Boundary integral equations in space and time: *Higher order Galerkin methods and applications*

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(joint with A. Aimi[2], G. Di Credico[1,2], C. Özdemir[3], E. P. Stephan[4])

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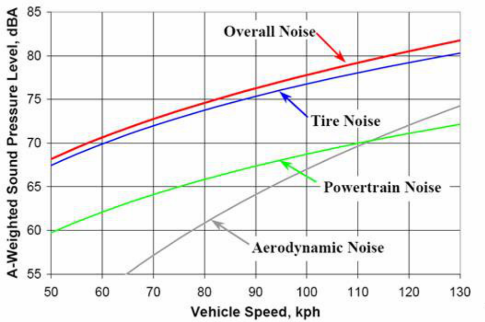
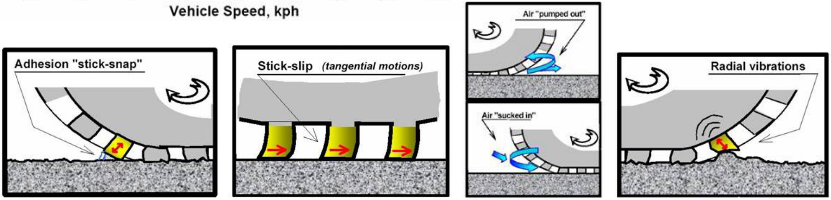
Zürich Colloquium in Applied and Computational Mathematics
September 27, 2023

Motivation: Sound radiation of tires

Noise of distant cars



Amplification: tire resonances and horn effect



Noise sources (+ resonances) $\rightsquigarrow$ data f

Beyond traffic noise

Deutsche Oper, Berlin
picture provided by M. Ochmann

Wave equation

$$\begin{aligned} u &= u(t,x) \quad \text{sound pressure} \\ \partial_t^2 u - \Delta u &= 0 \qquad \text{in } \mathbb{R}_t \times \mathbb{R}^3_x \setminus \Omega \\ u &= 0 \qquad \text{for } t \leq 0\ . \end{aligned}$$

Realistic: acoustic boundary conditions $\partial_\nu u - \alpha \partial_t u = g$ on $\Gamma = \partial\Omega$.

Simple: Dirichlet boundary conditions $u = g$ on $\Gamma = \partial\Omega$.

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Develop efficient methods to compute:

- time-domain wave propagation over large distances
- for *complex* geometries and boundary conditions
- with possibly *nonsmooth* solutions.

Wave equation

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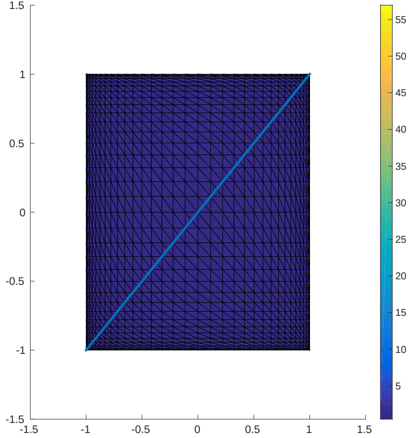
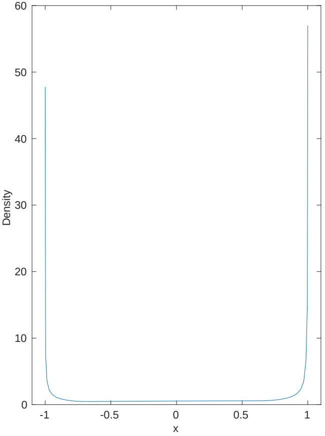
Space-time boundary integral formulation of Dirichlet problem (3d):

$$\mathcal{V}\phi(t,x) = \int_\Gamma \frac{\phi(x', t - |x - x'|)}{4\pi|x - x'|}\ d\Gamma_{x'} = (\mathcal{K} + \tfrac{1}{2})g(t,x)$$

What's this talk about?

$\Omega^c = \mathbb{R}^3 \setminus ([-1,1]^2 \times \{0\})$, $\mathcal{V}\phi = \sin(t)^5$ on $[-1,1]^2 \times \{0\}$.
solution near corner $r^{-0.703\ldots}$, near edge $r^{-\frac{1}{2}}$





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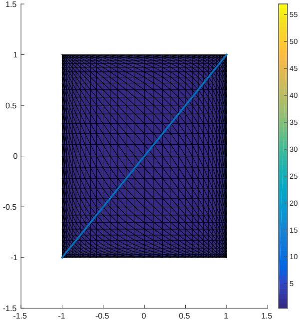
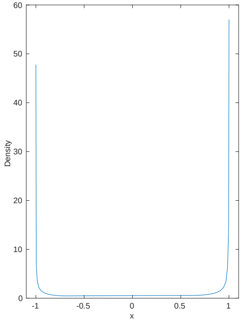
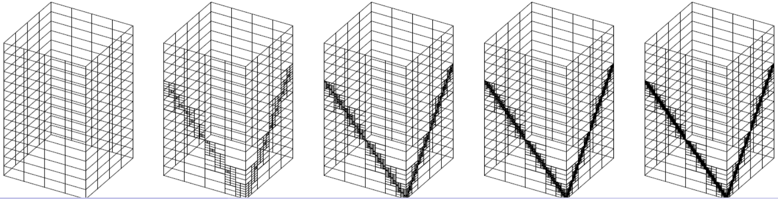






What's this talk about?

Time domain BEM for wave scattering off a knife's blade (screen)

$$\mathcal{V}\phi(t,x) = \int_\Gamma \frac{\phi(x', t - |x - x'|)}{4\pi|x - x'|}\, d\Gamma_{x'} = f$$

Convergence rates - theory and numerical experiments
(in DOF on a 2d screen, energy norm)

- 0.5: h-version, uniform
- 0.77: h-version, adaptive
  (ongoing + HG, Özdemir, Stark, Stephan, Numer. Math. 2020)
- 1.0: hp-version, uniform
  (HG, Özdemir, Stark, Stephan, CMAME 2019)
- $\beta/2$: h-version, β-graded
  (HG, Meyer, Özdemir, Stark, Stephan, Numer. Math. 2018)

Extensions: polygonal scatterers, elasticity. Rates $\sim$ angles, material parameters. (Aimi, Di Credico, HG, Stephan, Numer. Math. 2023)

Related work: old and new

$$u = u(t,x) \quad \text{sound pressure}$$
$$\partial_t^2 u - \Delta u = 0 \qquad \text{in } \mathbb{R}_t \times \mathbb{R}^3_x \setminus \Omega$$
$$u = 0 \qquad \text{for } t \leq 0\ .$$

Singular solutions: h, p, hp versions & adaptive methods:

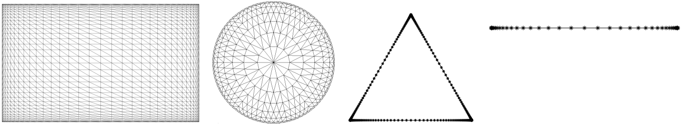
- Edge/corner/geometric singularities.
- Sharp travelling wave crests.
- (Nonlinear) contact problems.

