

Exercises 1 - Fourier transform

① a) Compute the Fourier transform of $f(x) = e^{-Ax}$

for $A \in \mathbb{C}^{n \times n}$, $\operatorname{Re} A$ positive definite.

b) Use this (with $n=1$) to show that if $\mathcal{F}: L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$ is continuous, then $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Remark: The Riesz-Thorin theorem shows that in this case \mathcal{F} is actually continuous.

② Prove the Paley-Wiener characterization of $\mathcal{F} L^1_{\text{loc}}(\mathbb{R}^n)$
 started in the lecture (see the section from Hörmander).

cont. vanishing at ∞

a) Recall the proof of $\mathcal{F} L^1(\mathbb{R}^n) \subseteq C_0(\mathbb{R}^n)$.

b) Give 2 proofs that $\mathcal{F} L^1 \neq C_0$:

i) Wlog $n=1$. Assume $f \in L^1(\mathbb{R})$ and f odd.

Show that $\left| \int_{-\infty}^b \frac{f(x)}{x} dx \right| \leq A \quad \forall b < 0$.

(Hint: $\left| \int_{-\infty}^b \frac{\sin(x)}{x} dx \right| \leq B \quad \forall x \in \mathbb{R} \setminus \{0\} \subset \mathbb{R}$)

• Show that $g(x) := \frac{\tanh(x)}{\log(1+|x|)} \in C_0(\mathbb{R}) \setminus \mathcal{F} L^1(\mathbb{R})$.

ii) It is well-known that $(L^1(\mathbb{R}^n), *)$ is a

Banach algebra, not a C^* -algebra.

Check that $\tau: \mathcal{F}: (L^1(\mathbb{R}^n), *) \rightarrow (C_0(\mathbb{R}^n), \circ)$

is an injective $*$ -algebra homomorphism.

Remark: $\mathcal{J}L'(\mathbb{R}^n)$ is often denoted by $A(\mathbb{R})$ and, for those following Hörmander's courses, associated with the $A(\mathbb{C})$ he defines in terms of Schur multipliers.

- ④ a) Show that if $f: \mathbb{R} \rightarrow \mathbb{C}$ extends to a holomorphic function on the strip $\mathbb{R} + i(-\delta, \delta)$ for some $\delta > 0$ s.t.

$$\forall y \in \mathbb{R} : |f(x+iy)| \leq C_0 g(x) \in L^1(\mathbb{R})$$

$$\Rightarrow \exists C \exists \varepsilon > 0 \quad |\hat{f}(\xi)| \leq C e^{-\varepsilon |\xi|} \quad \forall \xi \in \mathbb{R},$$

- b) Let $A := \{ f \in C_c(\mathbb{R}) : \exists \delta > 0 : f \in \mathcal{O}(\mathbb{R} + i(-\delta, \delta)) \cap C(\mathbb{R} + i[-\delta, \delta]) \}$
 and $\forall N \in \mathbb{N} : \sup_{\substack{|y| < \delta \\ x \in \mathbb{R}}} |f(x+iy)| \cdot e^{N|x|} < \infty \}$

Show $\mathcal{J}A = \{ f \in \mathcal{O}(\mathbb{C}) : \forall R > 0 \exists \varepsilon > 0 :$

$$\sup_{\substack{|y| < R \\ x \in \mathbb{R}}} |f(x+iy)| \cdot e^{\varepsilon |x|} < \infty \},$$

Hint: Use Cauchy's integral theorem.

$$C^0 = \mathcal{E} \ni S \ni D = C_0^0$$

$$\mathcal{E}' \subseteq S' \subseteq D'$$

1st Exercises for Diffuse

C₀

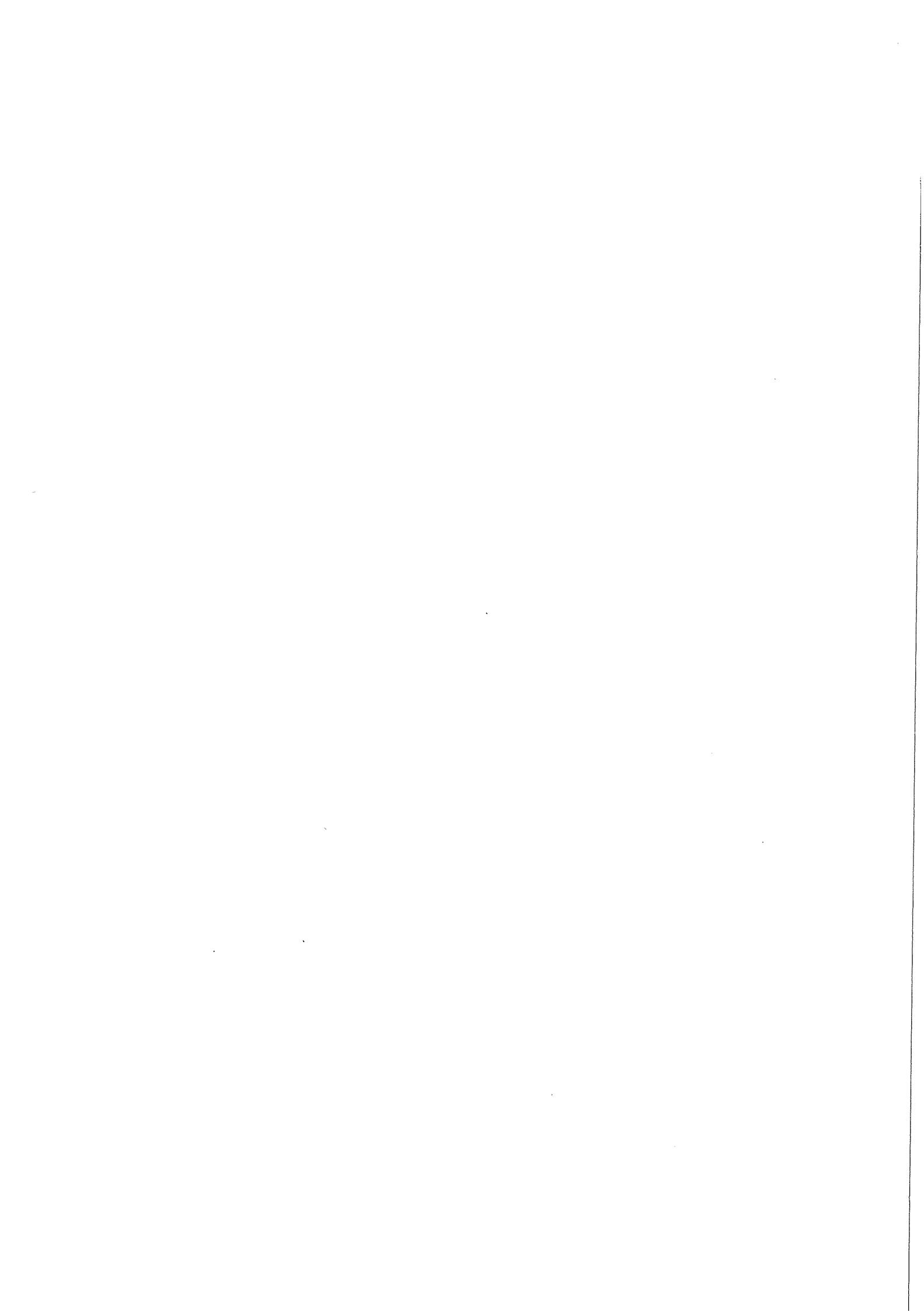
- ① a) Compute $\mathcal{F} e^{-\langle Ax, x \rangle} = \dots$ for $A \in \mathbb{C}^{n \times n}$, $\text{Re}(A) \subseteq (0, \infty)$.
 (see e.g. Hörmander)
- b) Use this to show that if $\mathcal{F}: L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$ continuous
 $\Rightarrow \frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p \leq 2$.

Remark: In class we are going to see that " \iff " holds.

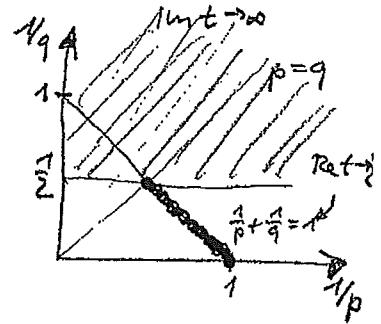
- ② a) Show that for $u \in \mathcal{E}'(\mathbb{R}^n)$, $\mathcal{F}u$ extends to an entire function $\tilde{\mathcal{F}}u \in \mathcal{O}(\mathbb{C}^n)$ and $\tilde{\mathcal{F}}u = (z \mapsto \sum_{k=0}^{\infty} \frac{1}{k!} u(e^{-i\langle z, k \rangle}))$.
 (for $z \in \mathbb{C}^n$)
- b) Extend the Paley-Wiener theorem from class to show that
 $u \in \mathcal{E}'(\mathbb{R}^n)$ w/ $\text{supp } u \subset \Omega \iff \dots$ (see Hörmander I)
 Section 7.3
- ③ a) Show that $\mathcal{F}: L^1(\mathbb{R}^n) \rightarrow C_0(\mathbb{R}^n)$, (Riemann-Lebesgue)
 Hörn: GG: $\mathcal{F}: S \rightarrow S$, $S \subseteq L^1$ please $\xrightarrow{\text{continuous}}$ $L^1 \subseteq \mathcal{F}(S)$. $f \in S \Rightarrow \exists k \in \mathbb{N}_0$ $\forall x \in \mathbb{R}^n$
- b) Show that not every function $u \in C_0(\mathbb{R}^n)$ is $= f_\theta$ for some $\theta \in L^1$.
 b1) (Stein/Weiss, Chapter 1) is odd, continuous, $\sim \frac{1}{\log(x)}$
 $\Rightarrow \not\exists \theta$. p. 38
 b2) (C^* -theory) It is well-known that $(L^1(\mathbb{R}^n), *)$ is a Banach algebra, but not a C^* -algebra.
 Check that $\mathcal{F}: L^1 \rightarrow C_0$ is injective (it is so on S' !)

If \mathcal{F} was surjective, it would be a \star -isomorphism
 and L^1 would be a C^* -algebra. p. 38

Remark: $\mathcal{F} L^1(\mathbb{R})$ is often denoted by $A(\mathbb{R})$ dual, for those following Haagerup's courses, coincides w/ the A he defined in terms of Schur multipliers.



$$\begin{aligned}
 26) \quad & \frac{\|\mathcal{F} e^{-t|x|^2}\|_q}{\|e^{-t|x|^2}\|_p} = \frac{\|e^{-t|x|^2/4t}/\sqrt{2\pi}\|_q}{\|e^{-t|x|^2}\|_p} \\
 & = \frac{\frac{1}{\sqrt{2\pi t}} \left(\int_{-\infty}^{\infty} |e^{-q|x|^2/4t}| dx \right)^{1/q}}{\left(\int_{-\infty}^{\infty} |e^{-pt|x|^2}| dx \right)^{1/p}} = \frac{\frac{1}{\sqrt{2\pi t}} \left(\frac{\sqrt{\pi}}{\sqrt{p} \sqrt{4t}} \right)^{1/q}}{\sqrt{\frac{\pi}{p \cdot 4t}}} \\
 & = \frac{\frac{1}{\sqrt{2\pi t}} \left(\frac{4\pi}{q \sqrt{4t}} \right)^{1/q}}{\left(\frac{\pi}{p \cdot 4t} \right)^{1/p}}
 \end{aligned}$$



Case 1: $t_{\text{real}} = \left(\frac{4\pi}{q}\right)^{1/q} \left(\frac{\pi}{p}\right)^{1/p} \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2p}$

$$\begin{aligned}
 -\frac{1}{2} + \frac{1}{2q} + \frac{1}{2p} &= 0 \\
 \Rightarrow \frac{1}{p} + \frac{1}{q} &= 1
 \end{aligned}$$

Case 2: $t = \alpha + i\beta \rightsquigarrow \frac{1}{t} = \frac{\alpha - i\beta}{\alpha^2 + \beta^2}$

$$\begin{aligned}
 &= C_{\alpha, \beta, q} \frac{1}{\sqrt{\alpha^2 + \beta^2}} \left(\frac{\alpha^2 + \beta^2}{\alpha}\right)^{1/q} \left(\frac{1}{\alpha}\right)^{1/p} \\
 &\xrightarrow{\beta \rightarrow \infty} C_{\alpha, \beta, q, \infty} \beta^{1/2} \beta^{1/q} \text{ if } q < 2 \Leftrightarrow p > 2
 \end{aligned}$$

V6: $\mathcal{F}: L^2 \xrightarrow{\sim} L^2$ isometrisch ($\|f\|_{L^2} = \|F\|_{L^2 \rightarrow L^2} = 1$)

$$\mathcal{F}: L^1 \rightarrow L^\infty : \left\| \left(\frac{1}{\sqrt{2\pi}} \right)^n \int f(x) e^{-ix\cdot} \right\|_\infty \leq \left(\frac{1}{\sqrt{2\pi}} \right)^n \int |f(x)| dx = \left(\frac{1}{\sqrt{2\pi}} \right)^n \|f\|_{L^1}$$

Interpolation

$$\mathcal{F}: L^p \rightarrow L^q \quad \frac{1}{p} + \frac{1}{q} = 1$$

$$\|\mathcal{F}\|_{L^p \rightarrow L^q} \leq (\dots) \|\mathcal{F}\|_{L^1 \rightarrow L^\infty}^{4 - \frac{2}{q}} \|\mathcal{F}\|_{L^2 \rightarrow L^2}^{\frac{2}{q}}$$

$$\text{Für } \frac{2}{p} = \frac{2}{q} \quad \text{d.h. } 2(1 - \frac{1}{p}) = \frac{2}{q} \quad \frac{1}{p} = \frac{1}{2} - \frac{1}{q}$$

$$1 - \Theta = 1 - \frac{2}{q} = 1 - \frac{1}{2} - \frac{1}{q}$$



can find $\mu \in M$ and a null sequence $\{\varepsilon_n\}$ such that $u_{\varepsilon_n} \rightarrow \mu$ as $k \rightarrow \infty$ in this topology. That is, for each $\varphi \in C_0$,

$$(i) \quad \lim_{k \rightarrow \infty} \int_{E_n} \varphi(x) u_{\varepsilon_k}(x) dx = \int_{E_n} \varphi(x) d\mu(x).$$

We now claim that μ , considered as a distribution, equals u .

We must show, therefore, that $u(\psi) = \int_{E_n} \psi(x) d\mu(x)$ for all $\psi \in \mathcal{S}$. Let $\psi_\varepsilon(x) = \int_{E_n} \psi(x-t) W(t, \varepsilon) dt$. Then, for all n -tuples $\alpha = (\alpha_1, \dots, \alpha_n)$ of nonnegative integers, we have $(D^\alpha \psi_\varepsilon)(x) = \int_{E_n} (D^\alpha \psi)(x-t) W(t, \varepsilon) dt$. It follows from Theorem 1.18 that $(D^\alpha \psi_\varepsilon)(x)$ converges to $(D^\alpha \psi)(x)$ uniformly in x . Thus, $\psi_\varepsilon \rightarrow \psi$ as $\varepsilon \rightarrow 0$ (in \mathcal{S}) and this implies that $u(\psi_\varepsilon) \rightarrow u(\psi)$. But, since $W(\cdot, \varepsilon) = \tilde{W}(\cdot, \varepsilon)$,

$$u(\psi_\varepsilon) = u(W(\cdot, \varepsilon) * \psi) = (u * W(\cdot, \varepsilon))(\psi) = \int_{E_n} \psi(x) u_\varepsilon(x) dx.$$

Thus, putting $\varepsilon = \varepsilon_n$, letting $k \rightarrow \infty$ and applying (i) with $\varphi = \psi$ we obtain the desired equality $u(\psi) = \int_{E_n} \psi(x) d\mu(x)$. The remainder of the theorem now follows easily.

We mentioned before that a simple characterization of the general (L^p, L^q) spaces is not known (and probably does not exist). We do have, however, the following general result of duality:

THEOREM 3.20. *If $1 \leq p, q \leq \infty$, $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$ then $(L^p, L^q) = (L^{q'}, L^{p'})$.*

PROOF. By the F. Riesz representation theorem we can identify the spaces $L^{q'}$ and $L^{p'}$ with the duals of L^p and L^q (this is true if $p, q < \infty$. If one of p or q is ∞ , a slight modification of the argument we give will prove the theorem in this case as well). Let $u \in (L^p, L^q)$ and $B: L^p \rightarrow L^q$ denote the unique bounded linear extension to L^p of the mapping $\varphi \mapsto u * \varphi$ ($\varphi \in \mathcal{S}$). If $B^*: L^{q'} \rightarrow L^{p'}$ denotes the adjoint to B we then have

$$\int_{E_n} (B\varphi)\psi dx = \int_{E_n} \varphi(B^*\psi) dx$$

for all $\varphi, \psi \in \mathcal{S}$. But, in terms of distributions, this is equivalent to the equality $(u * \varphi)(\psi) = (B^*\psi)(\varphi)$. That is, $(B^*\psi)(\varphi) = u((\tilde{\varphi} * \psi)')$ $= \tilde{u}(\tilde{\varphi} * \psi) = (\tilde{u} * \psi)(\varphi)$ for all $\varphi, \psi \in \mathcal{S}$. Thus, $B^*\psi = \tilde{u} * \psi$ for all $\psi \in \mathcal{S}$. Since $\|B^*\psi\|_{p'} \leq \|B^*\| \||\psi|_{q'}\|_{q'}$ we have $\tilde{u} \in (L^{q'}, L^{p'})$. Since $u \in (L^q, L^p)$ if and only if $\tilde{u} \in (L^{q'}, L^{p'})$ this shows that $(L^p, L^q) \subset (L^{q'}, L^{p'})$. The opposite inclusion relation follows by interchanging the roles of (p, q) and (q', p') .

