \qquad (g \in G)\,.$$

If we denote by $\mathcal{D}'(G)$ the space of distributions, resp. by $\mathcal{E}'(G)$ the subspace of compactly supported distributions, then the convolution above naturally extends to distributions provided one of them is compactly supported, i.e. lies in $\mathcal{E}'(G)$.

Denote by $\mathcal{V}(G)$ the space of left-invariant vector fields on G. It is common to identify the Lie algebra $\mathfrak{g}$ with $\mathcal{V}(G)$ where $X \in \mathfrak{g}$ corresponds to the vector field $\widetilde{X}$ given by

$$(\widetilde{X}f)(g) = \frac{d}{dt}\Big|_{t=0} f(g\exp(tX)) \qquad (g \in G, f \in C^\infty(G)).$$

We note that the adjoint of $\widetilde{X}$ on the Hilbert space $L^2(G)$ is given by

$$\widetilde{X}^* = -\widetilde{X} - \mathrm{tr}(\mathrm{ad}\, X),$$

and $\widetilde{X}^* = -\widetilde{X}$ in case $\mathfrak{g}$ is unimodular. Let us fix an orthonormal basis $\mathcal{B} = \{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ with respect to $\mathbf{g}$. Then the Laplace–Beltrami operator $\Delta = d^*d$ associated to $\mathbf{g}$ is given explicitly by

$$\Delta = \sum_{j=1}^{n} (-\widetilde{X_j} - \mathrm{tr}(\mathrm{ad}\, X_j))\ \widetilde{X_j}\,. \tag{2.1}$$

As $(G, \mathbf{g})$ is complete, Δ is essentially selfadjoint with spectrum contained in $[0, \infty)$. We denote by

$$\sqrt{\Delta} = \int \lambda\ dP(\lambda)$$

the corresponding spectral resolution. It provides us with a measurable functional calculus, which allows to define

$$f(\sqrt{\Delta}) = \int f(\lambda)\ dP(\lambda)$$

as an unbounded operator $f(\sqrt{\Delta})$ on $L^2(G)$ with domain

$$D(f(\sqrt{\Delta})) = \left\{ \varphi \in L^2(G) \mid \int |f(\lambda)|^2\ d\langle P(\lambda)\varphi, \varphi\rangle < \infty \right\}.$$

We are going to apply the above calculus to a certain function space. To do so, for $R' > 0$ we define a region

$$\mathcal{W}_{R',\vartheta} = \{z \in \mathbb{C} \mid |\operatorname{Im} z| < R'\} \cup \{z \in \mathbb{C} \mid |\operatorname{Im} z| < \vartheta |\operatorname{Re} z|\}.$$

For $R > 0$, $s \in \mathbb{R}$, the relevant function space is then defined as

$$\begin{aligned} \mathcal{F}_{R,s} \ &= \{f \in C^\infty(\mathbb{R}, \mathbb{C}) \mid f \text{ even}, \exists \vartheta > 0\ \exists R' > R : f \in \mathcal{O}(\mathcal{W}_{R',\vartheta}), \\ &\qquad \forall k \in \mathbb{N} : \sup\nolimits_{z \in \mathcal{W}_{R',\vartheta}} |\partial_z^k f(z)|(1+|z|)^{k-s} < \infty\}\,. \end{aligned}$$

See the Appendix to §2 in [3] for a related space of functions.

The resulting operators $f(\sqrt{\Delta})$ are given by a distributional kernel $K_f \in \mathcal{D}'(G \times G)$, $\langle f(\sqrt{\Delta})\ \varphi, \psi\rangle = \langle K_f, \varphi \otimes \psi\rangle$ for all $\varphi, \psi \in C_c^\infty(G)$. K_f has the following properties:

- smooth outside the diagonal: For $\Delta(G) = \{(g,g) | g \in G\}$, $K_f \in C^\infty(G \times G \setminus \Delta(G))$,
- left invariant: $K_f(gx, gy) = K_f(x, y)$,

- hermitian: $K_f(x,y) = \overline{K_f(y,x)}$.

