

Exercise sheet 1

Exercise class week 17

Fourier transform**Exercise 1:**

a) Compute the Fourier transform of $f(x) = e^{-\langle Ax, x \rangle}$ for $A \in \mathbb{C}^{n \times n}$, $\Re A$ positive definite.

b) Use this (with $n = 1$) to show that if the Fourier transform $\mathcal{F} : L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$ is continuous, then $1 \leq p \leq 2$ and $\frac{1}{q} + \frac{1}{p} = 1$.

Remark: The Riesz-Thorin theorem shows that in this case \mathcal{F} is actually continuous.

Exercise 2:

Prove the Paley-Wiener characterization of $\mathcal{F}\mathcal{E}'_{B_0(R)}(\mathbb{R}^n)$ stated in the lecture.

Hint: See the corresponding section from Hörmander.

Exercise 3:

a) Recall the proof of $\mathcal{F}L^1(\mathbb{R}^n) \subseteq C_0(\mathbb{R}^n)$ from DifFun.

b) Give two proofs that $\mathcal{F}L^1 \neq C_0$:

1. Wlog $n = 1$. Assume $f \in L^1(\mathbb{R})$, \hat{f} odd.

- Show that

$$\exists A : \left| \int_1^b \frac{\hat{f}(x)}{x} dx \right| \leq A, \forall b < \infty.$$

Hint: Fubini's theorem and

$$\left| \int_{\alpha}^{\beta} \frac{\sin(x)}{x} dx \right| \leq B, \forall 0 \leq |\alpha| \leq |\beta| < \infty.$$

- Show that $g(x) := \frac{\tanh(x)}{\log(1+|x|)} \in C_0(\mathbb{R}) \setminus \mathcal{F}L^1(\mathbb{R})$.

2. It is well-known that $(L^1(\mathbb{R}^n), *)$ is a Banach algebra, not a C^* -algebra.

Check that

- $\mathcal{F} : (L^1(\mathbb{R}^n), *) \rightarrow (C_0(\mathbb{R}^n), \cdot)$ is an injective $*$ -algebra homomorphism and
- it cannot be surjective.

Exercise 4:

a) Show that if $f : \mathbb{R} \rightarrow \mathbb{C}$ extends to a holomorphic function $\mathbb{R} + i(-\delta, \delta) \rightarrow \mathbb{C}$ for some $\delta > 0$, such that $\exists C_\delta > 0$ and $\exists \eta \in L^q(\mathbb{R})$ such that $\forall |y| < \delta: |f(x + iy)| \leq C_\delta \eta(x)$, then $\exists C > 0 \exists \varepsilon > 0: \left| \hat{f}(\xi) \right| \leq C e^{-\varepsilon|\xi|}$

b) Let $A := \{f \in C_0(\mathbb{R}) : \exists \delta > 0 : f \in \mathcal{O}(\mathbb{R} + i(-\delta, \delta)) \cup C(\mathbb{R} + i[-\delta, \delta])$
and $\forall N \in \mathbb{N} : \sup_{\substack{|y| < R, \\ x \in \mathbb{R}}} |f(x + iy)| e^{N|x|} < \infty\}$.

Show that:

$$\mathcal{FA} = \{f \in \mathcal{O}(\mathbb{C}) : \forall R > 0 \exists \varepsilon > 0 : \sup_{\substack{|y| < R, \\ x \in \mathbb{R}}} |f(x + iy)| e^{\varepsilon|x|} < \infty\}.$$

Hint: Use Cauchy's Integral Theorem.