4. Further Results

4.1. We remarked, after the proof of Theorem 1.2, that belonging to C_0 is not a sufficient condition for being the Fourier transform of an integrable function. Suppose, for simplicity, that $n = 1$; then one way of seeing this is the following: We first observe that if f is such a Fourier transform and, moreover, f is an odd function then, for $1 < b < \infty$, $|\int_1^b [f(x)]x dx| \leq A$, where A does not depend on b . This is a consequence of the well-known corresponding property for the integrals of $\sin x/x$; that is $|\int_a^\beta (\sin x/x) dx| \leq B < \infty$ independently of a, β ($0 \leq |a| < |\beta| < \infty$). Indeed, because of the oddness of f we have $f(x) = -i \int_{-\infty}^{\infty} f(t) \sin(2\pi x t) dt$. Thus, using Fubini's theorem one easily deduces that $|\int_1^b [f(x)]x dx| \leq A$. To give an example, therefore, of a function in C_0 that is not a Fourier transform of an integrable function all we need to do is exhibit a continuous odd function g vanishing at ∞ and such that $\int_1^b (g(x)/x) dx$ is not bounded for $b \rightarrow \infty$. But this is clearly the case if, for example, $g(x) = 1/\log x$ for x large.

4.2. For $g \in L^1(E_n)$ fixed let us consider the operator B mapping $f \in L^p(E_n)$ to $f * g$. From Theorem 1.3 we see that B is a bounded linear transformation from $L^p(E_n)$ into $L^p(E_n)$ whose norm, $\|B\|^{(p)}$, is majorized by $\|g\|_1$. When $p = 1$ it is easy to show that $\|B\|^{(1)} = \|g\|_1$ (for, by Theorem 1.18 we see that $f_\varepsilon(t) = P(t, \varepsilon)$ is a family of functions of L^1 norm equal to 1 such that $f_\varepsilon * g \rightarrow g$ in the L^1 norm as $\varepsilon \rightarrow 0$). We have seen (Theorem 3.18) that $\|B\|^{(2)} = \|\hat{g}\|_\infty$. It is natural to ask, therefore, how the norm $\|B\|^{(p)}$ can be expressed in terms of g for $1 \leq p \leq \infty$. A satisfactory answer is not known in general. If $g \geq 0$, however, then $\|g\|_1 = |\hat{g}(0)| \leq \|\hat{g}\|_\infty \leq \|g\|_1$; thus, $\|B\|^{(2)} = \|g\|_1$. From this, Theorem 3.20 and a result (the M. Riesz convexity theorem) in the theory of interpolation of operators (that we shall develop in Chapter V) it follows that $\|B\|^{(p)} = \|g\|_1$ for $1 \leq p \leq \infty$, provided $g \geq 0$.

4.3. Theorem 1.3 has the following extension: if $f \in L^p(E_n)$ and $g \in L^r(E_n)$, $1 \leq p, r$ and $(1/p) + (1/r) \geq 1$ then $h = f * g \in L^q(E_n)$, where $(1/q) = (1/p) + (1/r) - 1$, and

$$\|h\|_q \leq \|f\|_p \|g\|_r.$$

For a direct proof of this result (often called Young's inequality) see Zygmund [1], Chapter II, page 37. We shall show, in Chapter V, that this inequality also follows immediately from the M. Riesz convexity theorem. 4.4. We can obtain more insight into the meaning of the derivatives in the L^p norms by observing that (a) when the dimension n is 1, then $f \in L^p(E_1)$ has a derivative in the L^p norm if and only if f is equal almost

b2)

$$\mathcal{F} : (L^1(\mathbb{R}), *) \xrightarrow[\text{hom., resp.}]{\text{alg.}} (\text{Conv.}, *)$$

$$(f * g)^\wedge = \hat{f} \cdot \hat{g}$$

compatible w/ \dagger (star op.)

If surjective \Rightarrow isomorphism

$$\Rightarrow (L^1(\mathbb{R}), *) \subset \mathbb{K}\text{-algebra}$$

Elie Cartan

$$\begin{aligned} \partial_z \langle u, \varphi_z \rangle &= \lim_{z \rightarrow z_0} \langle u, \varphi_z \rangle - \langle u, \varphi_{z_0} \rangle \\ &= \lim_{z \rightarrow z_0} \frac{\varphi_z - \varphi_{z_0}}{z - z_0} \langle u, \varphi_{z_0} \rangle \\ &\rightarrow \partial_z \langle u, \varphi_z \rangle \text{ exists} \end{aligned}$$

\Rightarrow holomorphic

~~$\mathcal{F}(W_A)$~~

5.8 Satz (Holomorphe Abhängigkeit des Integrals von einem komplexen Parameter). Es sei $G \subset \mathbb{C}$ offen, und $f : G \times X \rightarrow \mathbb{C}$ habe folgende Eigenschaften:

- a) $f(z, \cdot) \in \mathcal{L}^1$ für alle $z \in G$.
- b) Für alle $x \in X$ ist $f(\cdot, x) : G \rightarrow \mathbb{C}$ holomorph.
- c) Zu jeder kompakten Kreisscheibe $K \subset G$ gibt es eine integrierbare Funktion $g_K \in \mathcal{M}^+$, so daß für alle $z \in K$ gilt: $|f(z, \cdot)| \leq g_K$ μ-f.ü.

Dann ist die Funktion $F : G \rightarrow \mathbb{C}$,

$$F(z) := \int_X f(z, x) d\mu(x) \quad (z \in G)$$

Beweis. Es sei $(t_n)_{n \geq 1}$ eine Folge von Punkten aus U mit $\lim_{n \rightarrow \infty} t_n = t_0$. Dann ergibt eine Anwendung des Satzes von der majorisierten Konvergenz auf die Folge der Funktionen $f_n := f(t_n, \cdot)$ ($n \in \mathbb{N}$) sogleich die Behauptung. \square

5.7 Satz (Differentiation unter dem Integralzeichen). Es seien $I \subset \mathbb{R}$ ein Intervall, $t_0 \in I$, und $f : I \times X \rightarrow \mathbb{K}$ habe folgende Eigenschaften:

- Für alle $t \in I$ gilt $f(t, \cdot) \in \mathcal{L}^1$.
- Die partielle Ableitung $\frac{\partial f}{\partial t}(t_0, x)$ existiert für alle $x \in X$.
- Es gibt eine Umgebung U von t_0 und eine integrierbare Funktion $g \in \mathcal{M}^+$, so daß für alle $t \in U \cap I$, $t \neq t_0$ gilt

$$\left| \frac{f(t, x) - f(t_0, x)}{t - t_0} \right| \leq g(x) \text{ μ-f.ü.}^9$$

Dann ist die Funktion $F : I \rightarrow \mathbb{K}$,

$$F(t) := \int_X f(t, x) d\mu(x) \quad (t \in I)$$

im Punkt t_0 (ggf. einseitig) differenzierbar, $\frac{\partial f}{\partial t}(t_0, \cdot)$ ist integrierbar, und es gilt

$$F'(t_0) = \int_X \frac{\partial f}{\partial t}(t_0, x) d\mu(x).$$

Zusatz. Die Aussage dieses Satzes bleibt bestehen, wenn man die Voraussetzung b), c) ersetzt durch:

- b*) Es gibt ein $\delta > 0$, so daß die partielle Ableitung $\frac{\partial f}{\partial t}(t, x)$ ($x \in X$) für alle $t \in U := [t_0 - \delta, t_0 + \delta] \cap I$ existiert.
- c*) Es gibt eine integrierbare Funktion $g \in \mathcal{M}^+$, so daß für alle $t \in U$ und $x \in X$ gilt:

$$\left| \frac{\partial f}{\partial t}(t, x) \right| \leq g(x).$$

Beweis. Es sei $(t_n)_{n \geq 1}$ eine Folge in U mit $\lim_{n \rightarrow \infty} t_n = t_0$, $t_n \neq t_0$ für alle $n \in \mathbb{N}$. Eine Anwendung des Satzes von der majorisierten Konvergenz auf $f_n := (f(t_n, \cdot) - f(t_0, \cdot)) / (t_n - t_0)$ ($n \in \mathbb{N}$) liefert unter den Voraussetzungen a)-c) sogleich die Behauptung. – Zum Beweis des Zusatzes wenden wir den Mittelwertsatz der Differentialrechnung an und erhalten zu jedem $n \in \mathbb{N}$ und $x \in X$ ein (i.a. von x abhängiges) $t'_n \in U$, so daß

$$|f_n(x)| = \left| \frac{\partial f}{\partial t}(t'_n, x) \right| \leq g(x) \quad (x \in X).$$

Wieder ergibt der Satz von der majorisierten Konvergenz das Gewünschte. \square

⁹Die Vereinigung der Ausnahme-Nullmengen braucht keine Nullmenge zu sein.

Beweis. Es seien $a \in G$ und $r > 0$ so klein, daß $K := \overline{K_{2r}(a)} \subset G$. Für alle $z \in K_{2r}(a)$ ist dann nach der Cauchyschen Integralformel für Kreisscheiben

$$f(z, x) = \frac{1}{2\pi i} \int_{\partial K_{2r}(a)} \frac{f(\zeta, x)}{\zeta - z} d\zeta,$$

wobei das Kurvenintegral im Riemannschen Sinn zu verstehen ist (s. Grundwissen-Band *Funktionentheorie I* von R. REMMERT). Für alle $z, w \in K_r(a)$, $z \neq w$ ist also

$$\frac{F(z) - F(w)}{z - w} = \int_X \frac{1}{2\pi i} \int_{\partial K_{2r}(a)} \frac{f(\zeta, x)}{(\zeta - z)(\zeta - w)} d\zeta d\mu(x).$$

Es sei nun $(w_k)_{k \geq 1}$ eine Folge in $K_r(a)$ mit $\lim_{k \rightarrow \infty} w_k = z$, $w_k \neq z$ für alle k und

$$\varphi_k(z, x) := \frac{1}{2\pi i} \int_{\partial K_{2r}(a)} \frac{f(\zeta, x)}{(\zeta - z)(\zeta - w_k)} d\zeta.$$

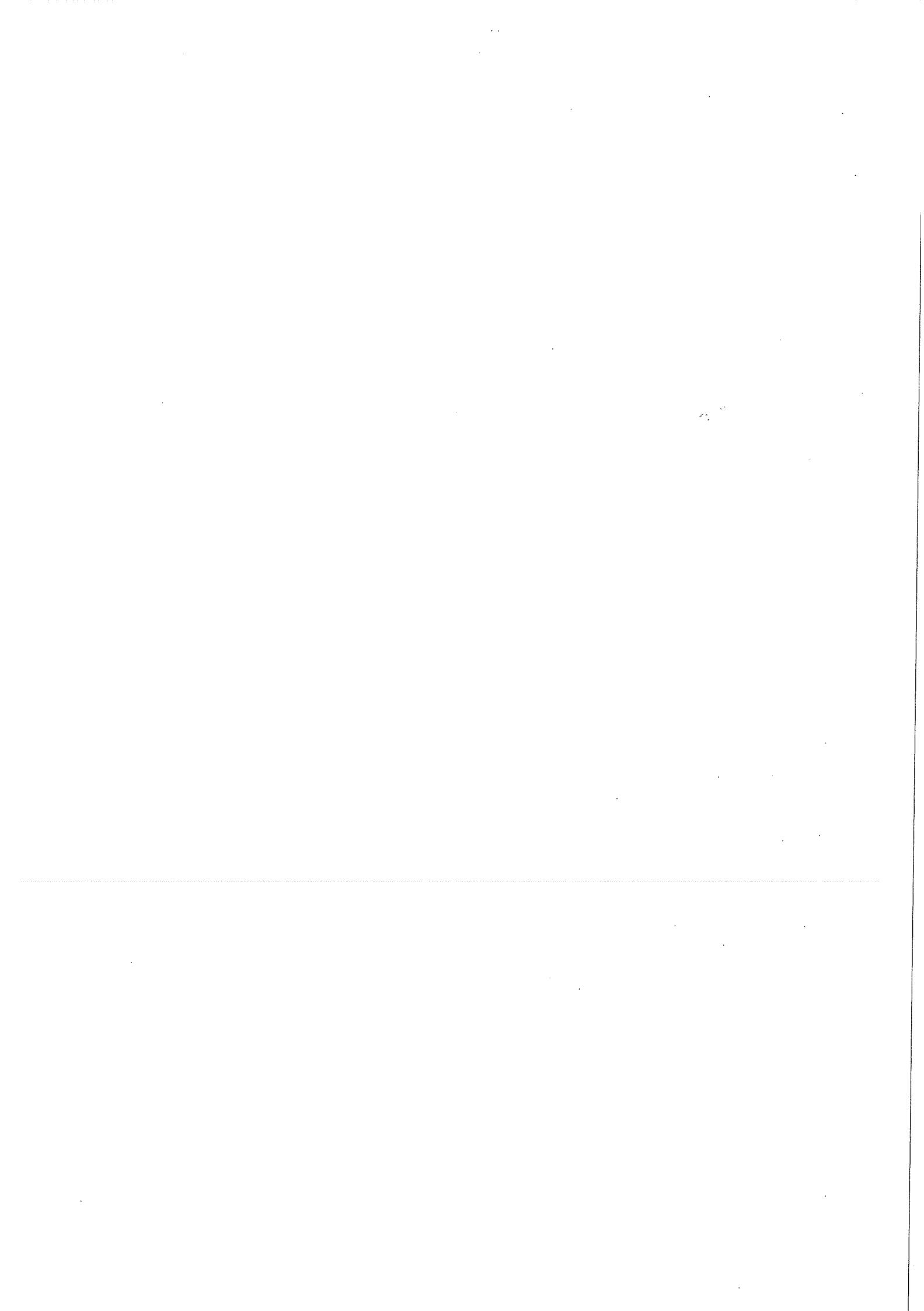
Dann ist $\varphi_k(z, \cdot) = (z - w_k)^{-1} (f(z, \cdot) - f(w_k, \cdot))$ messbar, genügt der Abschätzung

$$|\varphi_k(z, \cdot)| \leq \frac{2}{r} g(\cdot) \text{ μ-f.ü.,}$$

und es gilt wegen der gleichmäßigen Konvergenz des Integranden im Kurvenintegral

$$\lim_{k \rightarrow \infty} \varphi_k(z, x) = \frac{1}{2\pi i} \int_{\partial K_{2r}(a)} \frac{f(\zeta, x)}{(\zeta - z)^2} d\zeta = \frac{\partial f}{\partial z}(z, x);$$

die zweite Gleichheit beruht hier auf der Cauchyschen Integralformel für die Ableitung $\frac{\partial f}{\partial z}(\cdot, x)$. Der Satz von der majorisierten Konvergenz liefert nun die Behauptung für $n = 1$. Eine Fortsetzung dieser Schlußweise liefert unter Benutzung der Cauchyschen Integralformel für die höheren Ableitungen die Behauptung in vollem Umfang. \square



