Exercise sheet 1 — Fourier transform

Exercise class week 17

Exercise 1:

a) Compute the Fourier transform of $f(x) = e^{-\langle Ax,x\rangle}$ for $A \in \mathbb{C}^{n\times n}$, $\mathfrak{Re}\, A$ positive definite.

b) Use this (with $n = 1$) to show that if $\mathcal{F} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is continuous, then $1 \leq p \leq 2$ and $\frac{1}{q} + \frac{1}{p} = 1$.

Remark: The Riesz-Thorin theorem shows that in this case the Fourier transform $\mathcal{F}$ is actually continuous.

Exercise 2:

Prove the Paley-Wiener characterization of $\mathcal{F}\mathcal{E}'_{B_0(R)}(\mathbb{R}^n)$ stated in the lecture.

Hint: See the corresponding section from Hörmander.

Exercise 3:

a) Recall the proof of $\mathcal{F}L^1(\mathbb{R}^n) \subseteq C_0(\mathbb{R}^n)$ from DifFun.

b) Give two proofs that $\mathcal{F}L^1 \neq C_0$:

1. $\mathrm{Wlog}\ n = 1$. Assume $f \in L^1(\mathbb{R})$, $\hat{f}$ odd.
   - Show that
   $$\exists A : \left| \int_1^b \frac{\hat{f}(x)}{x}\, \mathrm{d}x \right| \leq A,\ \forall b < \infty.$$
   Hint: Fubini's theorem and
   $$\left| \int_\alpha^\beta \frac{\hat{f}(x)}{x}\, \mathrm{d}x \right| \leq B,\ \forall 0 \leq |\alpha| \leq |\beta| < \infty.$$
   - Show that $g(x) := \frac{\tanh(x)}{\log(1+|x|)} \in C_0(\mathbb{R}) \backslash \mathcal{F}L^1(\mathbb{R})$.
2. It is well-known that $\left(L^1(\mathbb{R}^n), *\right)$ is a Banach algebra, not a C^*-algebra. Check that
   - $\mathcal{F} : \left(L^1(\mathbb{R}^n), *\right) \longrightarrow (C_0(\mathbb{R}^n), \cdot)$ is an injective $*$-algebra homomorphism and
   - it cannot be surjective.

Exercise 4:

a) Show that if $f : \mathbb{R} \longrightarrow \mathbb{C}$ extends to a holomorphic function $\mathbb{R} + i(-\delta, \delta) \longrightarrow \mathbb{C}$ for some $\delta > 0$, such that $\exists C_\delta > 0$ and $\exists \eta \in L^q(\mathbb{R})$ such that $\forall\, |y| < \delta$: $|f(x+iy)| \leq C_\delta \eta(x)$, then $\exists C > 0\; \exists \varepsilon > 0$: $\left|\hat{f}(\xi)\right| \leq Ce^{-\varepsilon|\xi|}$

b) Let $A := \big\{ f \in C_0(\mathbb{R}) : \exists \delta > 0 :\; f \in \mathcal{O}(\mathbb{R} + i\,(-\delta, \delta)) \cup C\,(\mathbb{R} + i[-\delta, \delta])$ and $\forall N \in \mathbb{N} : \sup\limits_{\substack{|y|<R,\\ x\in\mathbb{R}}} |f(x+iy)|\, e^{N|x|} < \infty \big\}.$

Show that:

$$\mathcal{F}A = \Big\{ f \in \mathcal{O}(\mathbb{C}) :\; \forall R > 0\; \exists \varepsilon > 0 : \sup_{\substack{|y|<R,\\ x\in\mathbb{R}}} |f(x+iy)|\, e^{\varepsilon|x|} < \infty \Big\}.$$

Hint: Use Cauchy's Integral Theorem.