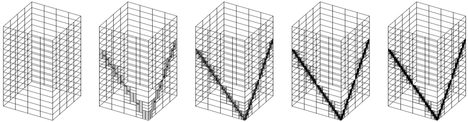
p and hp FEM / BEM: Long history since the 80's with Babuska, Dorr, Suri, Schwab, Melenk, Stephan, Bespalov, Heuer, ...

space or space-time adaptive BEM:
variable Δt: Sauter, Schanz, ... (gCQ), Steinbach, Zank, ...
variable Δx: Abboud, space-time (2d, brute force): Gläfke, Maischak.
related FEM: Chaumont-Frelet (2023), Steinbach, Zank

Outline

- Boundary integral formulation and space-time Galerkin approximation
- Screen problems:
  singular expansions of the solution near edges / corners
- approximation properties of graded h and of p versions
- experiments for hp version on geometrically graded meshes
- a posteriori error analysis and adaptive mesh refinements





(a) mesh 1 (b) mesh 4 (c) mesh 7 (d) mesh 10 (e) mesh 13

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Simple: Dirichlet boundary conditions $u = g$ on $\Gamma = \partial\Omega$.

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From PDE to boundary integral formulation

Fundamental solution G – wave of a point source

in $2d$: $G(x,x';t,\tau) = \frac{1}{2\pi}\frac{H(t-\tau-|x-x'|)}{\sqrt{(t-\tau)^2-|x-x'|^2}}$

in $3d$: $G(x,x';t,\tau) = \frac{1}{4\pi}\frac{\delta(t-\tau-|x-x'|)}{|x-x'|}$

H Heaviside function, δ Dirac point measure.

From PDE to boundary integral formulation

Fundamental solution G – wave of a point source

Representation formula – reduction to Γ

$$u(x,t) = \int_0^t \int_\Gamma G(x,x';t,\tau)\frac{\partial u}{\partial \nu}(x',\tau)\ d\Gamma_{x'} d\tau - \int_0^t \int_\Gamma \frac{\partial G}{\partial \nu}(x,x';t,\tau) u(x',\tau)\ d\Gamma_{x'} d\tau$$

ν outer unit normal to Γ.

From PDE to boundary integral formulation

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Dirichlet problem $u = g$ on Γ

Boundary integral equation

$$\mathcal{V}\phi(x,t) = \left(\mathcal{K} + \tfrac{1}{2}\right) g(x,t)$$

Weakly singular operator

$$\mathcal{V}\phi(\mathbf{x},t) = \int_0^t \int_\Gamma G(x,x';t,\tau)\Phi(x',\tau)d\Gamma_{x'}d\tau$$

Solution $\phi = \dfrac{\partial u}{\partial \nu}$

From PDE to boundary integral formulation

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Linear elastodynamics: analogous to wave equation

Boundary integral equation

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in $2d$ $G_{ij}(\mathbf{x},\mathbf{x}';t,\tau) :=$

$$\frac{H[c_\mathtt{P}(t-\tau)-r]}{2\pi\varrho c_\mathtt{P}}\left\{\frac{r_i r_j}{r^4}\frac{2c_\mathtt{P}^2(t-\tau)^2-r^2}{\sqrt{c_\mathtt{P}^2(t-\tau)^2-r^2}} - \frac{\delta_{ij}}{r^2}\sqrt{c_\mathtt{P}^2(t-\tau)^2-r^2}\right\}$$

$$-\frac{H[c_\mathtt{S}(t-\tau)-r]}{2\pi\varrho c_\mathtt{S}}\left\{\frac{r_i r_j}{r^4}\frac{2c_\mathtt{S}^2(t-\tau)^2-r^2}{\sqrt{c_\mathtt{S}^2(t-\tau)^2-r^2}} - \frac{\delta_{ij}}{r^2}\frac{c_\mathtt{S}^2(t-\tau)^2}{\sqrt{c_\mathtt{S}^2(t-\tau)^2-r^2}}\right\}$$

Two wave speeds: $c_P = \sqrt{(\lambda+2\mu)/\varrho}$, $c_\mathtt{S} = \sqrt{\mu/\varrho} > 0$.

Set-up a la Becache–Ha Duong

$$\mathcal{V}\phi(x,t) = \left(\mathcal{K} + \frac{1}{2}\right) g(x,t) =: f$$

space–time anisotropic Sobolev spaces $H^r_\sigma(\mathbb{R}^+, H^s(\Gamma))$, $\sigma > 0$:

$H^r_\sigma(\mathbb{R}^+, H^s(\mathbb{R}^2))$ defined using Fourier–Laplace transform

$$\Big\{\psi : \mathrm{supp}\ \psi \subset \overline{\mathbb{R}_+} \times \mathbb{R}^2,\ \int_{\mathbb{R}+i\sigma} d\omega \int_{\mathbb{R}^2} d\xi |\omega|^{2r}(|\omega|^2 + |\xi|^2)^s |\mathcal{F}\psi(\omega,\xi)|^2 < \infty$$

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Space-time variational formulation of Dirichlet problem:
Find $\phi \in H^1(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))$ such that $\forall \psi \in H^1(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))$:

$$\langle \mathcal{V}\partial_t \phi, \psi \rangle = \langle \partial_t f, \Psi \rangle$$

The solution ϕ exists for $f \in H^2(\mathbb{R}^+, H^{\frac{1}{2}}(\Gamma))$.

Key: $\mathcal{V}\partial_t$ coercive with loss (for wave eq.: Bamberger – Ha Duong '86)

$$\|\phi\|^2_{1,-\frac{1}{2},\Gamma} \gtrsim \langle \mathcal{V}\partial_t \phi, \phi \rangle \gtrsim \|\phi\|^2_{0,-\frac{1}{2},\Gamma}$$

Set-up a la Becache–Ha Duong

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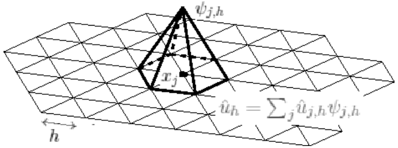
Loss avoided for bilinear form involving Hilbert transform in recent work by Urzua-Torres, Steinbach (+ Zank).