**Modul: 5010-B4-4f11;/Videregående Teori Og Anvendelser Af
Differentialoperatorer Og Funktionsrum (Difffun2)**
**Uge(r): 15-24 (11 apr
2011-19 jun 2011)**

mandag

M	13 - 15
W	13 - 15
T	13 - 15

tirsdag

Aktivitet	Beskrivelse	Type	Uge(r)	Start	Slut	Lokale	Underviser
5010-B4-Videregående teori og anvendelser af differentialoperatorer og funktionsrum NB!NB!undervisningstidspunkt er kun mandag den 14.4	Forelæsning NB!undervisningstidspunkt er kun mandag den 14.4	Forelæsning	15	8:15	10:00	aud - Aud 09 (HCØ)	Gimperlein, Heiko
5010-B4-Videregående teori og anvendelser af differentialoperatorer og funktionsrum NB! Se hjemmeside for lokale den 14.4	Forelæsning NB!se hjemmeside for lokale den 14.4	Forelæsning	17-24	10:15	12:00	aud - Aud 09 (HCØ)	Gimperlein, Heiko

onsdag

torsdag

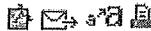
Aktivitet	Beskrivelse	Type	Uge(r)	Start	Slut	Lokale	Underviser
5010-B4-Videregående teori og anvendelser af differentialoperatorer og funktionsrum NB!! Undervisningstidspunkt er kun den 17.4.	Forelæsning NB! tidspunkt er kunne torsdag den 17.4	Forelæsning	15	8:15	10:00	aud - Aud 09 (HCØ)	Gimperlein, Heiko
5010-B4-Videregående teori og anvendelser af differentialoperatorer og funktionsrum NB!! Se hjemmeside for lokale den 14.4	Forelæsning	Forelæsning	17-25	10:15	12:00	aud - Aud 09 (HCØ)	Gimperlein, Heiko
5010-B4-Videregående teori og anvendelser af differentialoperatorer og funktionsrum	Øvelser	Øvelse	15, 17-25	13:15	15:00	øv - A104 (HCØ)	

fredag

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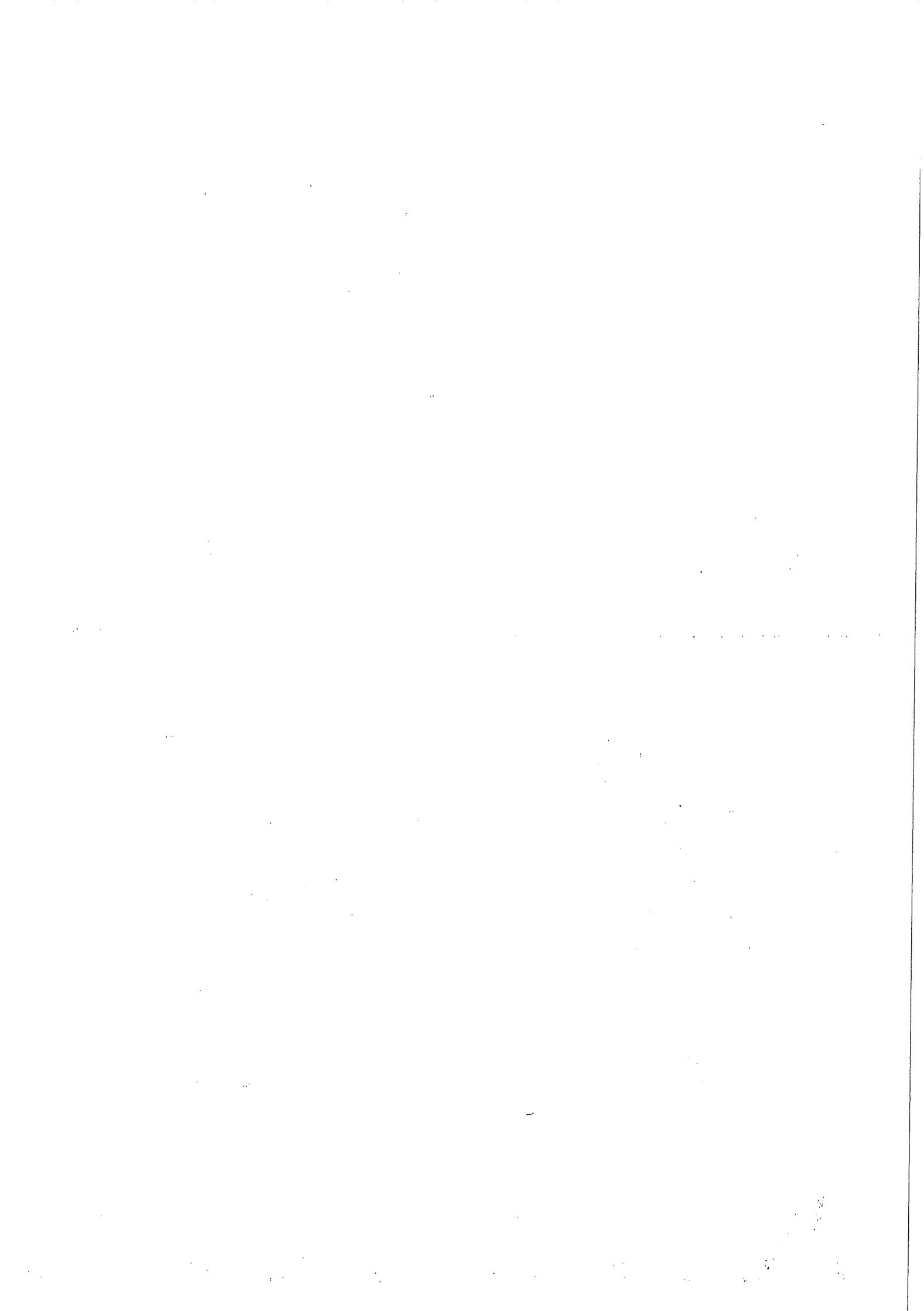
[SIS-hovedside](#) -> [Naturvidenskab](#) -> [De matematiske fag \(Matematik, Statistik, Forsikringsvidenskab og Matematik-Økonomi\)](#) -> [Kandidatuddannelsen](#) -> [Blok 4 \(11. april - 24. juni 2011\)](#) -> [Skemagruppe A](#)

Videregående teori og anvendelser af differentialoperatorer og funktionsrum (DiffFun2)

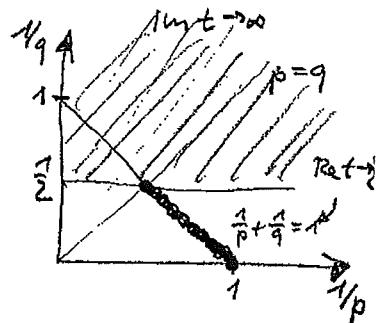


A 75

Udgave:	Forår 2011 NAT
Point:	7,5
Blokstruktur:	4. blok
Skemagruppe:	A
Fagområde:	mat Forår
Semester:	
Varighed:	9 uger
Institutter:	Institut for Matematiske Fag
Uddannelsesdel:	Kandidat niveau
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Skema-oplysninger:	Vis skema for kurset Samlet oversigt over tid og sted for alle kurser inden for Lektionsplan for Det Naturvidenskabelige Fakultet Forår 2011 NAT
Undervisningsperiode:	11. april – 24. juni 2011
Undervisningsform:	4 timers forelæsning og 2 timers øvelser pr. uge
Indhold:	I kurset videreudvikles emnerne fra DiffFun, og der gives en introduktion til harmonisk analyse. Det vil omfatte et antal effektive teknikker til vurdering af integraler (som stationær fase metoden), Littlewood og Paleys faserums analyse, singulære integraler og pseudo-differentialoperatorer, samt grundlæggende Fourieranalyse på lokalkompakte abelske grupper.
Målbeskrivelse:	At kunne løse konkrete og abstrakte problemer der vedrører kursets faglige område.
Lærebøger:	Udvalgte kapitler fra: <ul style="list-style-type: none"> • T. Tao, Graduate Fourier Analysis, lecture notes, • E. Stein, Singular integrals and differentiability properties of functions, Princeton University Press, • E. Stein and G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, suppleret med andet materiale.
Tilmelding:	Kursus- og eksamenstilmelding og afmelding sker på www.kunet.dk Tilmelding skal ske i perioden den 1. – 10. december 2010.
Faglige forudsætninger:	An2 og "Differentialoperatorer og funktionsrum" eller tilsvarende. Grundlæggende begreber fra KomAn og FunkAn vil være nyttige.
Eksamensform:	Løbende evaluering med ugentlige obligatoriske afleveringsopgaver og et afsluttende projekt, som bedømmes bestået/ikke bestået med intern censur. Reeksamen: 30 minutters mundtlig prøve uden forberedelse.
Eksamensform:	Løbende evaluering. Reeksamen: Mundtlig prøve den 1. september 2011.
Pensum:	Fastlægges løbende.
Undervisnings-sprog:	Engelsk
Sidst redigeret:	11/11-2010



$$\begin{aligned}
 26) \quad & \frac{\|\mathcal{F} e^{-t|x|^2}\|_q}{\|e^{-t|x|^2}\|_p} = \frac{\|e^{-t|x|^2/4t}/\sqrt{2t}\|_q}{\|e^{-t|x|^2}\|_p} \\
 & = \frac{\frac{1}{(2t)^{1/2}} \left(\int_{-\infty}^{\infty} |e^{-q|x|^2/4t}| dx \right)^{1/q}}{\left(\int_{-\infty}^{\infty} |e^{-pt|x|^2}| dx \right)^{1/p}} = \frac{1}{(2t)^{1/2}} \left(\frac{\pi}{p \operatorname{Re} t} \right)^{1/p} \\
 & = \frac{1}{(2t)^{1/2}} \left(\frac{4\pi}{q \operatorname{Re} t} \right)^{1/q} / \left(\frac{\pi}{p \operatorname{Re} t} \right)^{1/p}
 \end{aligned}$$



$$\text{Case 1: } t_{\operatorname{real}} = \left(\frac{4\pi}{q} \right)^{1/q} \left(\frac{\pi}{p} \right)^{1/p} \frac{1}{\sqrt{2}} \quad t = -\frac{1}{2} + \frac{1}{2q} + \frac{1}{2p}$$

$$\begin{aligned}
 -\frac{1}{2} + \frac{1}{2q} + \frac{1}{2p} &= 0 \\
 \Rightarrow \frac{1}{p} + \frac{1}{q} &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{Case 2: } t &= \alpha + i\beta \sim \frac{1}{t} = \frac{\alpha - i\beta}{\alpha^2 + \beta^2} \\
 &\Rightarrow C_{\pi, p, q} \frac{1}{4\sqrt{\alpha^2 + \beta^2}} \left(\frac{\alpha^2 + \beta^2}{\alpha} \right)^{1/q} \left(\frac{1}{\alpha} \right)^{1/p} \\
 &\xrightarrow{\beta \rightarrow \infty} C_{\pi, p, q} \alpha^{-1/2} \beta^{1/q} \xrightarrow{\text{if } q < 2 \Leftrightarrow p > 2} \infty
 \end{aligned}$$

V6: $\mathcal{F}: L^2 \rightarrow L^2$ isometrisch, $\Rightarrow \|\mathcal{F}\|_{L^2 \rightarrow L^2} = 1$

$$\begin{aligned}
 \mathcal{F}: L^1 \rightarrow L^\infty : \left\| \left(\frac{1}{(2\pi)} \int f(x) e^{-ix\cdot} \right) \right\|_\infty &\leq \left(\frac{1}{(2\pi)} \int |f(x)| dx \right) \\
 \Rightarrow \|\mathcal{F}\|_{L^1 \rightarrow L^\infty} &\leq \frac{1}{(2\pi)} \|\mathcal{F}\|_{L^1}
 \end{aligned}$$

Interpolation
\$\Rightarrow\$

$$\mathcal{F}: L^p \rightarrow L^q ; \quad \frac{1}{p} + \frac{1}{q} = 1$$

$$\|\mathcal{F}\|_{L^p \rightarrow L^q} \leq (\cdot) \|\mathcal{F}\|_{L^1 \rightarrow L^\infty}^{1 - \frac{2}{pq}} \|\mathcal{F}\|_{L^1 \rightarrow L^2}^{\frac{2}{pq}}$$

$$\text{for } \frac{2}{p} = 2 \Leftrightarrow \text{and } 2(1 - \frac{2}{pq}) = \frac{2}{q} \quad \text{and } 1 - \frac{2}{q} = 1 - \frac{1}{p} = \frac{1}{q}$$

$$1 - \frac{2}{q} = 1 - \frac{2}{q} = 1 - \frac{1}{p} = \frac{1}{q}$$

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Compare \mathcal{F}^f for f satisfying

- 1) f real analytic.
- 2) $\exists \varepsilon > 0$: $f \in \text{Hol}(\mathbb{R} + i(-\varepsilon, \varepsilon))$
- 3) $\sup_{x \in \mathbb{R} + i(-\varepsilon, \varepsilon)} e^{N|x|} |f(x)| < \infty \quad \forall N$

3) $\Rightarrow f \in \mathcal{S}_R \Rightarrow \mathcal{F}^f = \hat{f} \in \mathcal{S}$

$$\mathcal{F}^f(\xi) = \hat{f}(\xi) = \int e^{ix\xi} f(x) dx$$
$$\xi = u + iv \Rightarrow e^{ix\xi} f(x) = e^{ixu} e^{ivx} f(x)$$

3) $\Rightarrow |f(x)| \leq C_N e^{-N|x|} \quad \forall x \in \mathbb{R} + i(-\varepsilon, \varepsilon), \quad N$

b) Compute $\mathcal{F} = \mathcal{F}f$ for

$$f(x) = e^{-Ax}, \quad A \in \mathbb{C}^{n \times n}, \quad \operatorname{Re}(\lambda) \leq 0, \quad \text{for all } \lambda \in \sigma(A)$$

For simplicity let A be (diagonalizable) Hermitian / real symmetric

$$\Rightarrow A = UDU^*, \quad D = \operatorname{diag}(\lambda_1, \dots, \lambda_n), \quad \operatorname{Re} \lambda_j > 0$$

$$\begin{aligned}\mathcal{F}(g) &= \int e^{-ix\zeta} e^{-Ax} dx \\ &= \int e^{-iUx \cdot U\zeta} e^{-UDUx \cdot U\zeta} dx \\ &= \int e^{-iUx \cdot U\zeta} e^{-DUx \cdot U\zeta} dx \quad \zeta = U\zeta \\ &= \int e^{-i\zeta^T Ux} e^{-\sum \lambda_j z_j^2} dz \\ &\stackrel{j}{=} \prod_j \int e^{-iz_j \zeta_j - \lambda_j |z_j|^2} dz_j \\ \int e^{-iz_j \zeta_j - \lambda_j |z_j|^2} dz_j &= \int e^{-\lambda_j (x_j + i\zeta_j)^2 - \lambda_j |\zeta_j|^2} \end{aligned}$$

$$\begin{aligned}\int e^{-iz_j \zeta_j - \lambda_j |z_j|^2} dz_j &= \int e^{-\lambda_j (x_j + i\frac{1}{2}\lambda_j \zeta_j)^2 - \frac{1}{4}\lambda_j^2 \zeta_j^2} dx_j \\ &= e^{-\frac{1}{4}\lambda_j^2 \zeta_j^2} \int e^{-\lambda_j y_j^2} dy_j = (\frac{\pi}{2})^{\frac{1}{2}} e^{-\frac{1}{4}\lambda_j^2 \zeta_j^2} \end{aligned}$$

$n=1, \lambda > 0$

$$f \in \mathcal{S} \Rightarrow \hat{f} \in \mathcal{S}$$

$$f(x) = e^{-\lambda x^2} \text{ satisfies } \mathcal{F}'(x) = -2\lambda x f(x) \Leftrightarrow (\partial_x + 2\lambda x) f = 0$$

$$\hat{f}'(\zeta) = \int e^{-ix\zeta} e^{-\lambda x^2} dx$$

$$\begin{aligned}\hat{f}'(\zeta) &= \partial_{\zeta} \hat{f}(\zeta) = -i \int e^{-ix\zeta} x \underbrace{e^{-\lambda x^2}}_{-\frac{1}{2}\lambda \partial_x e^{-\lambda x^2}} dx \\ &= -\frac{i}{2\lambda} \int \underbrace{\partial_x e^{-ix\zeta}}_{-\zeta} e^{-\lambda x^2} dx = -\frac{\zeta}{2\lambda} \hat{f}(\zeta)\end{aligned}$$

$$\Leftrightarrow (\partial_{\zeta} + \frac{\zeta}{2\lambda}) \hat{f} = 0$$

Saint-feliu func type ODE $\Rightarrow \hat{f}(\zeta) = f(0) e^{-\frac{1}{2\lambda} \zeta^2}, \quad f(0) = \int e^{-\lambda x^2} dx = (\frac{\pi}{2})^{\frac{1}{2}}$

$$\therefore \hat{f}(\zeta) = \int e^{-ix\zeta} e^{-\lambda x^2} dx \stackrel{(*)}{=} \lambda^{-\frac{1}{2}} \pi^{\frac{1}{2}} e^{-\frac{1}{4\lambda} \zeta^2}$$

$n=1, \lambda \in \mathbb{C}, \operatorname{Re} \lambda > 0$! Choose the branch of the $\sqrt{\cdot}$, s.e. $\sqrt{\lambda} > 0$ for $\lambda > 0$

\Rightarrow Both sides of $(*)$ analytic in λ

\Rightarrow valid on the set $\{ \lambda \mid (\mathcal{F} \circ S \text{ only for } \operatorname{Re} \lambda > 0) \}$

$$n \geq 1, \text{ VR min } \Re A > 0 : \quad f(x) = e^{x \cdot A x}$$

Dagonalizze A : $A = S^T \Lambda S$, $\Lambda \in \text{diag}(\lambda_1, \dots, \lambda_n)$

$$\begin{aligned} f(x) &= \int e^{-i x \cdot S} e^{-x \cdot A x} dx \\ &= \int e^{-i S x \cdot S} e^{-S x \cdot \Lambda S x} d(S x) \quad \begin{matrix} y = Sx \\ S = S^T \end{matrix} \\ &= \int e^{-i \sum_j y_j \lambda_j} e^{-\frac{1}{4} \sum_j \lambda_j y_j^2} \pi dy_j \\ &\stackrel{\text{!}}{=} \pi \prod_j \int e^{-i y_j \lambda_j} e^{-\frac{1}{4} \lambda_j y_j^2} dy_j \\ &= \pi^{\frac{n}{2}} (\det \Lambda)^{\frac{1}{2}} e^{-\frac{1}{4} y \cdot \Lambda^{-1} y} \\ &= \pi^{\frac{n}{2}} (\det A)^{\frac{1}{2}} e^{-\frac{1}{4} x \cdot A^{-1} x} = \int e^{i x \cdot S} e^{-x \cdot A x} dx \quad \begin{matrix} (x \cdot x) \end{matrix} \end{aligned}$$

$$n \geq 1, A \in \mathbb{C}^{n \times n} \text{ symm}, \Re A > 0 :$$

The set H of symm. matrices A , $\Re A > 0$,

is an open convex set in the $n(n+1)/2 - \text{dim.}$ complex space of symm. $n \times n$ -matrices.