By left invariance $f(\sqrt{\Delta})$ is a convolution operator with kernel $\kappa_f(x^{-1}y) := K_f(\mathbf{1}, x^{-1}y) = K_f(x,y)$:

$$\langle f(\sqrt{\Delta})\,\varphi, \psi\rangle = \langle K_f, \varphi \otimes \psi\rangle = \langle \kappa_f(x^{-1}y), (\varphi\otimes\psi)(x,y)\rangle = \langle \varphi * \kappa_f, \psi\rangle \tag{2.2}$$

for all $\varphi, \psi \in C_c^\infty(G)$. The distribution $\kappa_f \in \mathcal{D}'(G)$ is a smooth function on $G \setminus \{\mathbf{1}\}$. Because K_f is hermitian, the kernel κ_f is involutive in the sense that

$$\kappa_f(x) = \overline{\kappa_f(x^{-1})} \qquad (x \in G). \tag{2.3}$$

In particular, κ_f is left differentiable at $x \in G \setminus \{\mathbf{1}\}$, if and only if it is right differentiable at x.

We define the weighted L^1-Schwartz space on G by

$$\mathcal{S}_R(G) := \{f \in C^\infty(G) \mid \forall u, v \in \mathcal{U}(\mathfrak{g}) :\ (\tilde{u}_l \otimes \tilde{v}_r) f \in L^1(G, e^{Rd(g)}dg)\}\ ,$$

where $\tilde{u}_l$, resp. $\tilde{v}_r$, is the left, resp. right, invariant differential operator on G associated with $u, v \in \mathcal{U}(\mathfrak{g})$.

A theorem by Cheeger, Gromov and Taylor [3] allows us to describe the global behavior of κ_f:

Theorem 2.3. *Let $R, \varepsilon > 0$, $s \in \mathbb{R}$ and $f \in \mathcal{F}_{R,s}$. Then $\kappa_f = \kappa_1 + \kappa_2$, where*

1. *$\kappa_1 \in \mathcal{E}'(G)$ is supported in $B_\varepsilon(\mathbf{1})$, and $K_1(x,y) = \kappa_1(x^{-1}y)$ is the kernel of a pseudodifferential operator on G of order s,*
2. *$\kappa_2 \in \mathcal{S}_R(G)$.*

Part (1) is the content of Theorem 3.3 in [3]. For (2), the pointwise decay of κ_2 is stated in (3.45) there, while the Schwartz estimates are obtained as in their Appendix to §2.

From part (1) and the kernel estimates for pseudodifferential operators, we obtain $\kappa_1 \in C_c^{-s-n-\varepsilon}(G)$ for $\varepsilon > 0$ small enough, provided $-s-n-\varepsilon > 0$. Here $C_c^\alpha(G)$ denotes the space of Hölder continuous functions of order $\alpha > 0$, with compact support.

Applying the theorem to the function $f(z) = (R'^2 + z^2)^{-m}$ for $m \in \mathbb{N}$, which lies in $\mathcal{F}_{R,-2m}$ for any $R < R'$, we conclude the following factorization of the Dirac distribution $\delta_{\mathbf{1}}$:

Proposition 2.4. *Let $R' > R > 0$, $m \in \mathbb{N}$. Then*

$$\delta_{\mathbf{1}} = [(R'^2 + \Delta)^m \delta_{\mathbf{1}}] * \kappa, \tag{2.4}$$

where $\kappa = \kappa_1 + \kappa_2$ has the properties from Theorem 2.3 with $s = -2m$.

Proof. Set $T := f(\sqrt{\Delta})$ and $S := \frac{1}{f}(\sqrt{\Delta})$. Notice that $S(\varphi) = (R'^2 + \Delta)^m \varphi \in C_c^\infty(G)$ and thus $TS(\varphi) = \varphi$ for all $\varphi \in C_c^\infty(G)$ by the functional calculus. In particular,

$$\varphi = [(R'^2 + \Delta)^m \varphi] * \kappa \tag{2.5}$$

since T is given by right convolution with $\kappa = \kappa_f$, see (2.2). Choose a Dirac sequence $\varphi_n \to \delta_{\mathbf{1}}$. Passing to the limit in (2.5) yields

$$\delta_{\mathbf{1}} = [(R'^2 + \Delta)^m \delta_{\mathbf{1}}] * \kappa, \tag{2.6}$$

as asserted. □

3. Banach representation of Lie groups

In this section we briefly recall some basics on Banach representation of Lie groups and apply Proposition 2.4 to the factorization of vectors in E^k.