Discretization

- $\Gamma = \cup_{i=1}^{M} \Gamma_i$ triangulation
- V_h^p piecewise polynomial functions of degree p on $\Gamma = \cup_{i=1}^{M} \Gamma_i$ (continuous if $p \geq 1$)
- $[0, T) = \cup_{n=1}^{L} [t_{n-1}, t_n)$, $t_n = n(\Delta t)$
- $V_{\Delta t}^q$ piecewise polynomial functions of degree q in time (continuous and vanishing at $t = 0$ if $q \geq 1$)
- simplest case: tensor products in space-time $V_{h,\Delta t}^{p,q} = V_h^p \otimes V_{\Delta t}^q$



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Time domain BEM: Find $\phi_{h,\Delta t} \in V_{h,\Delta t}^{p,q}$ such that $\forall \psi_{h,\Delta t} \in V_{h,\Delta t}^{p,q}$:

$$\langle \mathcal{V} \partial_t \phi_{h,\Delta t}, \psi_{h,\Delta t} \rangle = \langle \partial_t f, \psi_{h,\Delta t} \rangle$$

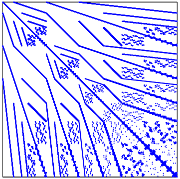
Time stepping for tensor product meshes

- exact wave *propagation*
- discretized *reflection* at Γ
- unconditionally stable

- sparse or easily compressible Galerkin matrix

causality $\rightsquigarrow$ block triangular Galerkin matrix $\rightsquigarrow$ backsubstitution:
compute 1 matrix per time step (for tensor product discretizations)

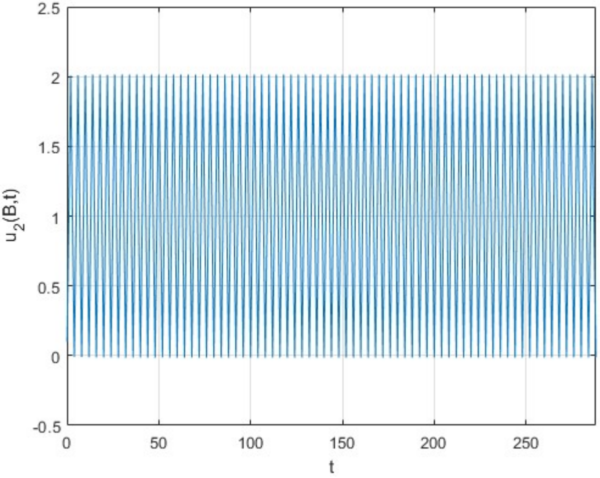
$$\forall n : \sum_{m=1}^{n} V^{n-m}\Phi^m = F^n$$



$$\iff \quad V^0\Phi^n = F^n - \sum_{m=1}^{n-1} V^{n-m}\Phi^m$$

V^0 for $\Gamma = \mathbb{S}^2 \subset \mathbb{R}^3$

Long-time stability



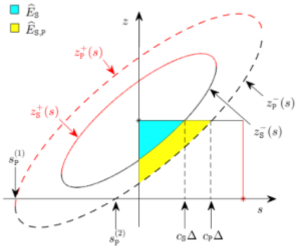
numerical solution for a mixed boundary problem in elastodynamics
(Aimi, Di Credico, HG, Guardasoni, Speroni, to appear in APNUM)

Numerical quadrature: Light cones & decompositions

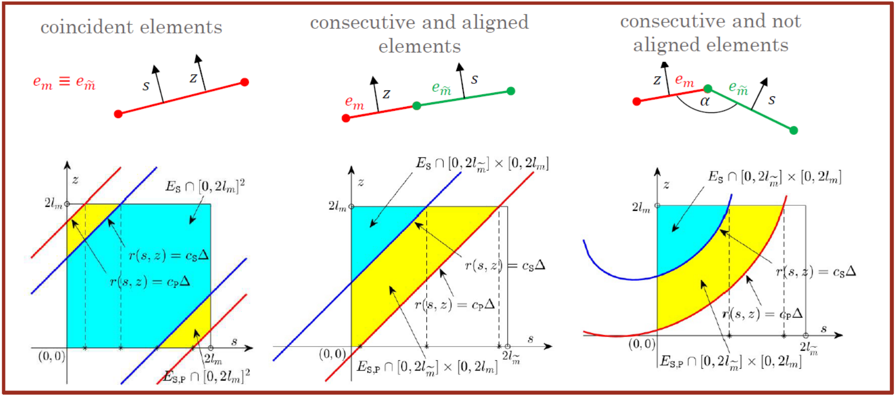
$$\int_\Gamma \int_\Gamma w^{\mathbf{p}}_{\tilde{m}}(\mathrm{x}) w^{\mathbf{p}}_{m}(\xi) \nu^{\mathcal{V}}_{ij}(\mathrm{r}, \Delta) d\xi d\mathrm{x} \qquad i,j = 1,2$$

$$\begin{aligned}
\nu^{\mathcal{V}}_{ij}(\mathrm{r}, \Delta) &= \left(\frac{r_i r_j}{r^4} - \frac{\delta_{ij}}{2r^2}\right)\left[\frac{H[c_\mathrm{p}\Delta - r]}{c_\mathrm{p}}\Delta\varphi_\mathrm{p}(r,\Delta) - \frac{H[c_\mathrm{s}\Delta - r]}{c_\mathrm{s}}\Delta\varphi_\mathrm{s}(r,\Delta)\right] \\
&+ \frac{\delta_{ij}}{2}\left[\frac{H[c_\mathrm{p}\Delta - r]}{c_\mathrm{p}^2}\hat{\varphi}_\mathrm{p}(r,\Delta) + \frac{H[c_\mathrm{s}\Delta - r]}{c_\mathrm{s}^2}\hat{\varphi}_\mathrm{s}(r,\Delta)\right]
\end{aligned}$$

$$\begin{aligned}
\varphi_\gamma &= \sqrt{c_\gamma^2\Delta^2 - r^2} \\
\hat{\varphi}_\gamma &= \log\left(\sqrt{c_\gamma^2\Delta^2 - r^2} + c_\gamma\Delta\right) - \log(r)
\end{aligned}$$



Numerical quadrature: Light cones & decompositions



$E_S = \{r(s,z) < c_S\Delta\}$

$E_{S,P} = \{c_S\Delta < r(s,z) < c_P\Delta\}$

Screen problems

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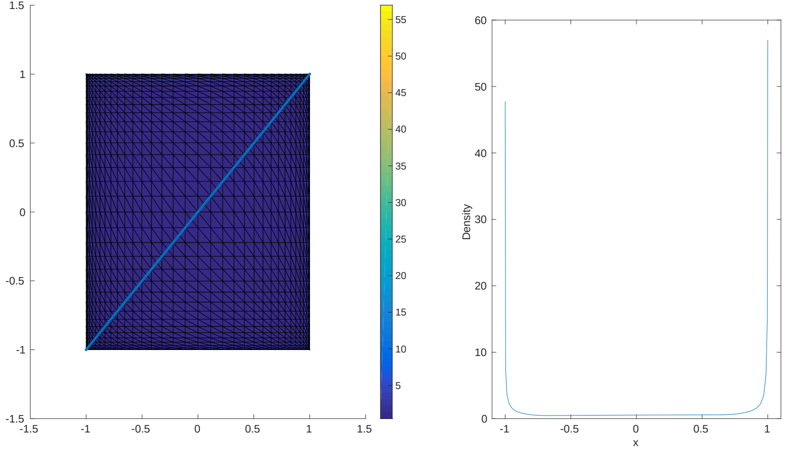
solution near corner $r^{-0.703\ldots}$, near edge $r^{-\frac{1}{2}}$