$A \in H \Rightarrow \det A \neq 0$ since

$$Ax = 0, x \in \mathbb{C}^n \Rightarrow 0 = \Re i x \cdot A x = \bar{x} \cdot (\Re A)x \Rightarrow x = 0$$

H convex \Rightarrow I unique analytic branch of

$$H \ni A \rightarrow (\det A)^{\frac{1}{2}} \in (\det A)^{\frac{1}{2}} > 0 \quad (\text{for } A \text{ real})$$

Both sides off (cross) analytic in $A \in H$

\Rightarrow valid for all $A \in H$.

b) Show that $\mathcal{F}: L^p(\mathbb{R}) \rightarrow L^q(\mathbb{R})$ is bounded.

then $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$

$$\text{Take } f(x) = e^{-\lambda x^2}, \|f\|_p = \left(\int_{\mathbb{R}} e^{-\lambda x^2} dx \right)^{\frac{1}{p}} = \left(\frac{\pi}{\lambda} \right)^{\frac{1}{2p}}$$

$$\Rightarrow f \in L^p$$

$$\mathcal{F}(f) = \hat{f}(\xi) = \left(\frac{\pi}{\lambda} \right)^{\frac{1}{2}} e^{-\frac{\pi}{\lambda} \xi^2}$$

$$\|\hat{f}\|_q = \left(\frac{\pi}{\lambda} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} e^{-\frac{\pi}{\lambda} \xi^2} d\xi \right)^{\frac{1}{q}} = \left(\frac{\pi}{\lambda} \right)^{\frac{1}{2}} \left(\frac{\pi}{q(4\pi)} \right)^{\frac{1}{q}}$$

$$= \left(\frac{\pi}{\lambda} \right)^{\frac{1}{2}} \left(\frac{4\pi}{q} \right)^{\frac{1}{q}} = \pi^{\frac{1}{2}} (4\pi)^{\frac{1}{q}} \lambda^{\frac{1}{2q} - \frac{1}{2}}$$

$$\frac{\|\hat{f}\|_q}{\|f\|_p} = \pi^{\frac{1}{2}} \left(\frac{4\pi}{q} \right)^{\frac{1}{q}} \left(\frac{\pi}{\lambda} \right)^{\frac{1}{2p}} \lambda^{\frac{1}{2q} - \frac{1}{2} + \frac{1}{2p}}$$

$$(\lambda^{\frac{1}{p} + \frac{1}{q} - 1})^{\frac{1}{2}}$$

If $\mathcal{F}: S \mapsto \hat{f}$ is continuous, i.e. $\|\mathcal{F}\| = \sup_{f \neq 0} \frac{\|\hat{f}\|}{\|f\|} < \infty$

then it cannot depend on λ

$$\Rightarrow \frac{1}{p} + \frac{1}{q} - 1 = 0 \Rightarrow p \geq 1$$

Hörmander Thm 7.6.6.: $\mathcal{F}: L^p \rightarrow \mathcal{D}'^k \Rightarrow k \geq n(\frac{1}{2} - \frac{1}{p})$

$$\text{So } \mathcal{F}: L^p \rightarrow L^q \subseteq \mathcal{D}'^{10} \Rightarrow 0 \geq \frac{1}{2} - \frac{1}{p} \Rightarrow p \leq 2$$

a)

Assume $p > 2$, $u \in L^p$, $\hat{u} \in L^q$, $q = (1 - \frac{1}{p})^{-1} \in (1, 2)$

$$\phi \in C_0^\infty(K) \Rightarrow |\langle \hat{u}, \phi \rangle| \leq \sup_K |\phi|$$

$$|\langle \hat{u}, \phi \rangle| \leq C_\phi \|u\|_p$$

$$\Rightarrow |\langle \hat{u}, \phi \rangle| \leq C \|u\|_p \sup_K |\phi| \quad (\text{Hörmander})$$

Choose $0 \neq u \in S$ s.t. $\text{supp } \hat{u} \subseteq K$

and take $u_t := u * ct^{-\frac{1}{2}} e^{-i \frac{x^2}{4t}} \Rightarrow \hat{u}_t = \hat{u} e^{i c t^2}$

$$\phi_c := \overline{\hat{u}_c}$$

$$\Rightarrow |\langle \hat{u}_c, \phi_c \rangle| = |\langle \hat{u}, \phi_c \rangle| = \|\hat{u}\|_2^2 \leq C_u$$

$$\begin{aligned}
 \text{but } \|u\|_p^p &= \int |u|^{p/2} |u|^2 dx \leq \|u\|_\infty^{p/2} \|u\|_2^2 \\
 &= \|u\|_\infty^{p/2} \|u\|_2^2 = \|u\|_2^2 \|u\|_\infty^{p-2} \\
 \|u\|_\infty &\leq \|u\|_1 \cdot C^{-\frac{1}{2}} \\
 \Rightarrow \|u\|_p &\leq C_u t^{-\frac{1}{2}} \cdot \frac{p-2}{p} \\
 (*) \Rightarrow C_u &\leq C C_n t^{-\frac{1}{2}} \frac{p-2}{p} \sup_K \|u\|_1 \\
 \Rightarrow C_u &\leq t^{-\frac{1}{2}} \frac{p-2}{p} \rightarrow 0 \text{ for } t \rightarrow \infty \text{ if } p > 2
 \end{aligned}$$

at x:

$$\lambda = \alpha + i\beta, \quad x > 0, \quad \lambda^{-1} = \frac{\alpha - i\beta}{\alpha^2 + \beta^2}$$

$$\begin{aligned}
 f(x) = e^{-\lambda x} \Rightarrow f(\xi) &= \left(\frac{\alpha}{\lambda}\right)^{\frac{1}{2}} e^{-\frac{\beta}{\lambda} \xi^2} \\
 \|f\|_q &= \frac{\pi^{\frac{1}{2}}}{12^{\frac{1}{2}}} \left(\int |e^{-\frac{\beta}{\lambda} \xi^2}| d\xi \right)^{\frac{1}{q}} \\
 &\quad \int e^{-\frac{q\alpha}{\lambda(\alpha^2+\beta^2)} \xi^2} d\xi = \left(\frac{\pi}{q\alpha M(\alpha^2+\beta^2)} \right)^{\frac{1}{2}} \\
 &= C_q (\alpha^2 + \beta^2)^{-\frac{1}{q}} \left(\frac{\alpha^2 + \beta^2}{\alpha^2} \right)^{\frac{1}{2q}} \\
 \|f\|_p &= \int |e^{-\lambda x}| dx = \int e^{-\lambda x} dx = C_{p,k} \\
 \Rightarrow \frac{\|f\|_q}{\|f\|_p} &= C_{q,p,k} (\alpha^2 + \beta^2)^{\frac{1}{2q} - \frac{1}{q}} = C_{q,p,k} (\alpha^2 + \beta^2)^{\frac{1}{2} \left(\frac{1}{q} - \frac{1}{p} \right)} \\
 &\quad \xrightarrow{q \rightarrow \infty, \beta \rightarrow \infty} \text{if } p > 2
 \end{aligned}$$

2. Show (Paley-Wiener-Schwartz)

$$\mathcal{F} \mathcal{E}'_{\bar{B}_R(0)}(\mathbb{R}^n) = \left\{ u \in \text{Hol}(\mathbb{C}^n) \mid \exists N \in \mathbb{C}, \quad |u(\zeta)| \leq C(1+|\zeta|)^N e^{R|\text{Im}\zeta|} \right\}$$

ori:

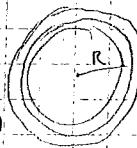
1) If $u \in \mathcal{E}'^N(\mathbb{R}^n)$, $\text{supp } u \subseteq \bar{B}_R(0)$ then $u \in \text{Hol}(\mathbb{C}^n)$ and $|u(\zeta)| \leq C(1+|\zeta|)^N e^{R|\text{Im}\zeta|} \quad \forall \zeta \in \mathbb{C}^n$

2) and NF $\hat{u} = \widehat{\mathcal{F}u}$ for some $u \in \mathcal{E}'(\mathbb{R}^n)$

then $\hat{u} = \mathcal{F}u$ for some $u \in \mathcal{E}'(\mathbb{R}^n)$, $\text{supp } u \subseteq \bar{B}_R(0)$.

3) Take $u \in \mathcal{E}'^N(\bar{B}_R(0))$

and let $x_\delta \in C_0^\infty(B_{R+\delta}(0))$ s.t. $x_\delta = 1$ on $B_{R+\frac{\delta}{2}}(0)$



Then $u(\phi) = u(x_\delta \phi)$ $\forall \phi \in C_0^\infty$ and

$$|u(\zeta)| = |u_x(e^{-ix\zeta})| = |u_x(x_\delta e^{-ix\zeta})|$$

$$\leq C \sum_{|\alpha| \leq N} \sup_{B_{R+\delta}(0)} |D^\alpha u_x e^{-ix\zeta}|$$

$$|D^\alpha x_\delta| \leq C \delta^{-|\alpha|}, \quad |D^\alpha e^{-ix\zeta}| \leq (1+|\zeta|)^{|\alpha|} e^{|\zeta||\text{Im}\zeta|}$$

$$\Rightarrow |\hat{u}(\zeta)| \leq C \sum_{|\alpha| \leq N} \sup_{B_{R+\delta}(0)} C \delta^{-|\alpha|} \frac{(1+|\zeta|)^{N-|\alpha|}}{e^{|\zeta||\text{Im}\zeta|}}$$

$$\leq C e^{(R+\delta)||\text{Im}\zeta|} \sum_{|\alpha| \leq N} \delta^{-|\alpha|} (1+|\zeta|)^{N-|\alpha|}$$

$$\text{Take } \delta = \frac{1}{1+|\zeta|} \Rightarrow |\hat{u}(\zeta)| \leq C'' e^{R|\text{Im}\zeta|} (1+|\zeta|)^N$$

$\hat{u} \in \text{Hol}(\mathbb{C}^n)$ from earlier

2) Take $U \in \text{Hol}(\mathbb{C}^n)$ s.t. $|U(\zeta)| \leq C(1+|\zeta|)^N e^{R|\text{Im}\zeta|} \quad \forall \zeta$

then $U|_{\mathbb{R}^n} \in C_0^\infty(\mathbb{R}^n) \cap S'$

$\Rightarrow u := \mathcal{F}^{-1} U \in S'$. Remains to show $\text{supp } u \subseteq \bar{B}_R(0)$.

Take a mollifier $\varphi_\delta \in \mathcal{B}_\delta(0)$. By Paley-Wiener for C_0^∞ :

$$\varphi_\delta \in \text{Hol}(\mathbb{C}^n), \quad |\varphi_\delta(\zeta)| \leq C_M (1+|\zeta|)^{-M} e^{S|\text{Im}\zeta|} \quad \forall M$$

Then for $u_s := u * \phi_s$

$$\hat{u}_s = \hat{u} \hat{\phi}_s = \hat{u} \phi_s \in \text{Hol}(\mathbb{C}^n)$$

$$|\hat{u}_s(z)| \leq C_{\phi_s} (1+|z|)^{-m} e^{(R+s) |z|^2} \quad \forall z \in \mathbb{C}^n$$

Again, by Paley-Wiener $u_s = F^{-1} \hat{u}_s \in C_0^\infty(B_{R+s}(0))$

When $s \rightarrow 0$, $u_s = u * \phi_s \rightarrow u \quad \Rightarrow \text{supp } u \subseteq B_R(0)$

3. a) Show that $\mathcal{F} L^1(\mathbb{R}^n) \subseteq C_0(\mathbb{R}^n)$

$u: \mathbb{R}^n \rightarrow \text{cont.}$
 $|u| \rightarrow 0, \text{ as } |z| \rightarrow \infty$
 $\|u\|_{L^\infty} < \infty$

ab 1: Riemann-Lebesgue $\hat{u}_{[a,b]}(z) \rightarrow 0, |z| \rightarrow \infty$

Mt 2: GG:

Use that $S \subseteq L^1$ dense and $\mathcal{F}: L^1 \rightarrow L^\infty$ cont.
 $u \in L^1$ and
take $\varepsilon > 0$. Then $\exists u_\varepsilon \in S : \|u - u_\varepsilon\|_1 < \frac{\varepsilon}{2}$
 $\Rightarrow \|u - \hat{u}_\varepsilon\|_\infty = \|\mathcal{F}(u - u_\varepsilon)\|_\infty$
 $\leq \|u - u_\varepsilon\|_1 < \frac{\varepsilon}{2}$

$u_\varepsilon \in S \Rightarrow |\hat{u}_\varepsilon(z)| \rightarrow 0, |z| \rightarrow \infty$
 $\Rightarrow |\hat{u}(z)| \leq \|u - u_\varepsilon\|_\infty + |\hat{u}_\varepsilon(z)| < \varepsilon$
for $|z| \text{ suff. large}$

b) Give 2 proofs that $\mathcal{F} L^1 \neq C_0$

i) Show that $g(x) := \frac{\tanh(x)}{\ln(1+x)} \in C_0(\mathbb{R}) \setminus \mathcal{F} L^1(\mathbb{R})$

Assume to the contrary that $g = \hat{f}, f \in L^1(\mathbb{R})$

Note: f odd $\Rightarrow \hat{f}(z) = \frac{1}{2}(f(z) - f(-z))$
 $= \frac{1}{2} \left(\int e^{-izx} f(x) dx - \int e^{izx} f(x) dx \right)$
 $= \int \frac{1}{2} (e^{izx} - e^{-izx}) f(x) dx$
 $= \int \sin x z f(x) dx$

 $\Rightarrow \left| \int_1^b \frac{\hat{f}(z)}{z} dz \right| = \left| \int_{z=1}^b \int_{x=-\infty}^{\infty} \frac{\sin x z}{z} f(x) dx dz \right|$

Fubini $\Rightarrow = \left| \int_{x=-\infty}^0 \int_{z=1}^b \frac{\sin x z}{x z} z dz f(x) dx \right|$

$\leq \int_{x=-\infty}^0 \left| \int_{y=x}^b \frac{\sin y dy}{y} \right| |f(x)| dx$

$\leq C \|f\|_1$ (generalized Riemann Integrable)
def. on $x \in \mathbb{R} \setminus \{0\}$

but we have

$$\int_1^b \frac{g(\xi)}{\xi} d\xi = \int_1^b \frac{\tanh(\xi)}{\xi \ln(1+|\xi|)} d\xi$$

$$2 \int_1^b \frac{\tanh \xi}{\xi \ln(1+|\xi|)} d\xi + 4 \int_0^b \frac{\xi \ln \xi}{\ln \xi} d\xi$$

b>1

$$(\ln \ln \xi)'$$

$$\ln b + \ln \ln b \rightarrow \infty$$

$b \rightarrow \infty$

$$h \Rightarrow g \# f$$

X

2) Use that $(L'(B^n), *)$ is a Banach algebra,

but not a C^* -algebra, while $(C_c(B^n), *)$ is C^*

Check that $\tilde{f}: (L'(B^n), *) \rightarrow (C_c(B^n), *)$

is an injective $*$ -algebra homomorphism

We know $\tilde{f}: L'(B^n) \xrightarrow[u \mapsto \tilde{u}]{} C_c(B^n)$ cont. & vanishing at ∞

$$\tilde{f}(u * v) = \tilde{u} \tilde{v}$$

$$\begin{aligned}\tilde{f}(\tilde{u})(\tilde{x}) &= \int e^{-ix \cdot \tilde{x}} \tilde{u}(x) dx \\ &= \int e^{-ix \cdot \tilde{x}} u(x) dx = \tilde{f}(u)(\tilde{x}) \\ &= \tilde{f}(u)(\tilde{x})\end{aligned}$$

take $u^* := \tilde{u}^*$ in C_c

$$\|uv\|_\infty \leq \|u\|_\infty \|v\|_\infty$$

$$\Rightarrow \tilde{f}(u) = \tilde{f}(u)^*$$

$$(uv)^* = u^* v^* = v^* u^*$$

$$\|u * v\|_\infty \leq \|u\|_\infty \|v\|_\infty$$

$$(u * v)^* = \tilde{u} * \tilde{v} = \tilde{v} * \tilde{u}$$

$\tilde{f}: L' \rightarrow C_c$ injective (ext since $\tilde{f}: S' \xrightarrow[u]{} S$ isomorphism)

\tilde{f} surjective $\Rightarrow \tilde{f}$ isomorphism of $*$ -algebras

$\Rightarrow L' \text{ } C^*$ -algebra

4. a) Show that if $f: \mathbb{R} \rightarrow \mathbb{C}$ extends to

$$f \in \text{Hol}(\mathbb{R} + i(-\delta, \delta)), \quad \delta > 0 \quad \text{s.t.}$$

$$\forall |y| < \delta \quad \exists C_y : \quad |f(x+iy)| \leq C_y e^{|x|}, \quad n \in \mathbb{N}$$

$$\text{then } \exists c, \varepsilon > 0 : \quad |\hat{f}(z)| \leq c e^{-\varepsilon |z|} \quad \forall z \in \mathbb{R}$$

pf:

$$\hat{f}(z) = \int_{x \in \mathbb{R}} e^{-izx} f(x) dx \quad \begin{matrix} f \in U \Rightarrow f \in C_0 \\ \text{if } |x| \rightarrow \infty \end{matrix}$$

$|f(x+iy)| \rightarrow 0 \text{ for } |y| < \delta$

$$\text{Cauchy} \Rightarrow \int_{x \in \mathbb{R} + iy} e^{-izx} f(x) dx \quad y \in (-\delta, \delta)$$

$$= \int_{x \in \mathbb{R}} e^{-i(x+z)} f(x+iy) dx$$

$$e^{yz} - e^{xz}$$

$$\Rightarrow |\hat{f}(z)| \leq e^{yz} \int_{x \in \mathbb{R}} |f(x+iy)| dx \leq C_y e^{yz} \int_{x \in \mathbb{R}}$$

$$\text{take } y = -\log \varepsilon, \quad 0 < \varepsilon < \delta \Rightarrow |\hat{f}(z)| \leq C_y \|f\|_n e^{-\varepsilon |z|}$$

$\forall z \in \mathbb{R}$

b) Let $A = \{f \in C(\mathbb{R}) : \exists \delta > 0 : f \in \text{Hol}(\mathbb{R} + i(-\delta, \delta)) \cap C(\mathbb{R} + i(-\delta, \delta))\}$

$$\text{and } A \cap C(\mathbb{R}) = \left\{ f \in C(\mathbb{R}) : \sup_{\substack{|y| < \delta \\ x \in \mathbb{R}}} |f(x+iy)| e^{N|x|} < \infty \right\}$$

$$\text{Show } \mathcal{F}A = \left\{ f \in \text{Hol}(\mathbb{C}) : \forall R > 0 \quad \exists \varepsilon > 0 : \sup_{\substack{|y| < \delta \\ x \in \mathbb{R}}} |f(x+iy)| e^{\varepsilon |y|} < \infty \right\}$$

pf: Take $f \in A$. $\forall |y| < \delta \quad |f(x+iy)| \leq \|f\|_n e^{-N|x|} \quad \forall n$

$$\Rightarrow f(x+iy) \in S$$

$$\Rightarrow \int_x f(x+iy) dx \in S$$

$$\hat{f}(z) = \int_x e^{-ixz} f(x) dx, \quad f \in S \Rightarrow \hat{f} \in \text{Hol}(\mathbb{C}) \quad (\text{cp. thin on sheet})$$

$$\text{Let } R > 0, \quad |n| < R$$

$$\hat{f}(z+in) = \int_x e^{-ix(z+in)} f(x) dx = \int_x e^{-ixz} e^{inx} f(x) dx \quad \begin{matrix} \text{if } n \rightarrow 0 \\ \text{then} \end{matrix}$$

$$\text{Cauchy} \Rightarrow \hat{f}(z+\eta) = \int_{\mathbb{R}} e^{-i(x+y)} \xi e^{(x+y)\eta} \hat{f}(x+\eta) dx$$

take $|y| < \delta$

$$\begin{aligned} \Rightarrow |\hat{f}(z+\eta)| &\leq \int_{\mathbb{R}} e^{\eta x} e^{x\eta} |\hat{f}(x+\eta)| dx \\ &\leq e^{\eta z} \int_{\mathbb{R}} e^{\eta x} |\hat{f}(x+\eta)| dx \\ &\leq e^{\eta z} C_R \end{aligned}$$

$$\text{take } y = -\operatorname{sgn} \xi \cdot \varepsilon, 0 < \varepsilon < \delta \Rightarrow |\hat{f}(z+\eta)| \leq C_R e^{\varepsilon |\xi|}$$

$$\Rightarrow \sup_{\substack{z \in \mathbb{C} \\ \eta \in \mathbb{R}}} |\hat{f}(z+\eta)| e^{\varepsilon |\xi|} \leq C_R < \infty$$

Exercises 2 - Applications of Paley-Wiener and Interpolation

① Let $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$, $a_\alpha \in \mathbb{C}$, wlog $a_{m,0,\dots,0} \neq 0$.

