Exercise sheet 2

Exercise class week 18

Applications of Paley-Wiener and Interpolation

Exercise 5:

Let $P(D) = \sum_{|a| \leq m} a_\alpha D^\alpha$, $a_\alpha \in \mathbb{C}$, wlog $a_{(m,0,\dots,0) \neq 0}$.

Show that the equation $P(D)u = f \in \mathcal{E}'(\mathbb{R}^n)$ has a solution $u \in \mathcal{E}'(\mathbb{R}^n)$ if and only if

$$\frac{\hat{f}(\xi)}{P(\xi)} \in \mathcal{O}(\mathbb{C}^n).$$

Hint: Show and use the following fact with $h(z) := \frac{\hat{f}(\xi_1 + z, \xi_2, \dots, \xi_n)}{P(\xi_1 + z, \xi_2, \dots, \xi_n)}$*:*

If $h(z) \in \mathcal{O}(\mathbb{C}),\ p(z) = p_m z^m + \cdots + p_1 z + p_0$

$$\Longrightarrow |p_m h(0)| \leq \max_{|z|=1} |h(z)p(z)|.$$

Exercise 6: "Schur's test"

Let (X, μ) and (Y, ν) be σ-finite measure spaces, $k : X \times Y \to \mathbb{C}$ measurable. Consider the integral operator

$$Kf(y) := \int_X k(x,y) f(x)\, \mathrm{d}\mu(x)$$

between suitable L^p-spaces, and let $1 \leq p_1,\ q_0 \leq \infty,\ p_0 = 1,\ q_1 = \infty,\ \frac{1}{p_i} + \frac{1}{p_i'} = 1.$

a) If $||k(x,y)||_{L^{q_0}(Y)} \leq B_0$ for all elements $x \in X \Rightarrow ||K||_{L^1(X) \to L^{q_0}(Y)} \leq B_0$

b) If $||k(x,y)||_{L^{p_1'}(X)} \leq B_1$ for all elements $y \in Y \Rightarrow ||K||_{L^{p_1}(X) \to L^\infty(Y)} \leq B_1$

c) Conclude that $K : L^{p_\theta}(X) \longrightarrow L^{q_\theta}(Y)$ is bounded and $||K||_{L^{p_\theta}(X) \to L^{q_\theta}(Y)} \leq B_0^{1-\theta} B_1^\theta$ for $\frac{1}{p_\theta} = 1 - \theta + \frac{\theta}{p_1}$ and $q_\theta = \frac{q_0}{1-\theta},\ 0 \leq \theta \leq 1.$

Hint: In a) and b) use Hölder's inequality and in c) the Riesz-Thorin theorem

Remark: According to c) $q_0 = p_1' = 1 \Rightarrow K : L^p(X) \to L^p(Y)$ *bounded for all* p*.*

Exercise 7: Young's inequality

Let $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L^p(\mathbb{R}^n),\ g \in L^q(\mathbb{R}^n)$.

Apply Schur's test to $k(x,y) = g(y - x)$ to conclude

$$||f * g||_r \leq ||f||_p\, ||g||_q$$

Remark: The argument applies to convolutions on any group with a biinvariant measure (and not just $\mathbb{R}^n$*).*

Exercise 8:
Let (X,μ) be a measure space, $f : X \to \mathbb{C}$ measurable. Define the distribution function $\lambda_f : [0,\infty) \to [0,\infty]$ by

$$\lambda_f(t) := \mu\left(\{x \in X : |f(x)| \geq t\}\right).$$

For $S \subset X$ let $\mathbb{1}_S(x) = \begin{cases} 1 \text{ for } x \in S \\ 0 \text{ for } x \notin S \end{cases}$.

a) Check that

$$|f(x)|^p = p \int_0^\infty \mathbb{1}_{\{|f|\geq t\}} t^p \frac{\mathrm{d}t}{t} \text{ and}$$

$$||f||_p^p = p \int_0^\infty \lambda_f(t) t^p \frac{\mathrm{d}t}{t} \qquad 1 \leq p < \infty,$$

$$||f||_\infty = \inf\{t \geq 0 : \ \lambda_f(t) = 0\}$$

b) Show Chebychev's inequality $\lambda_f(t) \leq t^{-p}\, ||f||_p^p$.

c) Show that for $1 \leq p \leq \infty$ $\exists \underline{c}_p, \overline{c}_p$:

$$\underline{c}_p\, ||f||_p \leq \left(\sum_{n\in\mathbb{Z}} \lambda_f(2^n) 2^{np}\right)^{\frac{1}{p}} = \left|\left|\left(2^n \lambda_f(2^n)^{\frac{1}{p}}\right)_{n\in\mathbb{Z}}\right|\right|_{l^p(\mathbb{Z})} \leq \overline{c}_p\, ||f||_p$$

d) Let $L^{p,\infty}(X) := \left\{ f : X \to \mathbb{C} \text{ measurable: } ||f||_{p,\infty} := \sup_{t>0} t\lambda_f(t)^{\frac{1}{p}} < \infty \right\}$. Use b) to show $L^p(X) \subseteq L^{p,\infty}(X)$. If $X = \mathbb{R}^n$ with the Lebesgue measure, show that $f(x) = |x|^{-\frac{n}{p}}$ belongs to $L^{p,\infty}(\mathbb{R}^n) \backslash L^p(\mathbb{R}^n),\ 1 \leq p < \infty$.