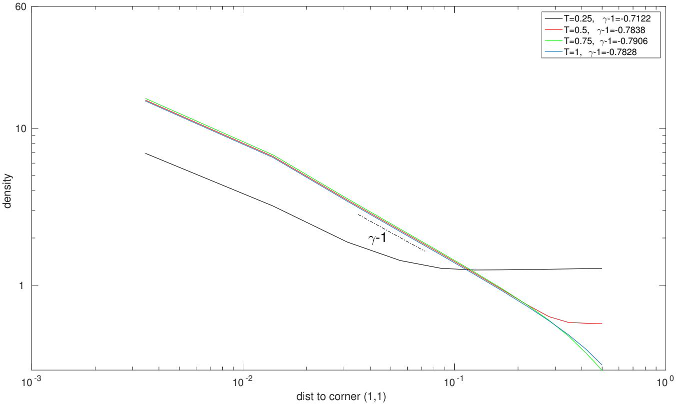


Screen problems: Corner exponents for waves

$\Omega^c = \mathbb{R}^3 \setminus ([-1,1]^2 \times \{0\})$, $\mathcal{V}\phi = \sin(t)^5$ on $[-1,1]^2 \times \{0\}$, $0 < t < 1$.

corner exponent: $-0.78 \sim \gamma - 1 = -0.703$ as in elliptic case

Plot: $\phi(t,r)$ as function of r along $x = y$



Screen problems: Edge exponents for waves

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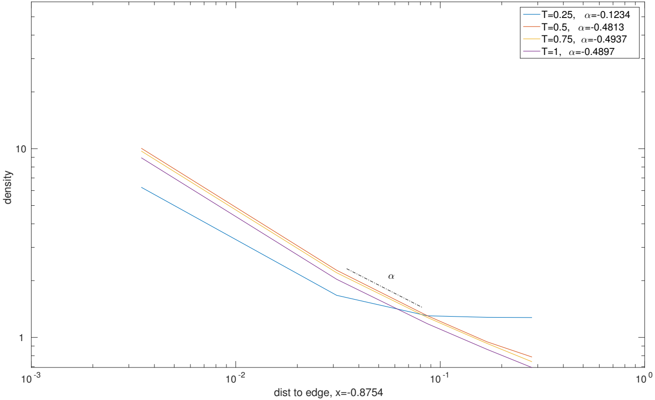
edge exponent: $-0.49 \sim -\frac{1}{2}$ as in elliptic case

Plot: $\phi(t,x,y)$ as function of y, at $x = 0.8754$



Singularities at edges and corners

Frequency domain: (Kondratiev, Dauge, Maz'ya, Nicaise, . . .)
Solution behaves like

- $r^{\nu-1}$ near edge.
- $r^{\gamma-1}$ near 3d corner

Near boundary $\nu = \frac{1}{2}$, for wave equation on square screen γ=0.29.

Singularities at edges and corners

von Petersdorff '89, +Stephan '90 for Helmholtz: precise tensor product decomposition, BEM on graded meshes
$\Longrightarrow$ optimal approximation on graded meshes.

Theorem ($r^{\gamma-1}$ in corner, $r^{-\frac{1}{2}}$ at edges, coeffs depend on ω)

Let $\mathcal{V}_\omega \psi_\omega = f_\omega \in H^2(\Gamma)$. *Then*

$$\psi_\omega = \psi_{0,\omega} + \chi_\omega(r) r^{\gamma-1} \alpha_\omega(\theta) + \tilde{\chi}_\omega(\theta) b_{1,\omega}(r) r^{-1} (\sin(\theta))^{-\frac{1}{2}}$$
$$+ \tilde{\chi}_\omega(\frac{\pi}{2} - \theta) b_{2,\omega}(r) r^{-1} (\cos(\theta))^{-\frac{1}{2}}$$

where $\psi_{0,\omega} \in H^{1-}(\Gamma)$, $\alpha_\omega(\theta) \in H^{1-}[0, \frac{\pi}{2}],\ b_{i,\omega} = c_{i,\omega} r^\gamma + d_{i,\omega}(r)$,
$r^{-\frac{1}{2}} d_{i,\omega}(r) \in H^1(\mathbb{R}^+),\ r^{-\frac{3}{2}} d_{i,\omega}(r) \in L_2(\mathbb{R}^+),\ c_{i,\omega} \in \mathbb{R}$.
(r, θ) *polar coordinates around (0,0),* $\chi_\omega,\ \tilde{\chi}_\omega \in C_c^\infty$, $= 1$ *near* 0.

γ eigenvalue: $\gamma \approx 0.2966$ for rectangle

Singularities at edges and corners

Work on wave equation and elastodynamics:

- frequency domain: Kondratiev, Dauge, Maz'ya, Nicaise, ...
  Dauge '87: singular expansions near corners and edges
  von Petersdorff '89, +Stephan '90:
  precise tensor product decomposition, BEM on graded meshes

- Plamenevskii et al. since '99: analysis of wave equation and elastodynamics in domains with singularities
- Müller – Schwab '15 / '16: $2d$ FEM on graded meshes
- HG, Özdemir, Stark, Stephan, '18 / '19: $3d$ BEM on graded meshes
- Aimi, Di Credico, HG, Stephan '23: $2d/3d$ BEM for elastodynamics

Singularities at edges and corners

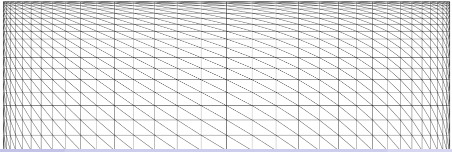
The next theorem in $2d$ goes back to Plamenevskii (a), Müller–Schwab (b) for FEM, 2d/3d Aimi, Di Credico, HG, Stephan.

Theorem

a) In $2d$, solution behaves like $r^{-\frac{1}{2}}$ at $\partial\Gamma$.
In $3d$ $r^{\gamma-1}$ near corner, $r^{-\frac{1}{2}}$ at $\partial\Gamma$.

b) Optimal approximation on β-graded mesh in energy norm ($\Delta t \leq h$):
Error of best approximation in $H^0_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) \lesssim h^{\min\{\frac{\beta}{2},\frac{3}{2}\}-\varepsilon}$.

$$x_j = 1 - \left(\frac{j}{N}\right)^\beta,\ j = 1, \dots, N\ .$$



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In $3d$ $r^{\gamma-1}$ near corner, $r^{-\frac{1}{2}}$ at $\partial\Gamma$.

b) Optimal approximation on β-graded mesh in energy norm $(\Delta t \leq h)$:
Error of best approximation in $H^0_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) \lesssim h^{\min\{\frac{\beta}{2},\frac{3}{2}\}-\varepsilon}$.