Show that the equation $P(D)u = f \in \mathcal{E}'(\mathbb{R}^n)$ has

$$\text{a solution } u \in \mathcal{E}'(\mathbb{R}^n) \iff \frac{\hat{f}(\xi)}{P(\xi)} \in \mathcal{O}(\mathbb{C}^n).$$

Hint: Use / show the following fact with

$$h(z) := \frac{\hat{f}(\xi_1 + z, \xi_2, \dots, \xi_n)}{P(\xi_1 + z, \xi_2, \dots, \xi_n)};$$

$$\left[\begin{array}{l} \text{If } h(z) \in \mathcal{O}(\mathbb{C}), p(z) = p_m z^m + \dots + p_1 z + p_0 \\ \Rightarrow |p_m h(0)| \leq \max_{|z|=1} |h(z) p(z)|, \end{array} \right]$$

Hörmander

② "Schur's test"
Let $(X, \mu), (Y, \nu)$ be measure spaces, $k: X \times Y \rightarrow \mathbb{C}$ measurable.

Consider the integral operator $Kf(y) := \int_X k(x, y) f(x) d\mu(x)$

between suitable L^p -spaces, and let $1 \leq p_1, q_0 \leq \infty$, $p_0 = 1, q_1 = \infty$,

$$\frac{1}{p_1} + \frac{1}{q_0} = 1.$$

a) If $\|k(x, y)\|_{L^{q_0}(Y)} \leq B_0$ for a.e. $x \in X \Rightarrow \|K\|_{L^{p_1}(X) \rightarrow L^{q_0}(Y)} \leq B_0$.

b) If $\|k(x, y)\|_{L^{p_1}(X)} \leq B_1$ for a.e. $y \in Y \Rightarrow \|K\|_{L^{p_1}(X) \rightarrow L^{q_0}(Y)} \leq B_1$.

c) Conclude that $K: L^{p_0}(X) \rightarrow L^{q_0}(Y)$ is bounded

$$\text{for } \frac{1}{p_0} = 1 - \Theta + \frac{\Theta}{p_1} \text{ and } q_0 = \frac{q_0}{1 - \Theta}, 0 \leq \Theta \leq 1.$$

Remark: $q_0 = p_1' = 1; \Theta = \frac{1}{2} \stackrel{?}{\Rightarrow} K: L^2(X) \rightarrow L^2(Y)$ bdd.

$$\Rightarrow \frac{1}{p_0} = \frac{1}{q_0} = 1 - \Theta \Rightarrow p_0 = q_0 = \frac{1}{1 - \Theta} \in [1, \infty) \quad K: L^{p_0} \rightarrow L^{p_0} \text{ bdd.}$$

Exercises 2 - Applications of Paley-Wiener and Interpolation

① Let $P(D) = \sum_{|\alpha| \leq m} \alpha_\alpha D^\alpha$, $\alpha_\alpha \in \mathbb{C}$, wlog $\alpha_{m,0,\dots,0} \neq 0$.

Show that the equation $P(D)u = f \in \mathcal{E}'(\mathbb{R}^n)$ has

a solution $u \in \mathcal{E}'(\mathbb{R}^n) \iff \frac{\hat{f}(\xi)}{P(\xi)} \in \mathcal{O}(\mathbb{C}^n)$.

Hint: Use / show the following fact with

$$h(z) := \frac{\hat{f}(\xi_1 + z, \xi_2, \dots, \xi_n)}{P(\xi_1 + z, \xi_2, \dots, \xi_n)};$$

$$\left[\begin{array}{l} \text{If } h(z) \in \mathcal{O}(\mathbb{C}), p(z) = p_m z^m + \dots + p_1 z + p_0 \\ \Rightarrow |p_m h(0)| \leq \max_{|z|=1} |h(z) p(z)|. \end{array} \right]$$

②

"Schur's test"

Let $(X, \mu), (Y, \nu)$ be measure spaces, $k: X \times Y \rightarrow \mathbb{C}$ measurable.

Consider the integral operator $Kf(y) = \int_X k(x, y) f(x) d\mu(x)$

between suitable L^p -spaces, and let $1 \leq p_1, q_0 \leq \infty$, $p_0 = 1, q_1 = \infty$, $\frac{1}{p_1} + \frac{1}{q_0} = 1$.

a) If $\|k(x, y)\|_{L^{q_0}(Y)} \leq B_0$ for a.e. $x \in X \Rightarrow \|K\|_{L^p(X) \rightarrow L^{q_0}(Y)} \leq B_0$.

b) If $\|k(x, y)\|_{L^{p_1}(X)} \leq B_1$ for a.e. $y \in Y \Rightarrow \|K\|_{L^{p_1}(X) \rightarrow L^\infty(Y)} \leq B_1$.

c) Conclude that $K: L^{p_0}(X) \rightarrow L^{q_0}(Y)$ is bounded

$$\text{for } \frac{1}{p_0} = 1 - \theta + \frac{\theta}{p_1} \text{ and } q_0 = \frac{q_0}{1 - \theta}, 0 \leq \theta \leq 1.$$

Remark: $q_0 = p_1' = 1; \theta = \frac{1}{2} \Rightarrow K: L^2(X) \rightarrow L^2(Y)$ bdd.

$$\alpha = \Theta(\xi_1 + \alpha) \alpha = \frac{1}{\xi_1 + \alpha} = \Theta\left(\frac{1}{\xi_1} + \frac{1}{\xi_1} + \frac{1}{q_0}\right) \quad 1 - \theta + \frac{\theta}{p_1} = \frac{1 - \theta}{q_0} + \frac{1}{p_1} + \frac{1}{q_0} = 1$$

$$\alpha(1 - \theta) = \frac{\Theta}{p_1} \cdot \frac{1}{\xi_1} + \frac{1}{q_0} \quad \alpha(1 - \theta) = \frac{1}{q_0} \quad \text{and } \frac{1}{p_1} + \frac{1}{q_0} = 1 \quad p_1, q_0 \in L^1, L^\infty$$

(3)

Young's inequality

Let $1 \leq p, q, r \leq \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$.

Apply Schur's test to $h(x,y) = g(y-x)$ to conclude

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

Remark: The argument applies to convolutions on any group with a bi-invariant measure (and not just \mathbb{R}^n).

(4)

Let (X, μ) be a measure space, $f: X \rightarrow \mathbb{C}$ measurable.

Define the distribution function $\lambda_f: [0, \infty) \rightarrow [0, \infty]$ by

$$\lambda_f(t) := \mu(\{x \in X : |f(x)| \geq t\}). \text{ For } S \subset X \text{ let } \mathbb{1}_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}.$$

a) Check that $|f(x)|^p = p \int_0^\infty \mathbb{1}_{\{|f| \geq t\}} t^{p-1} dt$ and

$$\|f\|_{L^p(X)}^p = p \int_0^\infty \lambda_f(t) t^{p-1} dt \quad (1 \leq p < \infty)$$

$$\|f\|_{L^\infty(X)} = \inf \{t \geq 0 : \lambda_f(t) = 0\}.$$

b) Show Chebychev's inequality $\lambda_f(t) \leq t^{-p} \|f\|_{L^p(X)}^p$.

c) Show that for $1 \leq p \leq \infty$ $\exists \zeta_p, \bar{\zeta}_p$,

$$\zeta_p \|f\|_{L^p(X)} \leq \left(\sum_{n \in \mathbb{Z}} \lambda_f(2^n) 2^{np} \right)^{\frac{1}{p}} = \left\| (2^n \lambda_f(2^n))^{\frac{1}{p}} \right\|_{l^p(\mathbb{Z})} \leq \bar{\zeta}_p \|f\|_{L^p(X)}$$

d) Let $L^{p,\infty}(X) := \{f: X \rightarrow \mathbb{C} \text{ measurable} : \|f\|_{L^{p,\infty}(X)} := \sup_{t > 0} t \lambda_f(t) < \infty\}$.

Use b) to show $L^p(X) \subseteq L^{p,\infty}(X)$.

If $X = \mathbb{R}^n$ with the Lebesgue measure, show that $f(x) = |x|^{-\frac{n}{p}}$ belongs to $L^{p,\infty}(\mathbb{R}^n) \setminus L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.

(3) Young's inequality

Let $1 \leq p, q, r \leq \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$.

Apply Schur's test to $h(x,y) = g(y-x)$ to conclude

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}$$

seen-works

Remark: The argument applies to convolutions on any group with a bi-invariant measure (and not just \mathbb{R}^n).

(4) Let (X, μ) be a measure space, $f: X \rightarrow \mathbb{C}$ measurable.

Define the distribution function $\lambda_f: [0, \infty) \rightarrow [0, \infty]$ by

$$\lambda_f(t) := \mu(\{x \in X : |f(x)| \geq t\}). \text{ For } S \subset X \text{ let } \mathbb{1}_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}$$

a) Check that 1) $|f(x)|^p = p \int_0^\infty \mathbb{1}_{\{|f| \geq t\}} t^{p-1} \frac{dt}{t}$ and

$$2) \|f\|_{L^p(X)}^p = p \int_0^\infty \lambda_f(t) t^{p-1} \frac{dt}{t} \quad (1 \leq p < \infty)$$

$$3) \|f\|_{L^\infty(X)} = \inf \{t \geq 0 : \lambda_f(t) = 0\}.$$

b) Show Chebychev's inequality $\lambda_f(t) \leq t^{-p} \|f\|_{L^p(X)}^p$.

c) Show that for $1 \leq p \leq \infty$ $\exists \bar{c}_p, \bar{C}_p$ such that $\bar{c}_p \leq \lambda_f(2^n) 2^{np} \leq \bar{C}_p$ for large n .

$$\bar{c}_p \|f\|_{L^p(X)} \leq \left(\sum_{n \in \mathbb{Z}} \lambda_f(2^n) 2^{np} \right)^{\frac{1}{p}} = \left\| (2^n \lambda_f(2^n))^{\frac{1}{p}} \right\|_{L^p(\mathbb{Z})} \leq \bar{C}_p \|f\|_{L^p(X)}$$

d) Let $L^{p,\infty}(X) := \{f: X \rightarrow \mathbb{C} \text{ measurable} : \|f\|_{L^{p,\infty}(X)} := \sup_{t > 0} t \lambda_f(t) < \infty\}$.

1) Use b) to show $L^p(X) \subseteq L^{p,\infty}(X)$.

2) If $X = \mathbb{R}^n$ with the Lebesgue measure, show that $f(x) = |x|^{-\frac{n}{p}}$ belongs to $L^{p,\infty}(\mathbb{R}^n) \setminus L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.

Note: $\|\cdot\|_{L^{p,\infty}}$ is not a norm

$$\|f+g\| \leq \|f\| + \|g\| \quad (\text{not a norm})$$

\exists norm equiv. to this ℓ^1 -norm, with this norm $L^{p,\infty}$ is a Banach space

$$\|f+g\| \leq C(\|f\| + \|g\|)$$

Ex: $\mathbb{X} = \mathbb{R}, \quad f \in L^{q,\infty}(\mathbb{R}) \setminus L^q(\mathbb{R})$

$$\text{Cheby shows: } \|f\|_{L^{q,00}} \leq \|f\|_{L^q} \quad \Leftrightarrow \quad L^q(\mathbb{R}) \hookrightarrow L^{q,00}(\mathbb{R})$$

$\Leftrightarrow: T \text{ strong-type } (p, q)$

$$T \text{ sublinear} \Leftrightarrow \begin{aligned} T: L^p &\rightarrow L^q \text{ cont.} \Leftrightarrow \exists B: \forall F \quad \|T.F\|_{L^q} \leq B \|F\|_{L^p} \\ T: L^p &\rightarrow L^{q,00} \text{ cont.} \Leftrightarrow \dots \quad \|T.F\|_{L^{q,00}} \leq B \|F\|_{L^p} \end{aligned}$$

$\Leftrightarrow: T \text{ weak-type } (p, q)$

$$\Leftrightarrow \forall \lambda_T^{q,p}(A) \leq B + \|F\|_U^q \quad (*)$$

$$\|F\|_U \simeq \left(\sum_{n \in \mathbb{Z}} \lambda_F(2^n) 2^{np} \right)^{1/p}$$

$$= \left\| (2^n \lambda_F(2^n))_{n \in \mathbb{Z}} \right\|_{U^{(q,p)}} \quad (**)$$

BSC 2

1. Let $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$, $a_\alpha \in \mathbb{C}$, $a_{(m, 0, 0, \dots, 0)} \neq 0$

Show that $P(D)u = f \in \mathcal{E}'(\mathbb{R}^n)$ has a solution in $\mathcal{E}'(\mathbb{R}^n)$

$$\Leftrightarrow \frac{\hat{f}(\xi)}{P(\xi)} \in \mathcal{O}(\mathbb{C}^n)$$

Let $f \in \mathcal{E}'$.