For a Banach space E we denote by $GL(E)$ the associated group of topological linear isomorphisms. By a *Banach representation* (π, E) of a Lie group G we understand a group homomorphism $\pi : G \to GL(E)$ such that the action

$$G \times E \to E, \quad (g, v) \mapsto \pi(g)v,$$

is continuous. For a vector $v \in E$ we denote by

$$\gamma_v : G \to E, \quad g \mapsto \pi(g)v,$$

the corresponding continuous orbit map. Given $k \in \mathbb{N}_0$, the subspace $E^k \subset E$ consists of all $v \in E$ for which $\gamma_v \in C^k$. We write $E^\infty = \bigcap_k E^k$ and refer to E^∞ as the space of smooth vectors. Note that all E^k for $k \in \mathbb{N}_0 \cup \{\infty\}$ are G-stable.

Remark 3.1. Let (π, E) be a Banach representation. The uniform boundedness principle implies that the function

$$w_\pi : G \to \mathbb{R}_+, \quad g \mapsto \|\pi(g)\|,$$

satisfies the assumptions of Lemma 2.1.

Let

$$c_\pi := \inf\{c > 0 \mid \exists C > 0 : \; w_\pi(g) \leq Ce^{cd(g)}\}\,.$$

For $R > 0$ we introduce the exponentially weighted spaces

$$\mathcal{R}_R(G) := L^1(G, w_R dg), \quad w_R(g) = e^{Rd(g)}\,.$$

Notice that $\mathcal{R}_R(G) \subset \mathcal{R}_{R'}(G)$ for $R > R'$ and that the corresponding Fréchet algebra $\mathcal{R}(G) := \bigcap_{R>0} \mathcal{R}_R(G)$ is independent of the particular choice of the metric $\mathbf{g}$.

Denote by π_l the left regular representation of G on $\mathcal{R}_R(G)$, and by π_r the right regular representation. A simple computation shows that $\mathcal{R}_R(G)$ becomes a Banach algebra under left convolution

$$\varphi * \psi(g) = \int_G \varphi(x)\; [\pi_l(x)\psi](g)\; dx \qquad (\varphi, \psi \in \mathcal{R}_R(G), g \in G)$$

for $R > c_G$.

More generally, whenever (π, E) is a Banach representation, Lemma 2.1 and Remark 3.1 imply that

$$\Pi(\varphi)v := \int_G \varphi(g)\; \pi(g)v\; dg \qquad (\varphi \in \mathcal{R}_R(G), v \in E)$$

defines an absolutely convergent Banach space valued integral for $R > R_E := c_\pi + c_G$. Hence the prescription

$$\mathcal{R}_R(G) \times E \to E, \quad (\varphi, v) \mapsto \Pi(\varphi)v,$$

defines a continuous algebra action of $\mathcal{R}_R(G)$ (here continuous refers to the continuity of the bilinear map $\mathcal{R}_R(G) \times E \to E$).

As an example, the left-right representation $\pi_l \otimes \pi_r$ of $G \times G$ also induces a Banach representation on $\mathcal{R}_R(G)$.

Our concern is now with the space of k-times differentiable vectors $\mathcal{R}_R(G)^k$ of $(\pi_l \otimes \pi_r, \mathcal{R}_R(G))$. It is clear that $\mathcal{R}_R(G)^k$ is a subalgebra of $\mathcal{R}_R(G)$ and that

$$\Pi(\mathcal{R}_R(G)^k)\ E \subset E^k \ ,$$

whenever (π, E) is a Banach representation and $R > R_E$.