Theorem extends to elastodynamics, polygonal domains and hypersingular boundary integral equation (Aimi, Di Credico, HG, Stephan '23).

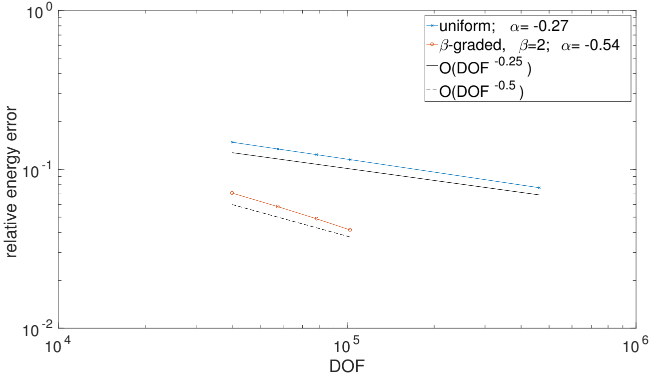
Screen problems: convergence rates for $3d$ wave equation

$\Omega^c = \mathbb{R}^3 \setminus ([-1,1]^2 \times \{0\})$, $\mathcal{V}\phi = \sin(t)^5$ on $[-1,1]^2 \times \{0\}$, $0 < t < 1$.

Energy norm^2 $= \langle \mathcal{V}\partial_t(\phi_h - \phi), \phi_h - \phi \rangle \sim h^2 \simeq DOF(\Gamma)^{-1}$ (2-graded)

$\sim h \simeq DOF(\Gamma)^{-1/2}$ (uniform)

similar results for W and for Dirichlet-to-Neumann operator

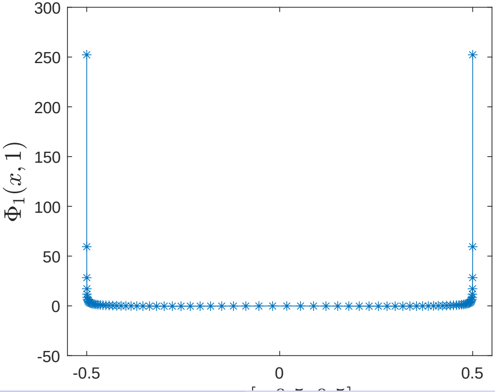


Screen problems: elastodynamics on graded meshes

$\Omega^c = \mathbb{R}^2 \setminus ([-\frac{1}{2}, \frac{1}{2}] \times \{0\})$, $\mathcal{V}\phi(t,x) = g(x,t)(1,1)^T$ on $[-\frac{1}{2}, \frac{1}{2}] \times \{0\}$.
Material parameters $\lambda = 2,\ \mu = 1,\ \varrho = 1$

$$g(x,t) = f(t)x^4, \quad f(t) = \sin^2(4\pi t).$$

solution near vertex $\sim r^{-\frac{1}{2}}$



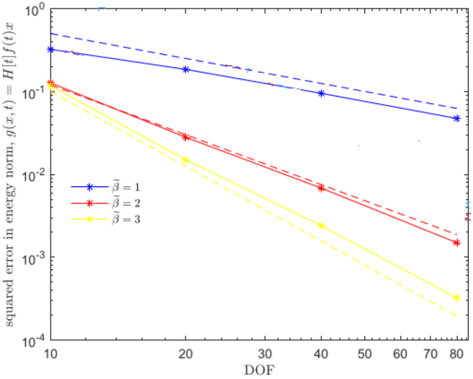
Screen problems: Convergence rates on β-graded meshes

$\Omega^c = \mathbb{R}^2 \setminus ([-\frac{1}{2}, \frac{1}{2}] \times \{0\})$, $\mathcal{V}\Phi(t,x) = g(x,t)(1,1)^T$ on $[-\frac{1}{2}, \frac{1}{2}] \times \{0\}$.

$$g(x,t) = f(t)x^4, \quad f(t) = \sin^2(4\pi t).$$

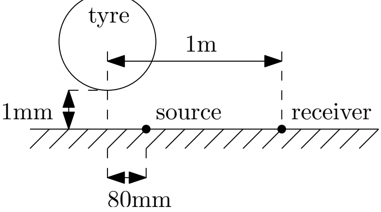
Energy norm^2 $= \langle \mathcal{V}\partial_t(\phi_h - \phi), \phi_h - \phi \rangle \sim h^\beta \simeq DOF(\Gamma)^{-\beta}$ (β-graded)

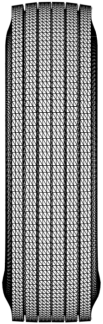
$\sim h \simeq DOF(\Gamma)^{-1}$ (uniform)

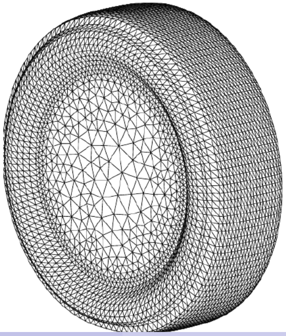


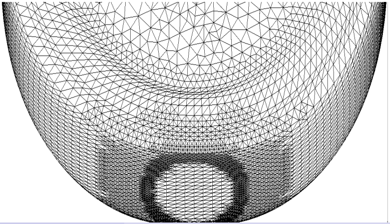
Traffic noise: Sound amplification in horn geometry

Support: "LeiStra3" programme of BASt, EPSRC IAA.



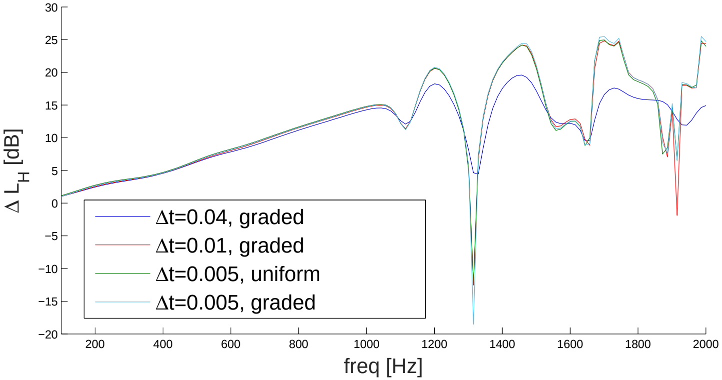






Traffic noise: Sound amplification in horn geometry

HG, Meyer, Özdemir, Stark, Stephan, Numer. Math. 2018.



Grading with various Δt compared to uniform tire mesh.