Assume that $\exists u \in \mathcal{E}'(\mathbb{R}^n)$ s.t. $P(D)u = f$. (K)

Then by Paley-Wiener-Schwartz

$$\hat{u} \in \mathcal{O}(\mathbb{C}^n) \cap S' \quad \& \quad \hat{f} \in \mathcal{O}(\mathbb{C}^n) \cap S'$$

and by taking the Fourier-Laplace transf. on both sides of (K)

$$P(\xi) \hat{u}(\xi) = \hat{f}(\xi)$$

$$\Rightarrow \frac{\hat{f}(\xi)}{P(\xi)} = \hat{u}(\xi) \in \mathcal{O}(\mathbb{C}^n),$$

\Leftarrow P.W.S $\Rightarrow \hat{f} \in \mathcal{O}(\mathbb{C}^n) \cap S' \quad \& \quad |\hat{f}(\xi)| \leq C(1+|\xi|)^N e^{R|Im \xi|}$

Assume that $\frac{\hat{f}(\xi)}{P(\xi)} \in \mathcal{O}(\mathbb{C}^n)$

Proof (short): Let $h(z) := \frac{\hat{f}(z_1 + z, z_2, \dots, z_n)}{P(z_1 + z, z_2, \dots, z_n)} \in \mathcal{O}(\mathbb{C})$

$$p(z) := P(z_1 + z, z_2, \dots, z_n) = \underbrace{a_{(m, 0, 0, \dots, 0)}}_{\neq 0} z^m + \underbrace{c_{m-1} z^{m-1}}_{\vdots} + \dots + c_0$$

Lemma: If $|h(z)| \in \mathcal{O}(\mathbb{C})$ & $p(z) = p_m z^m + \dots + p_0$

$$\text{then } |p_m h(0)| \leq \max_{|z|=1} |h(z)p(z)|.$$

Proof: Let $q(z) := z^m \bar{p}(\frac{1}{z}) = \bar{p}_m + \bar{p}_{m-1} z + \dots + \bar{p}_0 z^m \in \mathcal{O}(\mathbb{C})$

$$\text{then } q(z)h(z) \in \mathcal{O}(\mathbb{C})$$

and by the maximum modulus principle

$$|q(0)h(0)| \leq \max_{|z|=1} |q(z)h(z)|$$

$$\begin{aligned} \text{LHS} &= |\bar{p}_m h(0)|, \quad \text{RHS} = \max_{|z|=1} \underbrace{|\bar{p}(z) h(z)|}_{|\bar{p}(z) h(z)|} \\ &= |p_m h(0)| \end{aligned}$$

$$\text{Lemma} \Rightarrow |P_m h(z)| \leq \max_{|z|=1} |p(z)| \frac{|F(z_1, z_2, z_3, z_4)|}{|p(z)|}$$

$$\text{LHS} = \left| a_{(n_1, n_2, 0)} \frac{F(z)}{p(z)} \right| = |a| |U(z)|$$

$$\begin{aligned} \text{RHS} &= \max_{|z|=1} |F(z_1, z_2, z_3, z_4)| \\ &\leq \max_{|z|=1} C (1 + |z_1 + z_2|)^N e^{R |\operatorname{Im}(z_1 + z_2)|} \\ &\leq C (1 + |z|)^N e^{R |\operatorname{Im} z|} \end{aligned}$$

Hence $|U(z)| \in \mathcal{O}(\mathbb{C}^n)$ and $|U(z)| \leq \frac{C'}{|z|} (1 + |z|)^N e^{R |\operatorname{Im} z|}$

$PV-S \Rightarrow U = u$ where $u \in \mathcal{E}'(\mathbb{C}^n)$

and $(P(D)u, v) = \int_{\mathbb{C}^n} v \cdot \nabla u$ by inverse F-L of $P(z)u = \hat{f}$

2. Let (\mathbb{X}, μ) , (Y, ν) be measure spaces,

$k: \mathbb{X} \times Y \rightarrow \mathbb{C}$ measurable

Consider the integral operator $Kf(y) := \int_{\mathbb{X}} k(x, y) f(x) d\mu(x)$

between suitable L^p -spaces, and let

$$1 \leq p_1, q_0 \leq \infty, \quad p_0 = 1/q_0, \quad \frac{1}{p_1} + \frac{1}{p_0} = 1.$$

Show

$$(a) \|f\|_{L^{p_1}(\mathbb{X})} \|Kf(y)\|_{L^{q_0}(Y)} \leq B_0 \quad \text{for a.e. } y \in Y$$

then $\|K\|_{L(\mathbb{X}) \rightarrow L^{q_0}(Y)} \leq B_0$.

$$\|K\|_{L(\mathbb{X}) \rightarrow L^{q_0}(Y)} = \sup_{f \in L^p(\mathbb{X})} \frac{\|Kf\|_{L^{q_0}(Y)}}{\|f\|_{L^p(\mathbb{X})}} = \sup_{f \in L^p(\mathbb{X})} \|Kf\|_{L^{q_0}(Y)}$$

$$\text{Take } f \in L^p(\mathbb{X}). \text{ Then } \|Kf\|_{L^{q_0}(Y)} = \int_Y |(Kf)(y)|^{q_0} d\nu(y)$$

$f \in L^p$:

$$= \int_Y \left| \int_{\mathbb{X}} k(x, y) f(x) d\mu(x) \right|^{q_0} d\nu(y)$$

$$\text{Jensen's Ineq.} \leq \int_Y \left(\int_{\mathbb{X}} |k(x, y)|^{q_0} |f(x)| d\mu(x) \right)^{q_0} d\nu(y)$$

with $\|\cdot\|_{L^p}$ probability measure on \mathbb{X}

$$\mathbb{E}(\int_{\mathbb{X}} g d\mu) \leq \int_{\mathbb{X}} g d\mu$$

$$\text{We have } \int_{\mathbb{X}} \int_Y |(Kf)(y)|^{q_0} d\nu(y) |f(x)| d\mu(x)$$

$$\|Kf(y)\|_{L^{q_0}(Y)}^{q_0} \leq B_0^{q_0} \quad \text{a.e. } x \in \mathbb{X}$$

$$\leq B_0^{q_0} \int_{\mathbb{X}} |f(x)| d\mu = B_0^{q_0} \|f\|_{L^p(\mathbb{X})} = B_0^{q_0}$$

$$\Rightarrow \text{Fubini} \Rightarrow \int_Y \left(\int_{\mathbb{X}} |k(x, y)|^{q_0} |f(x)| d\mu(x) \right) d\nu(y)$$

$$= \int_{\mathbb{X}} \left(\int_Y |(Kf)(y)|^{q_0} d\nu(y) \right) |f(x)| d\mu(x) \leq B_0^{q_0}$$

$$\Rightarrow \|Kf\|_{L^{q_0}(Y)} \leq B_0$$

$$\Rightarrow \|K\|_{L(\mathbb{X}) \rightarrow L^{q_0}(Y)} \leq B_0$$

for $q_0 < \infty$

b) Show: If $\|k(\cdot, y)\|_{L^p(\mathbb{X})} \leq B_1$ for a.e. $y \in Y$
 then $\|K\|_{L^p(\mathbb{X}) \rightarrow L^\infty(Y)} \leq B_1$

$$q_0 = \infty: \|K\| = \sup_{\|f\|_p=1} \text{ess sup}_{y \in Y} |Kf(y)|$$

$$\|K\|_{L^\infty(Y)} \leq B_0 \Rightarrow |Kf(y)| \leq \int |k(x, y)| |f(x)| dm(dx) \leq B_0 \|f\|_p = B_0$$

a.e. x
 b.o.c.t.

b) if: Let $f \in L^p(\mathbb{X})$, $\|f\|_{L^p(\mathbb{X})} = 1$

$$\|Kf\|_{L^\infty(Y)} = \text{ess sup}_{y \in Y} |Kf(y)|$$

$$= \text{ess sup}_{y \in Y} \left| \int k(x, y) f(x) dm(dx) \right|$$

$$\leq \text{ess sup}_{y \in Y} \left\{ \int |k(x, y)| |f(x)| dm(dx) \right\} \leq B_0$$

$$\text{Hölder} \Rightarrow \leq \|k(\cdot, y)\|_{L^{p'}(\mathbb{X})} \|f\|_{L^p(\mathbb{X})} \leq B_1 \cdot 1$$

$$\Rightarrow \|K\|_{L^p(\mathbb{X}) \rightarrow L^\infty(Y)} = \sup_{\|f\|_p=1} \|Kf\|_{L^\infty(Y)} \leq B_1$$

c) Conclude that $K: L^{p_0}(\mathbb{X}) \rightarrow L^{q_0}(\mathbb{Y})$ is bounded for

$$p_0 := (1-\theta) \underbrace{\frac{1}{p}}_{p_0} + \theta \underbrace{\frac{1}{p'}}_{p_0} \quad \text{and} \quad \frac{1}{q_0} := (1-\theta) \underbrace{\frac{1}{q}}_{q_0} + \theta \underbrace{\frac{1}{q'}}_{q_0}, \quad \theta \in [0, 1]$$

We have $K: L^{p_0}(\mathbb{X}) + L^{p_1}(\mathbb{X}) \rightarrow L^{q_0}(\mathbb{Y}) + L^{q_1}(\mathbb{Y})$

$$\text{with } \|K\|_{L^{p_0}(\mathbb{X}) \rightarrow L^{q_0}(\mathbb{Y})} \leq B_0$$

$$\|K\|_{L^{p_1}(\mathbb{X}) \rightarrow L^{q_1}(\mathbb{Y})} \leq B_1$$

Riesz-Thorin $\Rightarrow K: L^{p_0}(\mathbb{X}) \rightarrow L^{q_0}(\mathbb{Y})$ bdd. with
 norm $\leq B_0^{1-\theta} B_1^\theta$ for $0 \leq \theta \leq 1$

3. Let $1 < p, q, r \leq \infty$ s.t. $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$
 Apply Schur's test to $k(x, y) := g(y-x)$ to conclude
 $\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}$ (Young's Ineq.)

Let $X = \mathbb{R}^n \setminus \{0\}$ measure space, $k: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$ measurable (g meas.).

$$R_g(x) := g * f, \quad f \in L^p(\mathbb{R}^n)$$

$$(R_g f)(y) = (g * f)(y) = \int_{X \setminus \{y\}} g(y-x) f(x) dx$$

$$\|k(x, y)\|_{L^{q_0}(X)} = \|g(x-y)\|_{L^{q_0}(X)} = \|g\|_{L^q(\mathbb{R}^n)} \quad \text{for } q_0 = q$$

$$\|k(x, y)\|_{L^{p_1}(X)} = \|g(x-y)\|_{L^{p_1}(X)} = \|g\|_{L^q(\mathbb{R}^n)} \quad \text{for } p_1 = q$$

$$\Rightarrow R_g: L^p \rightarrow L^q, \quad P_0 = (1 - \frac{1}{p})^{-1}, \quad P_1 = (1 - \frac{1}{q})^{-1}$$

$$P_0 = 1, \quad P_1 = (\frac{1}{p} - \frac{1}{q})^{-1}, \quad q_0 = q, \quad q_1 = \infty$$

$$\|R_g\|_{L^{p_0} \rightarrow L^{q_0}} \leq P_0^{1-\theta} P_1^\theta = \|g\|_{L^q} \quad \text{for } \theta \in [0, 1]$$

$$\text{i.e. } \|R_g\|_{L^{q_0}} \leq \|g\|_{L^q} \|f\|_{L^{p_0}}$$

$$\text{Take } P_0 = (1-\theta) + \theta \frac{1}{p}, \quad P_1 = (1-\theta) + \theta (\frac{1}{p} - \frac{1}{r}) = \frac{1}{p}$$

$$\Rightarrow \theta (\underbrace{\frac{1}{p} - \frac{1}{r}}_{-\frac{1}{q}} - 1) = \underbrace{\frac{1}{p} - 1}_{\frac{1}{q} + \frac{1}{r}} \quad \Rightarrow \theta = 1 + \frac{1}{r}$$

$$\Rightarrow \frac{1}{q_0} = (1-\theta) \frac{1}{q_1} = (1-\theta) \frac{1}{q} = \frac{1}{r}$$

$$\text{Hence, } \|f * g\|_{L^r} = \|R_g f\|_{L^{q_0}} \leq \|g\|_{L^q} \|f\|_{L^p}$$

$$\text{for } \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$$

4. Let (\mathcal{X}, μ) meas. space, $f: \mathcal{X} \rightarrow \mathbb{C}$ measurable.

Def. $\lambda_f: [0, \infty) \rightarrow [0, \infty]$

$$\lambda_f(\epsilon) := \mu(\{x \in \mathcal{X} : |f(x)| > \epsilon\})$$

a) Check: 1) $|f(x)|^p = p \int_0^\infty \mathbb{1}_{\{|f(x)| \geq t\}} t^{p-1} dt$

2) $\|\lambda_f\|_{L^p(\mathbb{R})}^p = p \int_0^\infty \lambda_f(\epsilon) \epsilon^{p-1} d\epsilon, \quad 1 \leq p < \infty$

3) $\|\lambda_f\|_{L^\infty(\mathbb{R})} = \inf \{\epsilon > 0 : \lambda_f(\epsilon) = 0\}$

1) RHS = $\int_0^\infty p \epsilon^{p-1} \mathbb{1}_{\{|f(x)| \geq \epsilon\}} d\epsilon$
 $= \int_0^{|f(x)|} p \epsilon^{p-1} d\epsilon = [p \epsilon^p]_{\epsilon=0}^{|f(x)|} = |f(x)|^p = \text{LHS}$
 $(\text{RHS} = 0 \iff |f(x)| = 0)$

2) $\|\lambda_f\|_{L^p(\mathbb{R})}^p = \int_{\mathcal{X}} |f(x)|^p d\mu(x) = \int_{\mathcal{X}} p \int_0^\infty \mathbb{1}_{\{|f(x)| \geq t\}} t^{p-1} dt d\mu(x)$
 $= \text{positive } \int_0^\infty p \int_{\mathcal{X}} \mathbb{1}_{\{|f(x)| \geq t\}} d\mu(x) t^{p-1} dt$
 $\lambda_f(\epsilon) = \int_0^\infty \lambda_f(t) t^{p-1} dt$
 $\|\lambda_f\|_{L^p(\mathbb{R})}^p = \infty \iff \int_0^\infty \lambda_f(t) t^{p-1} dt = \infty$

3) $\|\lambda_f\|_{L^\infty(\mathbb{R})} = \text{ess sup}_{x \in \mathcal{X}} |\lambda_f(x)| = \inf \{\epsilon > 0 : \mu(\{x \in \mathcal{X} : \lambda_f(x) > \epsilon\}) = 0\}$
 $= \inf \{\epsilon > 0 : \lambda_f(\epsilon) = 0\}$

b) Show Chebychev's Ineq. $\lambda_p(\epsilon) \leq \epsilon^{-p} \|f\|_{L^p(\mathbb{R})}^p$

$$\begin{aligned}\lambda_p(\epsilon) &= \left[\lambda_p(\epsilon) \epsilon^p \right]_{\epsilon=0}^{s=\epsilon} = \int_0^\epsilon \overbrace{\lambda_p(s)}^{\leq \lambda_p(s)} s^{p-1} ds \\ &\leq \int_0^\infty \lambda_p(s) s^{p-1} ds \stackrel{(a)}{=} \|f\|_{L^p(\mathbb{R})}^p\end{aligned}$$

c) Show that for $1 \leq p \leq \infty$ $\exists c_p, \bar{c}_p$

$$c_p \|f\|_{L^p(\mathbb{R})} \leq \left\| (2^n \lambda_p(2^n))_{n \in \mathbb{Z}} \right\|_{l^p(\mathbb{Z})} \leq \bar{c}_p \|f\|_{L^p(\mathbb{R})}$$

$$\left(\sum_{n \in \mathbb{Z}} 2^{np} \lambda_p(2^n) \right)^{\frac{1}{p}}$$

for $f(x)$

$$\text{def. } \sum_{n \in \mathbb{Z}} 2^{np} \lambda_p(2^n) = \sum_{n \in \mathbb{Z}} (2^n)^p \cdot \nu(\{x \in \mathbb{R} : |f(x)| \geq 2^n\})$$

$$= \sum_{n \in \mathbb{Z}} \int_{\{x \in \mathbb{R} : |f(x)| \geq 2^n\}} (2^n)^p \, d\nu(x)$$

$$\sum_{n \in N} \int_{\{x \in \mathbb{R} : |f(x)| \geq 2^n\}} (2^n)^p \, d\nu(x) = \sum_{n \in N} \dots \quad (1)$$

$$J_N := \sum_{n \in N} (2^n) \nu(\{x \in \mathbb{R} : |f(x)| \geq 2^n\})$$

$$= \sum_{k=n}^{\infty} \nu(\{x \in \mathbb{R} : 2^{k+1} |f(x)| \geq 2^k\})$$

$$\|f\|_p^p = \int_{\mathbb{R}} |f(x)|^p \, d\nu = \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} |f(x)|^p \mathbb{1}_{\{2^{n+1} |f(x)| \geq 2^n\}} \, d\nu$$

$$\leq \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} (2^{n+1})^p \mathbb{1}_{\{2^{n+1} |f(x)| \geq 2^n\}} \, d\nu$$

$$= 2^p \sum_{n \in \mathbb{Z}} 2^{np} \nu(\{2^{n+1} |f(x)| \geq 2^n\})$$

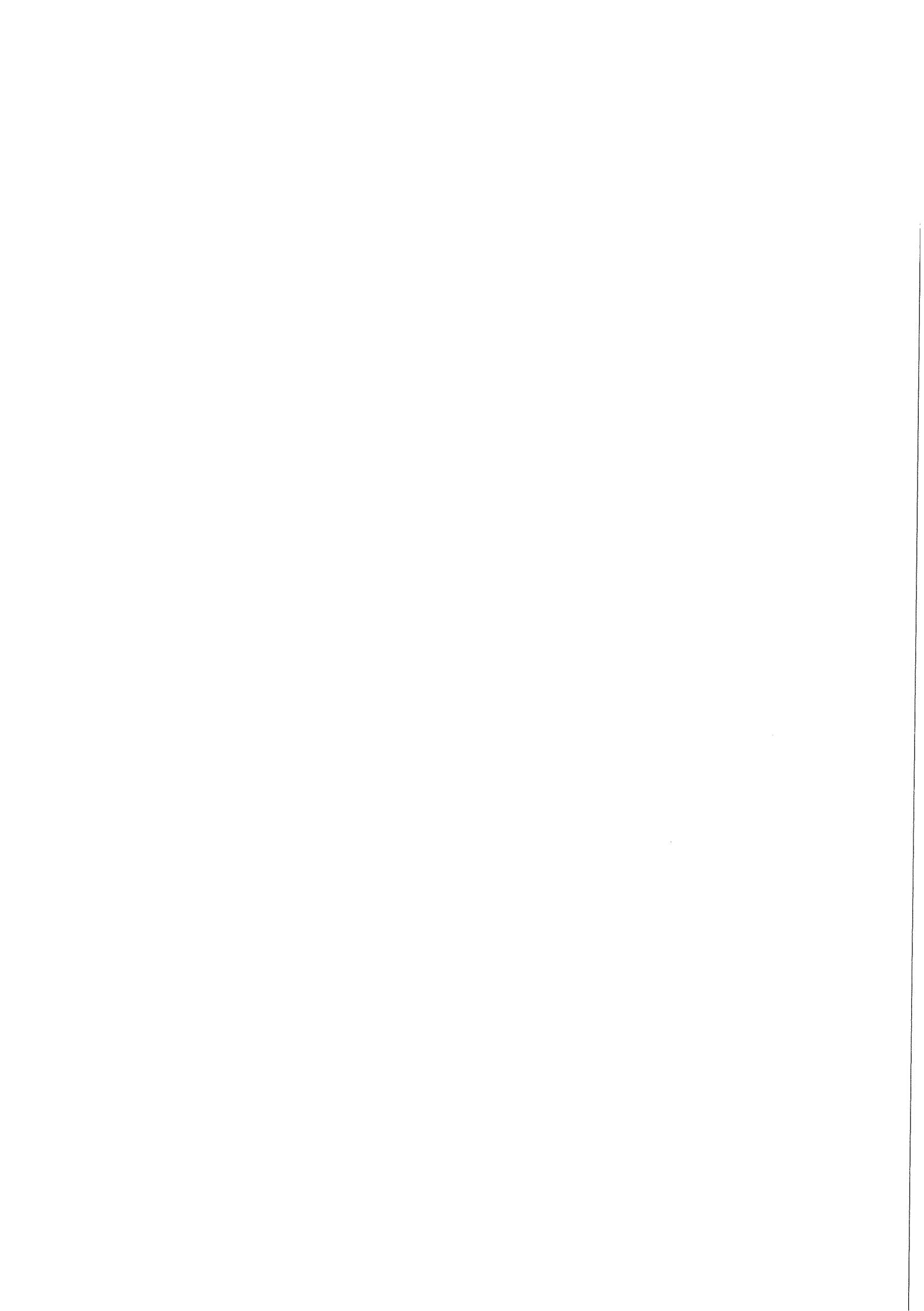
$$\leq 2^p \sum_{n \in \mathbb{Z}} 2^{np} \nu(\{|f(x)| \geq 2^n\}) = 2^p \| (2^n \lambda_p(2^n))_{n \in \mathbb{Z}} \|_p^p$$