Theorem 3.2. *Let $R > 0$ and $k = 2m$ for $m \in \mathbb{N}$. Set $k' := k - \dim G - 1 \geq 0$. Then there exists a $\kappa \in \mathcal{R}_R(G)^{k'}$ such that: For all Banach representations (π, E) with $R > R_E$ one has the following factorization of k-times differentiable vectors*

$$v = \Pi(\kappa) d\pi((R^2 + \Delta)^m) v \qquad (v \in E^k)\,. \tag{3.1}$$

Proof. Recall the factorization (2.4) of $\delta_{\mathbf{1}}$,

$$\delta_{\mathbf{1}} = [(R^2 + \Delta)^m \delta_{\mathbf{1}}] * \kappa \ .$$

We claim $\kappa \in \mathcal{R}_R(G)^{k'}$. Indeed, for $s = -2m$, $n = \dim G$ and $\varepsilon \in (0,1)$, Theorem 2.3 shows that $\kappa_1 \in C_c^{2m - \dim G - \varepsilon}(G) \subset \mathcal{R}_R(G)^{k'}$ and $\kappa_2 \in \mathcal{S}_R(G) \subset \mathcal{R}_R(G)^{k'}$. We then obtain that

$$\gamma_v = [(R^2 + \Delta)^m \gamma_v] * \kappa \ ,$$

see also (2.5), and evaluation at $g = \mathbf{1}$ gives

$$v = \gamma_v(\mathbf{1}) = \int_G \kappa(g^{-1}) \pi(g) d\pi((R^2 + \Delta)^m) v \ dg \ .$$

Now recall from (2.3) that $\kappa(g) = \overline{\kappa(g^{-1})}$ and that with our choice of $f(z) = (R^2 + z^2)^{-m}$ from before the kernel κ is even real. Hence

$$v = \Pi(\kappa) d\pi((R^2 + \Delta)^m) v \ ,$$

as asserted. $\square$

Corollary 3.3. *Let $R > R_E$. Then*

$$d\pi\left(R^2 + \Delta\right) : E^\infty \to E^\infty$$

is invertible.

Remark 3.4. (Spectral gap for Banach representations) We can interpret Corollary 3.3 as a spectral gap theorem for Banach representations in terms of $R_E = c_G + c_\pi$. However, we note that the bound $R > R_E$ can be improved for special classes of representations. For example, if (π, E) is a unitary representation, then

$$\operatorname{Re}\langle d\pi(\Delta) v, v\rangle \geq 0$$

for all $v \in E^\infty$, and hence $d\pi(\Delta) + R^2$ is injective for all $R > 0$. Moreover, the Lax-Milgram theorem implies that $d\pi(\Delta) + R^2$ is in fact invertible. On the other hand, our bound in Corollary 3.3 gives information about the convolution kernel of the inverse of $d\pi(\Delta) + R^2$ for $R > c_G$.

4. Sobolev norms for Banach representations

4.1. **Standard and Laplace Sobolev norms.** As before, we let (π, E) be a Banach representation. On E^∞, the space of smooth vectors, one usually defines Sobolev norms as follows. Let p be the norm underlying E. We fix a basis $\mathcal{B} = \{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ and set

$$p_k(v) := \Big[\sum_{m_1+\ldots+m_n \le k} p(d\pi(X_1^{m_1} \cdot \ldots \cdot X_n^{m_n})v)^2 \Big]^{\frac{1}{2}} \qquad (v \in E^\infty)\,.$$

Strictly speaking this notion depends on the choice of the basis $\mathcal{B}$ and $p_{k,\mathcal{B}}$ would be the more accurate notation. However, a different choice of basis, say $\mathcal{C} = \{Y_1, \ldots, Y_n\}$ leads to an equivalent family of norms $p_{k,\mathcal{C}}$, i.e. for all k there exist constants $c_k, C_k > 0$ such that

$$c_k \cdot p_{k,\mathcal{C}}(v) \le p_{k,\mathcal{B}}(v) \le C_k \cdot p_{k,\mathcal{C}}(v) \qquad (v \in E^\infty)\,. \tag{4.1}$$

Having said this, we now drop the subscript $\mathcal{B}$ in the definition of p_k and simply refer to p_k as the *standard k-th Sobolev norm* of (π, E). Note that p_k is Hermitian (i.e. obtained from a Hermitian inner product) if p was Hermitian.