Screen problems: Convergence rates of p and hp-versions

Theorem (HG, Özdemir, Stark, Stephan, CMAME '19, Aimi, Di Credico, HG, Stephan, Numer. Math. '23)

Approximation error in energy norm on a quasi-uniform mesh ($\Delta t \leq h$):

Error of best approximation in $H^0_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) \lesssim \left(\frac{h}{p^2}\right)^{\frac{1}{2}-\varepsilon} + \left(\frac{h}{p}\right)^{\frac{1}{2}+\eta}$

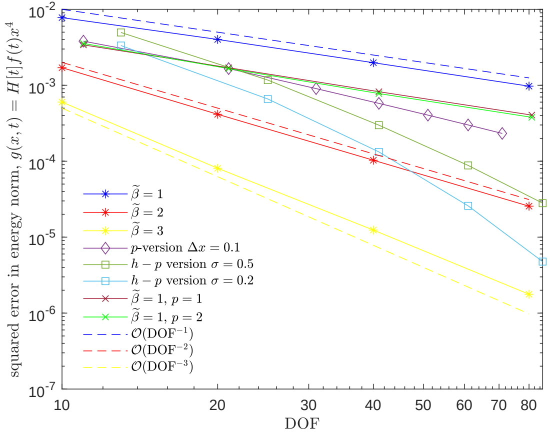
Here η depends on the regularity of rhs.

Screen problems: Convergence rates of p and hp-versions

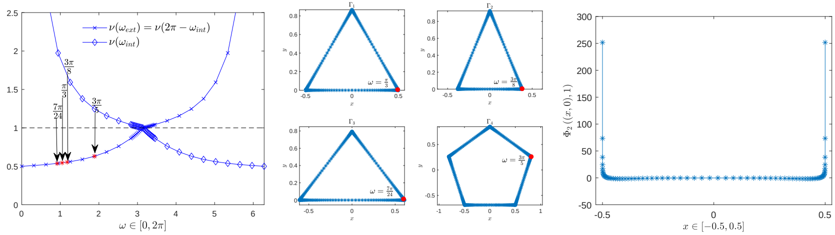
as above: $\mathcal{V}\phi(t,x) = g(x,t)(1,1)^T$ on $[-\frac{1}{2}, \frac{1}{2}] \times \{0\}$.

Fix mesh with $h = 0.1$, increase p.

hp-version on geometrically graded mesh:
$\sigma \in (0, 1/2]$, N intervalls in $[-\frac{1}{2}]$: $x_0 = -\frac{1}{2}$, $x_k = \frac{1}{2}\left(\sigma^{N+1-k} - 1\right)$.

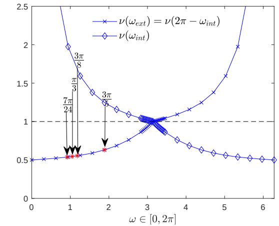


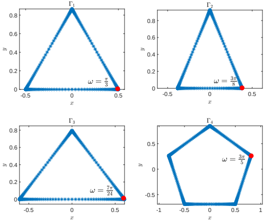
Polygonal screens: Singular exponents

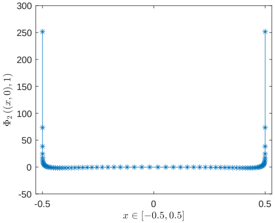


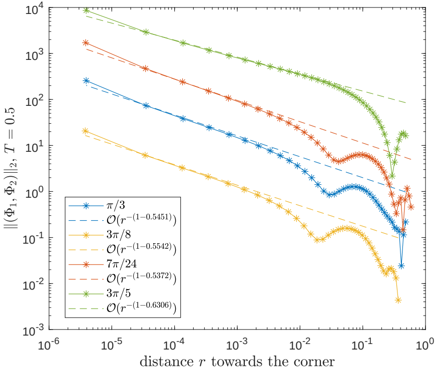
Polygonal meshes and expected exponent with dependence on ω_{int}

Polygonal screens: Singular exponents









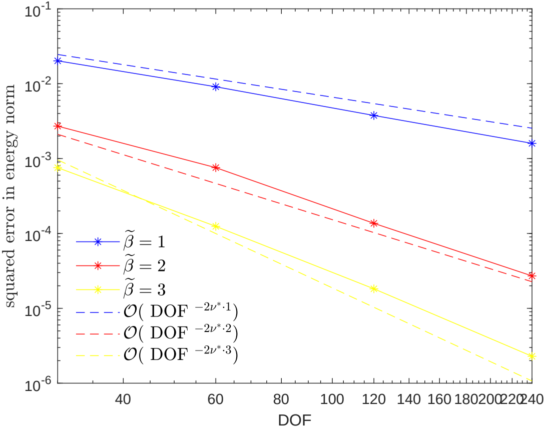
$$\mathcal{V}\Phi(t,x) = 100 f(t)|x|^n (0,1)^T.$$

Singular exponents ν^* depend on angle and material parameters

Polygonal screens: Convergence on β-graded meshes

$\mathcal{V}\Phi(t,x) = 100 f(t)|x|^n(0,1)^T$ on equilateral triangle. $\lambda = 2$, $\mu = \varrho = 1$.

Energy norm^2 = $\langle \mathcal{V}\partial_t(\phi_h - \phi), \phi_h - \phi\rangle \sim h^{2\nu^*\beta} \simeq DOF(\Gamma)^{-2\nu^*\beta}$



Extension to hypersingular integral equation

Theorem (Aimi, Di Credico, HG, Stephan, Numer. Math. '23)

Approximation error in energy norm on screen ($\Delta t \leq h$):

a) graded meshes:

Error of best approximation in $H^0_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) \lesssim h^{\min\{\frac{\beta}{2},\frac{3}{2}\}-\varepsilon}$.

b) p-version:

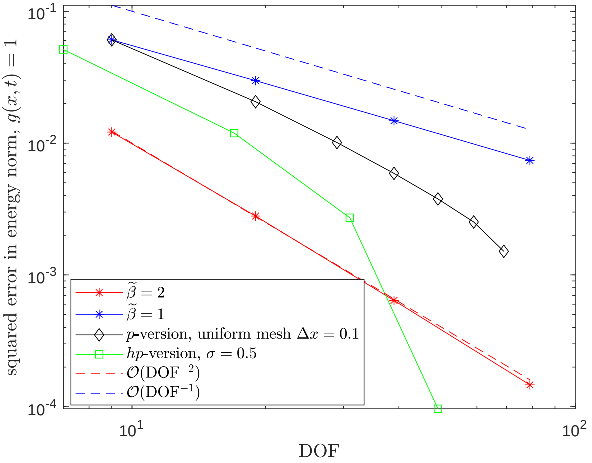
Error of best approximation in $H^0_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma)) \lesssim \left(\frac{h}{p^2}\right)^{\frac{1}{2}-\varepsilon} + \left(\frac{h}{p}\right)^{\frac{1}{2}+\eta}$

Here η depends on the regularity of rhs.