$$\geq \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} (2^{np}) \mathbb{1}_{\{2^{n+1} |f(x)| \geq 2^n\}} \, d\nu$$

$$\mathbb{1}_{\{2^{n+1} |f(x)| \geq 2^n\}} \mathbb{1}_{\{|f(x)| \geq 2^n\}}$$

$$\Rightarrow c_p = 2^{-1}$$





$$\sum_{n \in \mathbb{Z}} 2^{np} \mathbb{1}_{\{\|f(x)\| \geq 2^n\}} = \sum_{n \in \mathbb{Z}} 2^{np} \sum_{k \geq n} \mathbb{1}_{\{2^{k+1} \|f(x)\| \geq 2^{k+1}\}}$$

$$\left(\sum_{k \in \mathbb{Z}} 2^{kp} \mathbb{1}_{\{\|f(x)\| \geq 2^k\}} \right) = \sum_{k \in \mathbb{Z}} \sum_{n \geq k} 2^{np} \mathbb{1}_{\{2^{k+1} \|f(x)\| \geq 2^{k+1}\}}$$

$$= \sum_{k \in \mathbb{Z}} 2^{kp} \sum_{n=0}^{\infty} 2^{np} \mathbb{1}_{\{(1-2^{-p})^n \leq 2^{k+1} \|f(x)\| < 2^{k+2}\}}$$

$$\sum_{n \in \mathbb{Z}} 2^{np} \mathbb{1}_{\{\|f(x)\| \geq 2^n\}} = (1-2^{-p})^{-1} \sum_{k \in \mathbb{Z}} 2^{kp} \mathbb{1}_{\{2^{k+1} \|f(x)\| \geq 2^{k+1}\}}$$

$$\leq (1-2^{-p})^{-1} \sum_{k \in \mathbb{Z}} \left(\sum_{n \geq k} 2^{np} \mathbb{1}_{\{2^{k+1} \|f(x)\| \geq 2^{k+1}\}} \right)^p$$

$$\leq (1-2^{-p})^{-1} \|f\|_p^p$$

here we take
have done case

alt:

$$\|f\|_{L_p}^p = p \int_0^\infty t^p \lambda_p(t) \frac{dt}{t} = \int_{\ln 2}^\infty e^{-ts} \left\{ \begin{array}{l} t = e^s \\ dt = e^s ds \end{array} \right\} = p \ln 2 \int_{-\infty}^0 2^{sp} \lambda_p(2^s) ds$$

$$\leq p \ln 2 \cdot 2^p \sum_{n \in \mathbb{Z}} 2^{np} \lambda_p(2^n)$$

$$\leq p \ln 2 \cdot 2^p \sum_{n \in \mathbb{Z}} 2^{np} \mathbb{1}_{\{\|f(x)\| \geq 2^n\}}$$

$\Rightarrow \|f\|_{L_p} \leq C \left(\ln 2 \cdot \lambda_p(2^m) \right)^{\frac{1}{p}}$

a) Let $L^{p,\infty}(\mathbb{R}) := \{ f: \mathbb{R} \rightarrow \mathbb{C} \text{ measurable} : \|f\|_{L^{p,\infty}(\mathbb{R})} := \sup_{t>0} t \lambda_p(t) < \infty \}$

D) Show $L^p(\mathbb{R}) \subseteq L^{p,\infty}(\mathbb{R})$

Let $\rho \in L^p \Rightarrow \|f\|_{L^p} < \infty \quad (f: \mathbb{R} \rightarrow \mathbb{C} \text{ measurable})$

Then by b) (Chebychev):

$$t \lambda_p(t) \leq \|f\|_p^p < \infty$$

$$\Rightarrow \|f\|_{L^{p,\infty}} = \sup_{t>0} t \lambda_p(t)^p = \sup_{t>0} (t \lambda_p(t))^p \leq \|f\|_p^p < \infty$$

2) If $\mathbb{X} = \mathbb{R}^n$ with Leb. meas., show that

$$f(x) := |x|^{-\frac{n}{p}} \in L^{p,\infty}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n), \quad 1/p < \infty$$

(f is σ -algebra measurable)

$f \notin L^p(\mathbb{R}^n)$:

$$\begin{aligned} \|f\|_{L^p}^p &= \int_{\mathbb{R}^n} |f|^p dx = \int_{\mathbb{R}^n} |x|^{-np} dx = \int_{\mathbb{R}^n} r^{-np} dr \\ &= \int_{\mathbb{R}^n} r^{n-1} r^{-np} dr = \int_0^\infty r^{n-1-p} dr = \infty \end{aligned}$$

$f \in L^{p,\infty}(\mathbb{R}^n)$:

$$\lambda_p(t) = \mu(\{x \in \mathbb{R}^n : |f(x)| \geq t\})$$

$$= \mu(\{x \in \mathbb{R}^n : |x|^{-\frac{n}{p}} \geq t\})$$

$$= \mu(\{x \in \mathbb{R}^n : M \leq t^{-\frac{p}{n}}\})$$

$$= \mu(B_{t^{-\frac{p}{n}}}(\mathbf{0})) = |\mathbb{B}^n| (t^{-\frac{p}{n}})^n = |\mathbb{B}^n| t^{-p}$$

$$\Rightarrow \|f\|_{L^{p,\infty}(\mathbb{R}^n)} = \sup_{t>0} t \lambda_p(t)^p = \sup_{t>0} t \cdot |\mathbb{B}^n|^{\frac{1}{p}} t^{-p} = |\mathbb{B}^n|^{\frac{1}{p}} < \infty$$

$\therefore f \in L^{p,\infty}(\mathbb{R}^n)$

Smooth cut-offs

Def: $\alpha \in \mathcal{S}(\mathbb{R}^d)$, open

ϕ_α at $\rightarrow \infty$ invertible affine

ϕ_α is called a bump func adapted to $L(\mathbb{R})$ if

$$\phi_\alpha \in C^\infty(\mathbb{R}^d)$$

$$\text{supp } \phi_\alpha \subseteq L_\alpha(\mathbb{R})$$

$$\text{and } \forall k \in \mathbb{N} \exists C_{k,\alpha}, \forall x: \sup_{x \in \mathbb{R}^d} |\partial_k \phi_\alpha(x)| \leq C_{k,\alpha}$$

Prop:

$$k \in L^1(\mathbb{R}^d \times \mathbb{R}^{d'})$$

$$f, g \in [1, \infty]$$

(deg. of ac family char.)

$$(T_k: L^p(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^{d'})) \text{ bounded}$$

$$L \subset \mathbb{R}^d, L' \subset \mathbb{R}^{d'} \text{ bdl open, } L' \text{ affine, } \phi: \mathbb{R}^{d+d'} \rightarrow \mathbb{C}$$

$$\text{bump func adapted to } (L \times L') \Rightarrow \|T_k \phi\|_{p \rightarrow q} \leq C_{p,q,d,d'} \|T_k\|_{p \rightarrow q}$$

Notice: unlike char. fun. considered before, the smooth function do not affect boundedness properties

Proof: Considering instead of $\mathbb{R}^d: L(\mathbb{R}), L(\mathbb{R}')$ \Rightarrow WLOG $L, L' = \mathbb{R}$

(ch. of basis)

$$\text{Wlog } L \times L' \subseteq [\frac{-1}{4}, \frac{1}{4}]^{d+d'}$$

$$\sup_{(x,y)} \phi \subseteq L \times L' \subseteq [-\frac{1}{4}, \frac{1}{4}]^{d+d'} \rightarrow \text{exacts of periodically with period 1}$$

call it $\tilde{\phi}$

$\tilde{\phi}$ periodic on $\mathbb{R}^{d+d'}$ \rightarrow Fourier series

$$\tilde{\phi}(x,y) = \sum_{n,m} c_{nm} e^{2\pi i(nx+my)}, c_{nm} \text{ Fourier coeff. of } \tilde{\phi}$$

Standard theory (int. by parts / $S \xrightarrow{*} S'$): $|c_{nm}| \leq C_k (1+n+l+m)^{-k}$

since $\tilde{\phi} \in C_0^\infty$

2) Weierstrass test for uniform conv.

$$\phi(x,y) = \sum_{n,m} c_{nm} e^{2\pi i(nx+my)} \text{ conv. uniformly}$$

$n=0$ holds for all $(x,y) \in \mathbb{R}^{d+d'}$

$$k(x,y) \phi(x,y) = \sum_{n,m} c_{nm} \prod_{l=1}^{d+d'} [\frac{x_l}{a_l} + \frac{y_l}{b_l}]^{d(l)} e^{2\pi i n x} \phi(x,y) \prod_{l=1}^{d+d'} [\frac{x_l}{a_l} + \frac{y_l}{b_l}]^{d(l)} e^{2\pi i m y}$$

$$T_k \phi = \sum_{n,m} c_{nm} C_m T_k C_n$$

$$C_m f(y) = e^{2\pi i m y} \prod_{l=1}^{d+d'} [\frac{y_l}{b_l}]^{d(l)}$$

correction up to
sm. for C_n

$\|T_{\text{req}}\| \leq E_{\text{lenm}} \|C_m\| \quad \|T_{\text{all}}\| \|C_m\| \leq$

$\|C_m\|$

$\leq \epsilon$

$< \infty$

Exercises 3 – Integral operators and inequalities

①

For $\varphi \in L^1(-1,1)$ let $(F\varphi)(\lambda) := \int_{-1}^1 e^{i\lambda x} \varphi(x) dx$.

a) If $\varphi \in C_c^\infty(-1,1)$, show that $|F\varphi(\lambda)| \leq C_k (1+|\lambda|)^{-k}$ $\forall k \in \mathbb{N}$.

b) If φ is the restriction of a function $\in C^\infty(\mathbb{R})$, show that

$$\left| F\varphi(\lambda) - \frac{e^{i\lambda}}{i\lambda} \varphi(1) + \frac{e^{-i\lambda}}{i\lambda} \varphi(-1) \right| \leq C |\lambda|^2 \quad \text{for } |\lambda| \geq 1.$$

Can you find an approximation $\sum_{k=1}^N \frac{f(\varphi)}{\lambda^k}$ to $F\varphi(\lambda)$, which is correct up to an error $\leq \lambda^{-N-1}$?

Hint: Integration by parts!

Remark: Kim will discuss related issues later in this course.

②

Let $\Omega = \{(x,y) \in \mathbb{R}^2 : \operatorname{Re} e^{-ixy} > 0\}$. Show that the integral operator associated to the kernel $k(x,y) = e^{-ixy} \mathbf{1}_{\Omega}(x,y)$ is a truncation of the Fourier transform, which is not bounded on $L^2(\mathbb{R})$.

Hint: Show that $\frac{\operatorname{Re} \langle T_k \mathbf{1}_I, \mathbf{1}_I \rangle_{L^2(\mathbb{R})}}{\langle \mathbf{1}_I, \mathbf{1}_I \rangle_{L^2(\mathbb{R})}}$ is large for large intervals $I \subset \mathbb{R}$.

(③) a) Let $K: (0, \infty)^2 \rightarrow \mathbb{C}$ be a measurable function satisfying

$$K(\lambda x, \lambda y) = \lambda^p K(x, y) \text{ for } \lambda > 0 \text{ and } \int_0^\infty |K(x, t)| t^{-p} dt < \infty.$$

Show that the operator $T_K f(y) = \int_0^\infty K(x, y) f(x) dx$

is a bounded operator on $L^p(0, \infty)$ and $\|T_K\|_{p \rightarrow p} \leq C$.

Hint: Write $T_K f(y) = \int_0^\infty K(x, 1) f(xy) dx$ and use the triangle inequality for L^p -norms (Minkowski's inequality).

b) Use a) to show that the "Hilbert integral" given by $K(x, y) = \frac{1}{x+y}$ defines a continuous operator on $L^p(0, \infty)$.

c) Use a) to deduce Hardy's inequalities

$$\left(\int_0^\infty dx \left(\int_0^x dy f(y) \right)^p x^{-r-1} \right)^{1/p} \leq \frac{p}{r} \left(\int_0^\infty dy (y f(y))^p y^{-r-1} \right)^{1/p}$$

$$\left(\int_0^\infty dx \left(\int_x^\infty dy f(y) \right)^p x^{r+1} \right)^{1/p} \leq r \left(\int_0^\infty dy (y f(y))^p y^{r+1} \right)^{1/p}$$

where $f \geq 0, p \geq 1, r \geq 0$.

Notes: $p=2, r=1, f(y)=y^{\alpha} g(y)$
 $g(0)=0$
or $r=p-1, p>1$

(4) Slightly more sophisticated than ③ and Ex. 2.2 is the weak-type Schur test (use Marcinkiewicz instead of Riesz-Thorin): If $k: X \times Y \rightarrow \mathbb{C}$ is measurable and $\|k(x, y)\|_{L^{q_0/\infty}(Y)} \leq B_0$,

for a.e. $x \in X$, $\|k(x, \cdot)\|_{L^{p_1, \infty}(Y)} \leq B_1$, for a.e. $y \in Y$, $p_1, q_0 \in (1, \infty)$

$\Rightarrow \forall \theta \in (0, 1) : T_k : L^{p_\theta}(X) \rightarrow L^{q_\theta}(Y)$ bounded for

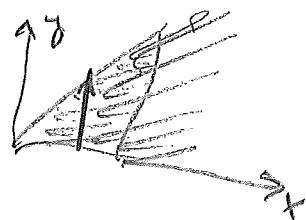
$$\frac{1}{p_\theta} = 1 - \theta + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} \quad \text{and} \quad \|T_k\|_{p_\theta \rightarrow q_\theta} \leq C_{p_1, q_0, \theta} B_0^{1-\theta} B_1^\theta.$$

Use this to show that if $\mapsto |x|^{-\alpha} * f$ defines a bounded operator $L^p(\mathbb{R}^n) \rightarrow L^r(\mathbb{R}^n)$ for $1 \leq p, r < \infty$, $0 < \alpha < n$, $\frac{\alpha}{p} + \frac{n}{r} = \frac{1}{p} + 1$.

$$\star \int_0^x dy f(y) x^{-\frac{r-1}{p}}$$

$$\frac{\int_0^\infty}{\int_0^\infty x^{-\frac{r-1}{p}} \prod_{j=1}^r (x-y_j)^{-\frac{1}{p}} f(y_j) dy_j}$$

$$\Omega = \{(x, \beta) : 0 < \beta < \alpha\}$$



$$\int_0^x$$

$$K(x, y) = y^{-\frac{r-1}{p}} \mathbb{1}_{\Omega}(x, y) = y^{-\frac{r-1}{p}} \mathbb{1}_{\Omega}(x, y) \cdot y^{-\frac{r-1}{p}}$$

$$\Rightarrow K(\lambda x, \lambda y) = \lambda^{-\frac{r-1}{p}} K(x, y)$$

either: a) generalizes to operators of this sort
of homogeneity

b)

Take this as a weight.

$$p > 1$$

$$p = r + 1$$

$$\left[\frac{1}{p-1} \right]$$

$|x|^{-\alpha} \in L^{p, \infty}$? stated in Exercise 2.4

Box 3

1. For $\varphi \in L^1(-1,1)$ let $(F\varphi)(z) := \int_{-1}^1 e^{izx} \varphi(x) dx$

a) If $\varphi \in C_0^\infty(-1,1)$, show that $|F\varphi(z)| \leq C_k (1+|z|)^{-k}$ $\forall k \in \mathbb{N}$