The completion of (E^∞, p_k) yields E^k. In particular, (E^k, p_k) is a Banach space for which the natural action $G \times E^k \to E^k$ is continuous, i.e. defines a Banach representation.

The family $(p_k)_{k\in\mathbb{N}}$ induces a Fréchet structure on E^∞ (in view of (4.1) of course independent of the choice of $\mathcal{B}$) such that the natural action $G \times E^\infty \to E^\infty$ becomes continuous.

Now we introduce a family of *Laplace Sobolev norms*, first of even order $k \in 2\mathbb{N}_0$, as follows. Let $R > R_E$ and set

$${}^\Delta p_k(v) := p\left(d\pi\left((R^2 + \Delta)^{k/2}\right) v\right) \qquad (v \in E^\infty)\ .$$

Of course, a more accurate notation would include $R > 0$, i.e. write ${}^{\Delta,R}p_k$ instead of ${}^\Delta p_k$. In addition, Δ also depends on the basis $\mathcal{B}$. For purposes of readability we decided to suppress this data in the notation.

Proposition 4.1. (Comparison of the families $(p_{2k})_{k\in\mathbb{N}_0}$ and $({}^\Delta p_{2k})_{k\in\mathbb{N}_0}$)
For all $k \in \mathbb{N}_0$ there exist c_k, $C_k > 0$ such that for all $v \in E^\infty$

$$c_k \cdot {}^\Delta p_{2k}(v) \le p_{2k}(v) \le C_k \cdot {}^\Delta p_{2k+m}(v)\,,$$

where m is the smallest even integer greater or equal to $1 + \dim G$.

Proof. The first inequality follows directly from the definitions of p_{2k}, ${}^\Delta p_{2k}$. The second is a consequence of the factorization (3.1). □

Remark 4.2. In general it is not true that p_{2k} is smaller than a multiple of ${}^\Delta p_{2k}$. In other words, an index shift as in Proposition 4.1, is actually needed. As an example we consider $E = C_0(\mathbb{R}^2)$ of continuous functions on $\mathbb{R}^2$ which vanish at infinity, endowed with the sup-norm $p(f) = \sup_{x\in\mathbb{R}^2} |f(x)|$. Then E becomes a Banach representation for the regular action of $G = (\mathbb{R}^2, +)$ by translation in the arguments. In this situation there exists a function $u \in E$ such $\Delta u \in E$ but $\partial_y^2 u \notin E$, see [6, Problem 4.9]. Hence $p_2(u) = \infty$, while ${}^\Delta p_2(u) < \infty$.

4.2. **Sobolev norms of continuous order $s \in \mathbb{R}$.**

4.2.1. *Induced Sobolev norms.* In [2] Sobolev norms for a Banach representation (π, E) were defined for all parameters $s \in \mathbb{R}$. We briefly recall their construction.

We endow the continuous dual E' of E with the dual norm

$$p'(\lambda) := \sup_{p(v)\leq 1} |\lambda(v)| \qquad (\lambda \in E') .$$

For $\lambda \in E'$ and $v \in E^\infty$ we define the matrix coefficient

$$m_{\lambda,v}(g) = \lambda(\pi(g)v) \qquad (g \in G) ,$$

which is a smooth function on G. Given an open relatively compact neighborhood $B \subset G$ of $\mathbf{1}$, diffeomorphic to the open unit ball in $\mathbb{R}^n$, we fix $\varphi \in C_c^\infty(G)$ such that $\operatorname{supp}(\varphi) \subset B$ and $\phi(\mathbf{1}) = 1$. The function $\phi \cdot m_{\lambda,\phi}$ is then supported in B and upon identifying B with the open unit ball in $\mathbb{R}^n$, say $B_{\mathbb{R}^n}$, we denote by $\|\phi \cdot m_{\lambda,v}\|_{H^s(\mathbb{R}^n)}$ the corresponding Sobolev norm. We then set

$$Sp_s(v) := \sup_{\substack{\lambda\in E'\\ p'(\lambda)\leq 1}} \|\phi \cdot m_{\lambda,v}\|_{H^s(\mathbb{R}^n)} \qquad (v \in E^\infty) .$$

In the terminology of [2] this defines a G-continuous norm on E^∞.