Extension to hypersingular integral equation

$\mathcal{W}\Psi(t,x) = (1,1)^T$ constant on $[-\frac{1}{2}, \frac{1}{2}] \times \{0\}$. $c_p = 2, 3$, $c_s = \varrho = 1$.

Convergence of β-graded h-version, p- and hp-versions



A posteriori error estimate for $\mathcal{V}\phi = f$

Theorem (HG, Özdemir, Stark, Stephan, Numer. Math. '20)

Let $\phi_{h,\Delta t} \in H^0_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))$ such that
$\mathcal{R} = \partial_t\ g - \mathcal{V}\partial_t \phi_{h,\Delta t} \in H^0_\sigma(\mathbb{R}^+, H^1(\Gamma)) \Longrightarrow$

$$\|\phi - \phi_{h,\Delta t}\|^2_{0,-\frac{1}{2}} \lesssim \sum_{i,\Delta} \max\{\Delta t, h_\Delta\}\ \|\mathcal{R}\|^2_{0,1,[t_i,t_{i+1})\times\Delta}$$

$$\max\{\Delta t, h\}\|\mathcal{R}\|^2_{0,1-\epsilon} \lesssim \|\phi - \phi_{h,\Delta t}\|^2_{2,-\frac{1}{2}}$$

The upper bound follows from a function-space argument (Carstensen '96), for large classes of meshes.

The lower bound holds on quasi-uniform meshes.

Extension to $\mathcal{W}$ with Aimi, Di Credico, Guardasoni (in preparation).

A posteriori error estimate for $\mathcal{V}\phi = f$

Theorem (HG, Özdemir, Stark, Stephan, Numer. Math. '20)

Let $\phi_{h,\Delta t} \in H^0_\sigma(\mathbb{R}^+, H^{-\frac{1}{2}}(\Gamma))$ such that
$\mathcal{R} = \partial_t \, g - \mathcal{V}\partial_t\phi_{h,\Delta t} \in H^0_\sigma(\mathbb{R}^+, H^1(\Gamma)) \Longrightarrow$

$$\|\phi - \phi_{h,\Delta t}\|^2_{0,-\frac{1}{2}} \lesssim \sum_{i,\Delta} \max\{\Delta t, h_\Delta\} \; \|\mathcal{R}\|^2_{0,1,[t_i,t_{i+1})\times\Delta}$$

$$\max\{\Delta t, h\}\|\mathcal{R}\|^2_{0,1-\epsilon} \lesssim \|\phi - \phi_{h,\Delta t}\|^2_{2,-\frac{1}{2}}$$

Residual error indicators:

$$\eta^2(\Delta, i) = \max\{\Delta t, h_\Delta\} \; \|\mathcal{R}\|^2_{0,1,[t_i,t_{i+1})\times\Delta}$$

(Simple) Proof of upper bound

$$\|\phi - \phi_{h,\Delta t}\|^2_{H^{0,-\frac{1}{2}}([0,T],\Gamma)}$$
$$\lesssim \int_0^T dt \int_0^t ds \int_\Gamma d\Gamma \; \mathcal{V}(\dot{\phi} - \dot{\phi}_{h,\Delta t})(\phi - \phi_{h,\Delta t})$$
$$= \int_0^T dt \int_0^t ds \int_\Gamma d\Gamma \; (\dot{f} - \mathcal{V}\dot{\phi}_{h,\Delta t})(\phi - \phi_{h,\Delta t})$$
$$\lesssim_T \; \|\mathcal{R}\|_{H^{0,\frac{1}{2}}([0,T],\Gamma)} \|\phi - \phi_{h,\Delta t}\|_{H^{0,-\frac{1}{2}}([0,T],\Gamma)} \; .$$

- interpolation inequality:

$$\|\mathcal{R}\|^2_{H^{0,\frac{1}{2}}} \lesssim \|\mathcal{R}\|_{H^{0,1}} \|\mathcal{R}\|_{L^2 L^2} \; .$$

- residual orthogonal: $\mathcal{R} \perp \psi_{h,\Delta t}$.
- interpolation $\rightsquigarrow$ $h, \Delta t$.

Lower bound for Dirichlet bvp in $\mathbb{R}^3$

dumb estimate:

$$\|\mathcal{R}\|_{H^{r-1,s+\frac{1}{2}}} = \|\mathcal{V}(\dot{\phi} - \dot{\phi}_{h,\Delta t})\|_{H^{r-1,s+\frac{1}{2}}} \lesssim \|\phi - \phi_{h,\Delta t}\|_{H^{r+1,s-\frac{1}{2}}} .$$

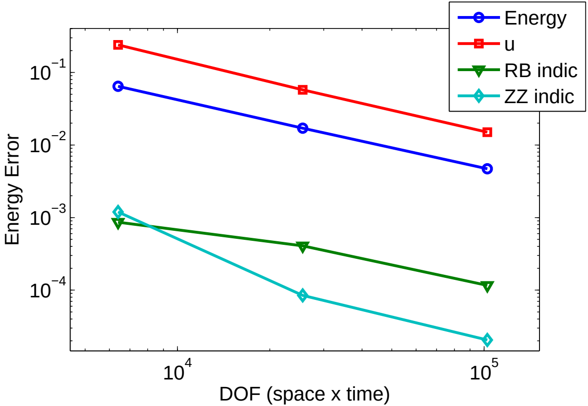
$$\|\mathcal{R}\|_{H^{0,1-\varepsilon}} \lesssim \|\phi - \phi_{h,\Delta t}\|_{H^{2,-\varepsilon}} .$$

Using the singular expansion for $\partial\Gamma \neq \emptyset$, we estimate $\|\phi - \phi_{h,\Delta t}\|_{H^{2,-\varepsilon}}$ on quasi-uniform meshes to obtain the efficiency of the estimator.

Error indicator $\sim$ energy error on uniform mesh

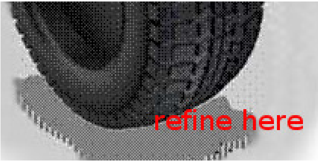
$f(t, \mathbf{x}) = \sin^5(t) z^2$ on $\Gamma = \{x, y, z \mid x^2 + y^2 + z^2 = 1\}$, $0 < t < 2.5$.

We consider residual and ZZ indicators on a uniform series of meshes. Compare to error in energy norm and sound pressure (with respect to benchmark).



- Efficient: Indicators scale like error in energy norm.

