Exercise sheet 7 — Useful facts

Exercise class week ?

Exercise 22:

a) Let H be a Hilbert space and $[\cdot,\cdot] : H \times H \to \mathbb{C}$ sesquilinear, such that $[h,h] \geq 0 \forall\, h \in H$. Show the Cauchy-Schwartz inequality $|[f,g]| \leq [f,f]^{1/2}[g,g]^{1/2}\ \forall f,g \in H$.

b) Let M be a subset of a Banach space X and $\operatorname{span} M := \{\sum a_i m_i :\ a_i \in \mathbb{C},\ m_i \in M\}$. Show that $\operatorname{span} M$ is dense in X if and only if every continuous linear functional on X, which vanishes on M is 0.

c) Let A be a Banach algebra, $\varphi : A \to \mathbb{C}$ linear and multiplicative, i.e. $\varphi(ab) = \varphi(a)\varphi(b)$. Show that φ is continuous and $||\varphi|| \leq 1$. If A is unital $\Rightarrow\ ||\varphi|| \in \{0,1\}$.

d) Let G be a locally compact abelian group (say $G = \mathbb{R}$) and $h \in L^1(G)$. Show that $||\underbrace{h * \cdots * h}_{n \text{ factors}}||_{L^1(G)}^{1/n} \xrightarrow{n\to\infty} \left|\left|\hat{h}\right|\right|_{L^\infty(\hat{G})}$.

Hint: In a (complex) Banach algebra $\lim\limits_{n\to\infty} ||a^n||^{1/n} = \sup|\sigma(a)|$.

Exercise 23:

a) Prove that $\mathbb{R}$ is a locally compact abelian group.

b) Prove that if $G_1, \ldots, G_n$ are locally compact abelian groups, so is $G_1 \times \cdots \times G_n$ (with the product topology and componentwise multiplication).

c) Prove that an infinite-dimensional Banach space is not locally compact.

Remark: A vector space with a Hausdorff topology, such that multiplication with scalars and vector addition are continuous is locally compact $\Leftrightarrow$ *it is homeomorphic to* $\mathbb{R}^n$ *for some* n.

Exercise 24:

Let G be a locally compact abelian group and m the Haar measure on G. Show:

a) If $\emptyset \neq V \subset G$ open $\Rightarrow m(V) > 0$.

b) $E \subset G$ measurable $\Rightarrow m(E) = m(-E)$.

Exercise 25:

Let $\gamma : \mathbb{R} \to \mathbb{C}$ continuous, $\gamma(x+y) = \gamma(x)\gamma(y)\ \forall x,y \in \mathbb{R}$ and γ not identically 0. By continuity $\exists \delta > 0 : \int\limits_0^\delta \gamma(x)\,\mathrm{d}x = \alpha \neq 0$. Show $\alpha\gamma(x) = \int\limits_x^{x+\delta} \gamma(t)\,\mathrm{d}t$. Conclude that γ is defferentiable and $\gamma'(x) = \gamma'(0)\gamma(x)$. Show that the only bounded solutions are $\gamma_y(x) := e^{iyx}$ for some $y \in \mathbb{R}$.

a) Endow the space of all such γ with the compact-open topology (as described in Rudin, 1.2.6). Show that $y \longleftrightarrow \gamma_y$ is a homeomorphism.

b) Show that all bounded γ as in a), with $\mathbb{R}$ replaced by $\mathbb{R}/\mathbb{Z}$ are of the form $y_n(x) = e^{2\pi i n x},\ n \in \mathbb{Z}$.

Exercise 26:
Let G be a locally compact abelian group. $\varphi : G \to \mathbb{C}$ is said to be positive definite, if $\sum\limits_{n,m=1}^{N} c_n \overline{c}_m \varphi(x_n - x_m) \geq 0$ for all $x_1, \dots, x_N \in G$ and $c_1, \dots c_N \in \mathbb{C}$.
Let $f \in L^2(G),\ \tilde{f} = \overline{f(-x)}$. Show that $f * \tilde{f}$ is positive definite and $\in C_0(G)$.

Hint: Use Young's inequality to deduce the last assertion.