Exercise sheet 5

Exercise class week 21

Exercise 16:

Let $\varphi_1(x) := c_1\left(1+|x|^2\right)^{-\frac{n+1}{2}}$, $\varphi_2(x) := c_2 e^{-\frac{|x|^2}{4}}$ with c_1, c_2 such that $\int \varphi_{1/2} = 1$. As in Exercise 13, write $\varphi_t(x) := t^{-n}\varphi(\frac{x}{t})$. Given $f \in L^p(\mathbb{R}^n)$, let $u_1(t,x) := f * (\varphi_1)_t(x)$ and $u_2(t,x) := f * (\varphi_2)_{\sqrt{t}}(x)$. Then $\partial_t^2 u_1(t,x) + \Delta_x u_1(t,x) = 0$ and $(\partial_t - \Delta)u_2(t,x) = 0$ for $t > 0,\ x \in \mathbb{R}^n$, and $\lim\limits_{t\to 0^+} u_{1/2}(t,x) = f(x)$ for almost every $x \in \mathbb{R}^n$.

Hint: This is a continuation of Exercise 13 on Sheet 4.

Exercise 17:

Let

$$S^m := \left\{a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) :\ \forall \alpha, \beta \in \mathbb{N}_0^n \exists C_{\alpha\beta} :\ \left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \le C_{\alpha\beta}\left(1+|\xi|\right)^{m-|\alpha|}\right\}$$

The operator $L_\xi := \left(1+|x|^2\right)^{-1}(1-\Delta_\xi)$ satisfies $L_\xi^N e^{ix\xi} = e^{ix\xi}\ \forall N.$

a) Check that for $a \in S^m$

$$\begin{aligned}\mathrm{op}(a)f(x) &:= \int \mathrm{d}\xi a(x,\xi)\hat{f}(\xi)e^{ix\xi}\\ &= \int \mathrm{d}\xi L_\xi^N\left[a(x,\xi)\hat{f}(\xi)\right]e^{ix\xi}\end{aligned}$$

defines an operator $\mathrm{op(a)} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$. If a is a polynomial in ξ, $\mathrm{op}(a)$ is a differential operator.

b) Let $A := \sum_\alpha a_\alpha \partial^\alpha$ be a differential operator on $\mathbb{R}^n$. Show that $Af(x) = \sum_\alpha a_\alpha(x)((\partial)^\alpha \delta * f)(x)$ and conclude that A is a singular integral operator. What is its kernel? Find an explicit distribution K on $\mathbb{R}^n \times \mathbb{R}^n$ such that $\langle g, Af\rangle = \langle K(x,y), f(x)g(y)\rangle$ for all test functions f, g.

Remark: Operators of the form $\mathrm{op}(a),\ a \in S^m$, are called "pseudodifferential operators". In DifFun 1, only x-independent symbols were considered.

Exercise 18:

Let A be a singular integral operator on $\mathbb{R}^n$ with kernel $k(x,y) > c\,|x-y|^{-n}$. Show that A does not extend to a bounded operator on $L^2(\mathbb{R}^n)$ and, in particular, is not a Calderon-Zygmund operator.

Hint: Let $Q = (-\frac{1}{4}, \frac{1}{4})^n$, $Q_y = Q + y$, $S_R = \bigcup\limits_{y\in\mathbb{Z}^n, |y|<R} Q_y$ and $f = \mathbb{1}_{S_R}$. Note that both of S_R and $\mathbb{R}^n \setminus S_R$ have a volume of at least $c_n R^n$ and show that $\frac{||Af||_2}{||f||_2}$ is unbounded as R goes to infinity, using that $||Af||_{L^2(\mathbb{R}^n)} \ge ||Af||_{L^2(\mathbb{R}^n\setminus S_R)}$.