A first adaptive method: space-adaptivity

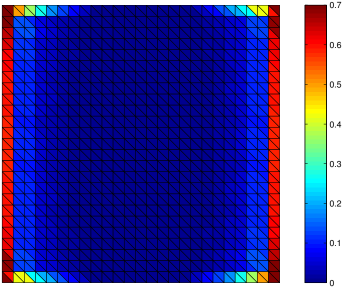


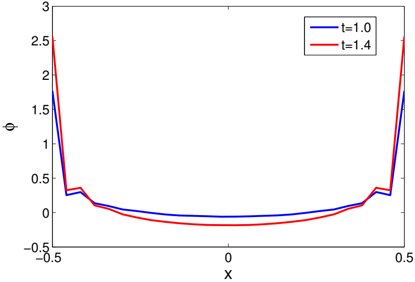
1. Start with coarse space-time grid: $(\Delta t)_i \simeq (\Delta x)_i \simeq h_0 \ \forall \Delta_i$
2. Solve discretisation of $\mathcal{V}\dot{\phi} = \dot{g}$.
3. Compute time-integrated error indicator $\eta(\Delta_i)$
4. $\sum_i \eta(\Delta_i) < \varepsilon \implies$ STOP
5. $\eta(\Delta_i) > \delta\eta_{max} \implies \Delta_i \to \Delta/4$, $(\Delta t)_i \to \frac{(\Delta t)_i}{2}$
6. GO TO 2.

Space-adaptive refinements on screen

$\mathcal{V}\phi = \sin^5(t)x^2$ on $\Gamma = [-0.5, 0.5]^2 \times \{z = 0\}$, $0 < t < 2.5$, $\Delta t = 0.1$.

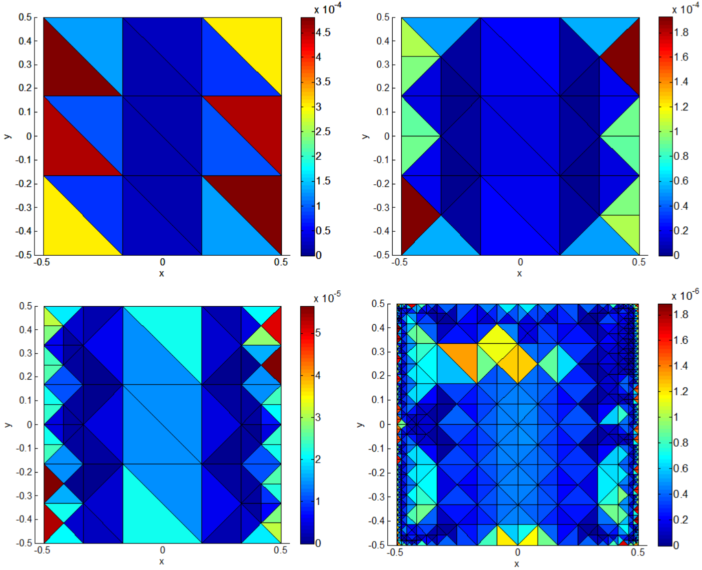
Compare residual indicators, energy, and sound pressure for uniform / adaptive mesh refinements.





- Uniform method: Density ϕ at $t = 1.0, 1.4$

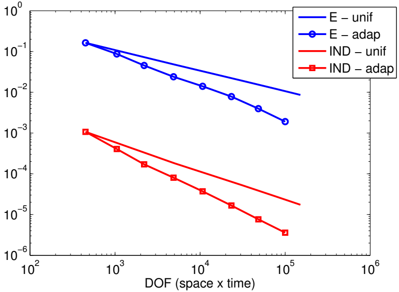
Space-adaptive refinements on screen: meshes

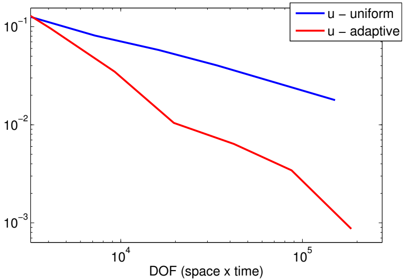


Space-adaptive refinements on screen: convergence

$\mathcal{V}\phi = \sin^5(t)x^2$ on $\Gamma = [-0.5, 0.5]^2 \times \{z = 0\}$, $0 < t < 2.5$, $\Delta t = 0.1$.

Compare residual indicators, energy, and sound pressure for uniform / adaptive mesh refinements.



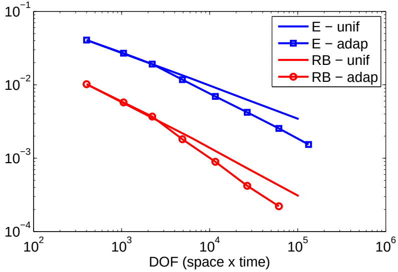


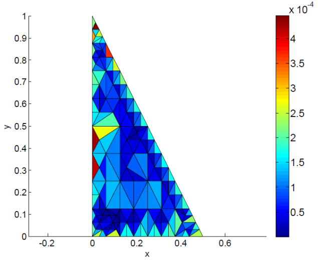
- Convergence rate 0.5 (uniform), 0.77 adaptive reproduces rates for time-independent BEM.

Space-adaptive refinements on triangular screen

$\mathcal{V}\phi = \sin^5(t)$ on $\Gamma = 30 - 60 - 90$ triangle, $0 < t < 2.5$.

Compare residual indicators, energy, and sound pressure for uniform / adaptive mesh refinements.



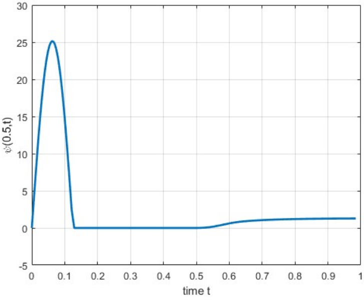


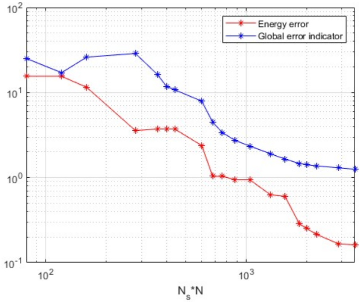
- Convergence rate 0.45 (uniform), 0.65 adaptive.

Adaptive time stepping

$$\mathcal{V}\phi = f,\ f(x,t) = \begin{cases} \sin^2(4\pi t) & \text{for} \quad 0 \le t \le \frac{1}{8} \\ 0 & \text{for} \quad t \ge \frac{1}{8} \end{cases}$$

on $\Gamma = (0,1) \times \{0\}$ slit, $0 < t < 1$, $h = \frac{1}{40}$.





Conclusions: Time domain BEM + mesh refinements

- Geometric singularities of wave equation and elastodynamics at edges/corners, resolved by time-independent meshes
- A posteriori analysis for elliptic BEM partly generalizes to space-time, (well-known) “loss” of time derivatives compared to elliptic case
- Static meshes optimal for geometric singularities $\rightsquigarrow$ space-only adaptive refinements sufficient
- Temporal singularities $\rightsquigarrow$ adaptive time stepping for convex scatterers

Outlook: Space-time adaptive mesh refinements.

Scattering off a knife's blade (screen problems), in h

(Convergence rates in energy norm)

- 0.5: h-version, uniform mesh
- 0.77: h-version, adaptive
- 1.0: p-version, uniform mesh
- $\beta/2$: h-version, β-graded mesh