4.2.2. *Laplace Sobolev norms.* For $R > R_E$ and $s \in \mathbb{R}$, on the other hand the functional calculus for $\sqrt{\Delta}$ also gives rise to a G-continuous norm on E^∞: We define

$$ {}^\Delta p_s(v) := p((R^2 + \Delta)^{s/2}\gamma_v(g)|_{g=\mathbf{1}}) \qquad (v \in E^\infty) . \tag{4.2}$$

4.2.3. *Comparison results.*

Proposition 4.3. (Comparison of the families $(Sp_s)_{s\geq 0}$ and $({}^\Delta p_s)_{s\geq 0}$) *Let $R > R_E$. Then for all $s \geq 0$, $\varepsilon > 0$, there exist $c_s, C_s > 0$ such that for all $v \in E^\infty$*

$$c_s \cdot Sp_s(v) \leq {}^\Delta p_s(v) \leq C_s \cdot Sp_{s+\frac{n}{2}+\varepsilon}(v) \ .$$

Proof. The first inequality was shown in [2] for $k \in 2\mathbb{N}$. It follows for all $s \geq 0$ by interpolation.

For the second inequality, we apply the standard Sobolev embedding theorem for $\mathbb{R}^n$ and obtain that

$$\|\phi \cdot m_{\lambda,v}\|_{H^{s+\frac{n}{2}+\varepsilon}(\mathbb{R}^n)} \gtrsim \|\phi \cdot m_{\lambda,v}\|_{C^s(B_{\mathbb{R}^n})} \gtrsim |\lambda((R^2+\Delta)^{s/2}\pi(g)v)|_{g=\mathbf{1}}| \ .$$

The assertion follows by taking the supremum over $\lambda \in E'$ with $p'(\lambda) \leq 1$. $\square$

4.3. **Sobolev norms of order $s \leq 0$.** The natural way to define negative Sobolev norms is by duality. For a Banach representation (π, E) with defining norm p and $k \in \mathbb{N}_0$ we let p_k' be the norm of $(E')^k$ and define p_{-k} as the dual norm of p_k', i.e.

$$p_{-k} := (p_k')' .$$

The norm p_{-k} is naturally defined on $((E')^k)'$. Now observe that the natural inclusion $(E')^k \hookrightarrow E'$ is continuous with dense image and thus yields a continuous dual inclusion $E'' \hookrightarrow ((E')^k)'$. The double-dual E'' contains E in an isometric fashion. Hence p_{-k} gives rise to a natural norm on E, henceforth denoted by the same symbol, and the completion of E with respect to p_{-k} will be denoted by E^{-k}.

Remark 4.4. In case E is reflexive, i.e. $E'' \simeq E$, the space E^{-k} is isomorphic to the strong dual of $(E')^k$.

On the other hand we have seen that the families $(p_k)_k$ and $\left({}^{\Delta}p_k\right)_k$ are equivalent. In this regard we note that ${}^{\Delta}p_{-k}$ as defined in (4.2) coincides with the dual norm of ${}^{\Delta}p_k'$.

As a corollary of Proposition 4.1 (and interpolation to also non-even indices $k \in \mathbb{N}_0$) we have:

Corollary 4.5. *For all $k \in \mathbb{N}_0$ there exist constants $c_k, C_k > 0$ such that*

$$c_k \cdot p_{-k}(v) \leq {}^{\Delta}p_{-k}(v) \leq C_k \cdot p_{-k+n+1}(v) \qquad (v \in E^\infty)\,. \tag{4.3}$$

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