PF: Note: $\varphi \in C_0^\infty(\mathbb{R}) \subseteq \mathcal{S}$

$$\cdot (\mathcal{F}\varphi)(z) = \int_{-\infty}^{\infty} e^{-ixz} \varphi(x) dx = \int_{-1}^1 e^{-ixz} \varphi(x) dx$$

$$\Rightarrow (F\varphi)(z) = (\mathcal{F}\varphi)(z) = \int_{-1}^1 e^{izx} \varphi(x) dx \quad \forall z \in \mathbb{C}$$

$$f: \mathcal{S} \rightarrow \mathcal{S} \Rightarrow |F\varphi(z)| = |\mathcal{F}\varphi(z)| \leq C_k (1+|z|)^{-k} \text{ when } \varphi \in C_0^\infty(\mathbb{R})$$

b) Let φ be the restriction of a function $\in C^\infty(\mathbb{R})$, i.e. $\varphi = \Phi|_{[-1,1]}$
 $\Phi \in C^\infty(\mathbb{R})$

$$\text{Show that } |F\varphi(z)| = \frac{e^{iz}}{iz} \varphi(1) + \frac{e^{-iz}}{iz} \varphi(-1) \leq C |z|^2$$

for $|z| \geq 1$

PF: $F\varphi(z) = \int_{-1}^1 e^{izx} \varphi(x) dx = \int_{-1}^1 \frac{1}{iz} \frac{d}{dx}(e^{izx}) \varphi(x) dx$

$$= - \int_{-1}^1 \frac{1}{iz} \frac{d}{dx}(e^{izx}) dx + \left[\frac{1}{iz} e^{izx} \varphi(x) \right]_{x=-1}^1$$

$$= - \frac{1}{iz} \int_{-1}^1 e^{izx} \varphi'(x) dx + \frac{1}{iz} e^{iz} \varphi(1) - \frac{1}{iz} e^{-iz} \varphi(-1)$$

$$\Rightarrow F\varphi(z) = \frac{e^{iz}}{iz} \varphi(1) + \frac{e^{-iz}}{iz} \varphi(-1) = - \frac{1}{iz} \int_{-1}^1 e^{izx} \varphi'(x) dx$$

Repeat (*) with $\varphi' \rightarrow \varphi$ $\Rightarrow \int_{-1}^1 e^{izx} \varphi'(x) dx$

$$= - \frac{1}{iz} \int_{-1}^1 e^{izx} \varphi''(x) dx + \frac{1}{iz} e^{iz} \varphi'(1) - \frac{1}{iz} e^{-iz} \varphi'(-1)$$

$$\Rightarrow |F\varphi(z)| = \left| \frac{e^{iz}}{iz} \varphi(1) + \frac{e^{-iz}}{iz} \varphi(-1) \right| \leq \frac{1}{|z|} \underbrace{\left| \int_{-1}^1 e^{izx} \varphi'(x) dx \right|}_{\leq \frac{1}{|z|} \left(\int_{-1}^1 |\varphi''(x)| dx + |\varphi'(1)| + |\varphi'(-1)| \right)} \leq \frac{C}{|z|^2}$$

Comparing with $(*)$, $(***)$, etc.

$$\int_{-1}^1 e^{izx} \varphi^{(k)}(x) dx = \frac{1}{iz} \left(\int_{-1}^1 e^{izx} \varphi^{(k+1)}(x) dx - e^{iz} \varphi^{(k)}(1) + e^{-iz} \varphi^{(k)}(-1) \right)$$

$$\Rightarrow F\varphi(z) = \sum_{k=0}^N \left(\frac{1}{iz} \right)^{k+1} \left(-e^{iz} \varphi^{(k)}(1) + e^{-iz} \varphi^{(k)}(-1) \right) + \left(\frac{1}{iz} \right)^{N+1} \int_{-1}^1 e^{izx} \varphi^{(N+1)}(x) dx \xrightarrow{\text{error}} O(|z|^{-N-1})$$

2. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : \operatorname{Re} e^{-\alpha xy} > 0\}$.

Show that the integral operator T_k associated

to the kernel $k(x, y) = e^{-\alpha xy} \mathbb{1}_{\Omega}(x, y)$

is a truncation of the Fourier transf.

which is not bounded on $L^2(\mathbb{R})$.

Hint: consider $f = \mathbb{1}_I$, $I \subseteq \mathbb{R}$ interval $\rightarrow \infty$

$$\text{say } f = \mathbb{1}_{[0, L]}$$

$$\|f\|_2^2 = \int_{-\infty}^{\infty} \mathbb{1}_{[0, L]}(x) dx = \int_0^L dx = L$$

$$(T_k f)(y) = \int_{-\infty}^{\infty} k(x, y) f(x) dx$$

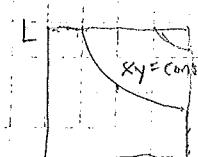
$$\langle f, T_k f \rangle_{L^2} = \int_{-\infty}^{\infty} f(y) T_k f(y) dy = \int_{-\infty}^{\infty} \mathbb{1}_{[0, L]}(y) \int_{-\infty}^{\infty} k(x, y) \mathbb{1}_{[0, L]}(x) dx dy$$

$$\operatorname{Re} \langle f, T_k f \rangle_{L^2} = \int_{-\infty}^{\infty} \mathbb{1}_{[0, L]}(y) (\underbrace{\int_{-\infty}^{\infty} (\operatorname{Re} e^{-\alpha xy}) \mathbb{1}_{\Omega}(x, y) dx}_{\cos xy} dy) \geq 0$$

To show

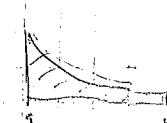
$$= \int_{\mathbb{R}^2} \mathbb{1}_{[0, L]}(y) (\operatorname{Re} e^{-\alpha xy}) \mathbb{1}_{\Omega}(x, y) \mathbb{1}_{[0, L]}(x) dx dy$$

$$= \int_{\substack{0 \leq x, y \leq L \\ \cos xy > 0}} \cos xy dx dy$$

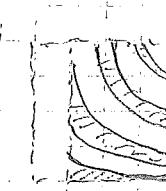


$$\geq \int_{\substack{0 \leq x, y \leq L \\ xy < 0}} \cos 1 dx dy$$

$$\geq \frac{1}{2} \int_{x=0}^L \int_{y=0}^{L-x} \cos 1 dy dx = \frac{1}{2} \int_{x=0}^L \frac{1}{x} dx = \frac{1}{2} \ln L$$



$$\geq \sum_{k=0}^{\infty} \int_{\substack{0 \leq x, y \leq L \\ -\frac{\pi}{4} \leq xy - 2\pi k \leq \frac{\pi}{4}}} \cos \frac{\pi}{4} dy dx$$



$$\geq \frac{1}{2} \int_{x=1}^L \sum_{k=0}^{\infty} \int_{\substack{0 \leq y \leq L \\ -\frac{\pi}{4} \leq xy - 2\pi k \leq \frac{\pi}{4}}} dy dx$$

$$\Rightarrow y \in [\frac{1}{x}(2\pi k - \frac{\pi}{4}), \frac{1}{x}(2\pi k + \frac{\pi}{4})]$$

$$\geq \sum_{k=0}^{\infty} \frac{xL}{2\pi} \frac{\pi L}{2x} = \frac{\pi L}{2\pi} = \frac{L}{4}$$

$$= \frac{1}{2} \cdot \frac{L}{4} \int_1^L dx \geq c \cdot L(L-1)$$

$$\Rightarrow \frac{|\operatorname{Re} \langle f, T_k f \rangle_2|}{\|T_k\|_2^2} \geq \frac{c(L-1)}{L} = c(L-1) \rightarrow \infty \text{ as } L \rightarrow \infty$$

But if $\|T_k\|_{L^2 \times L^2} < \infty$ then by Cauchy-Schwarz

$$|\operatorname{Re} \langle f, T_k f \rangle_2| \leq |\langle f, T_k f \rangle_2| \leq \|f\|_2 \|T_k f\|_2$$

$$\leq \|T_k\|_{L^2 \times L^2} \|f\|_2^2 \Rightarrow \frac{|\operatorname{Re} \langle f, T_k f \rangle_2|}{\|T_k\|_2^2} \leq \|T_k\|_{L^2 \times L^2} < \infty$$

$\Rightarrow T_k$ not a bounded operator $L^2 \times L^2$

T_k truncation of \mathcal{T} : iteration

$$(T_k f)(y) = \int_{-\infty}^{\infty} e^{-ixy} \frac{1}{\pi} k(x, y) f(x) dx$$

kernel for \mathcal{T}

What-type Schur test:

If $\kappa \in C_c^\infty(\mathbb{R})$ is measurable

$$\cdot \|\kappa(\cdot, \cdot)\|_{L^{q_0, \infty}(Y)} \leq B \quad \text{for a.e. } x \in \mathbb{R}$$

$$\cdot \|\kappa(\cdot, y)\|_{L_1^{p_0, \infty}(x)} \leq B \quad \text{for a.e. } y \in Y$$

then $\forall \theta \in (0, 1)$ $T_\theta : L^{p_0}(\mathbb{R}) \rightarrow L^{q_0}(Y)$ bdd.

$$\text{for } \theta_0 = 1 - \theta + \frac{\theta}{p_1}, \quad \frac{1}{q_0} = \frac{1-\theta}{\theta_0} \quad \text{with } \|T_\theta\|_{L^{p_0, q_0}} \leq C \frac{B^{1-\theta}}{\theta_0^{\theta_0}}$$

Use this to show that $T : \mathcal{F} \mapsto L^{1-\alpha} * \mathcal{F}$

defines a bounded operator $L^p(\mathbb{R}^n) \rightarrow L^r(\mathbb{R}^n)$

for $1 < p, r < \infty$, $0 < \alpha < n$, $\frac{1}{p} + \frac{n}{r} = \frac{1}{1-\alpha} + 1$

Note: $T = T_k$ with $k(x, y) = |x-y|^{1-\alpha}$

$$\begin{aligned} \text{as } (T_k f)(y) &= (1 \cdot |x|^{1-\alpha} * f)(y) = \int_{\mathbb{R}^n} |y-x|^{1-\alpha} f(x) dx \\ &= \int_{\mathbb{R}^n} |x-y|^{1-\alpha} f(x) dx \end{aligned}$$

$k: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \in \mathbb{C}$ - elliptically measurable
 $k \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$

$$\|k(x, \cdot)\|_{L^{q_0, \infty}(\mathbb{R}^n)} = \sup_{t > 0} t^{-\frac{1}{q_0}} \|k(x, \cdot)\|_{L^{q_0, \infty}}$$

$$\begin{aligned} \lambda_{k(x, \cdot)}(t) &= \mu\left(\{y \in \mathbb{R}^n : |k(x, y)| > t^{\frac{n}{\alpha}}\}\right) \\ &\Leftrightarrow |x-y|^{n-\alpha} > t \Leftrightarrow |x-y| \leq t^{\frac{\alpha}{n}} \end{aligned}$$

$$= \mu(B_{t^{\frac{\alpha}{n}}}(x)) = c_n t^{-\frac{n}{\alpha}}$$

$$\Rightarrow \|k(x, \cdot)\|_{L^{q_0, \infty}(\mathbb{R}^n)} = \sup_{t > 0} t \cdot c_n^{\frac{1}{q_0}} e^{-\frac{n}{\alpha} \frac{1}{q_0}} = c_n^{\frac{1}{q_0}} \sup_{t > 0} t^{-1 + \frac{n}{\alpha} \frac{1}{q_0}}$$

$$\text{Let } q_0 = \frac{n}{\alpha} \in (1, \infty)$$

$$\begin{aligned} \|k(\cdot, y)\|_{L^{p_1, \infty}(\mathbb{R}^n)} &= \sup_{t > 0} t \cdot \lambda_{k(\cdot, y)}(t)^{p_1} \quad \lambda_{k(\cdot, y)} = \lambda_{k(x, \cdot)} \\ &= \sup_{t > 0} c_n^{p_1} t^{-1 + \frac{n}{\alpha} p_1} < \infty \text{ iff. } p_1 = \frac{n}{\alpha} \\ &\Rightarrow \frac{1}{p_0} = 1 - \frac{1}{p_1} \in (0, 1) \end{aligned}$$

Hence, $\|k(x, \cdot)\|_{L^{q_0, \infty}(\mathbb{R}^n)}, \|k(\cdot, y)\|_{L^{p_1, \infty}(\mathbb{R}^n)} < \infty$

$$\text{for } \frac{1}{q_0} = \frac{\alpha}{n}, \quad \frac{1}{p_1} = 1 - \frac{\alpha}{n}$$

Weak-type Schur $\Rightarrow T_k: L^{p_0}(\mathbb{R}^n) \hookrightarrow L^{q_0}(\mathbb{R}^n)$ bounded

$$\text{for } \frac{1}{p_0} = 1 - \theta + \frac{\theta}{p_1} = 1 - \theta + \theta(1 - \frac{\alpha}{n}) \Rightarrow \frac{1}{p} \in (1 - \frac{\alpha}{n}, 1)$$

$$\frac{1}{q_0} = (1 - \theta) \frac{1}{q_0} = (1 - \theta) \frac{\alpha}{n} \Rightarrow \frac{1}{r} \in (0, \infty)$$

$$\frac{1}{p} + \frac{\alpha}{n} = 1 - \theta + \theta(1 - \frac{\alpha}{n}) + \frac{\alpha}{n} = 1 + (1 - \theta) \frac{\alpha}{n} = 1 + \frac{1}{r}$$

- 3.a) Let $K: (0, \infty)^2 \rightarrow \mathbb{C}$ be a measurable function satisfying
- $K(2x, 2y) = 2^{-1} K(x, y) \quad \forall x > 0$
 - $\int_0^\infty |K(x, 1)| x^{1/p} dx \leq C < \infty$

Show that the operator $T_K S(y) = \int_0^\infty K(x, y) f(x) dx$

is a bounded operator on $L^p(0, \infty)$ and $\|T_K\|_{L^p \rightarrow L^p} \leq C$.

(Let $f \in L^p(0, \infty)$,

$$(18) \quad T_K f(y) = \int_{x=0}^{\infty} K(x, y) f(x) dx \quad (y \in (0, \infty))$$

$$= \int_{x=0}^{\infty} K(y/x, 1) f(y/x) y^{-1} dx \quad t = y/x$$

$$= \int_{t=0}^{\infty} K(t, 1) f(yt) dt$$

$$\|T_K f\|_{L^p(0, \infty)} = \left(\int_{y=0}^{\infty} |T_K f(y)|^p dy \right)^{1/p}$$

$$\leq \left(\int_{y=0}^{\infty} \left(\int_{t=0}^{\infty} |K(t, 1) f(yt)| dt \right)^p dy \right)^{1/p}$$

Minkowski's
integral inequality $\leq \int_{t=0}^{\infty} \left(\int_{y=0}^{\infty} |K(t, 1) f(yt)|^p dy \right)^{1/p} dt$

$$= \int_{t=0}^{\infty} |K(t, 1)| \left(\int_{y=0}^{\infty} |f(yt)|^p dy \right)^{1/p} dt$$

$$t^{-1} \int_{y=0}^{\infty} |\mathcal{F}(ty)|^p dy = e^{-i \int_{y=0}^{\infty} \mathcal{F}(y) dy}$$

$$= \int_{t=0}^{\infty} |K(t, 1)| t^{-1/p} \|\mathcal{F}\|_{L^p(0, \infty)} dt$$

$$= C \|\mathcal{F}\|_{L^p(0, \infty)}$$

$$\Rightarrow \|T_K\|_{L^p(0, \infty) \rightarrow L^p(0, \infty)} \leq C$$

Minkowski: $\left[\int_Y \left(\int_X |F(x, y)| d\nu(x) \right)^p d\nu(y) \right]^{1/p}$

$$\leq \int_X \left[\int_Y |F(x, y)|^p d\nu(y) \right]^{1/p} d\nu(x)$$

(with equality iff $|F(x, y)| = \varphi(x)\psi(y)$ a.e., $\varphi, \psi \geq 0$)
then both sides finite.

(the "Hilbert Integral")

- b) Use a) to show that $T_k : L^p(0, \infty) \rightarrow L^p(0, \infty)$ defined by

$K(x, y) = \frac{1}{x+y} \quad \forall y \in (0, \infty)$

$p > 1$

$K(x, y) = \frac{1}{x+y}$ is a measurable function $(0, \infty)^2 \rightarrow (0, \infty) \subseteq \mathbb{C}$

$$K(x, y) = \frac{1}{x+y} = x^{-1} K(x, y) \quad \forall x > 0$$

$$C := \int_0^\infty |K(x, 1)| x^{-\frac{1}{p}} dx = \int_0^\infty \left| \frac{1}{x+1} \right| x^{-\frac{1}{p}} dx$$

$$= \int_0^\infty \underbrace{(x+1)^{-1}}_{g(x)} x^{-\frac{1}{p}} dx$$

$$\underset{x \rightarrow 0}{\lim} : g(x) \sim x^{-\frac{1}{p}} \quad \int_0^{\infty} x^{-\frac{1}{p}} dx = \frac{1}{1-p} [x^{1-p}]_0^\infty = \frac{1-p}{1-p} < \infty$$

$$\underset{x \rightarrow \infty}{\lim} : g(x) \sim x^{-\frac{1}{p}} \quad \int_R^\infty x^{-\frac{1}{p}} dx = -p [x^{-\frac{1}{p}}]_R^\infty = p R^{\frac{1}{p}} < \infty$$

$$\Rightarrow C < \infty$$

By a) $T_k : L^p(0, \infty) \rightarrow L^p(0, \infty)$ cont., $\|T_k f\|_p \leq C$

measures related to

c) Use a) to deduce Hardy's inequalities

$$(i) \left(\int_0^\infty \left(\int_x^\infty f(y) dy \right)^p x^{r-1} dx \right)^{\frac{1}{p}} \leq \frac{1}{r} \left(\int_0^\infty (y f(y))^p y^{-r} dy \right)^{\frac{1}{p}}$$

$$(ii) \left(\int_0^\infty \left(\int_x^\infty e^{cy} dy \right)^p x^{r-1} dx \right)^{\frac{1}{p}} \leq \frac{1}{r} \left(\int_0^\infty (y f(y))^p y^{-r} dy \right)^{\frac{1}{p}}$$

where $r > 0$, $p > 1$, $c > 0$.

$$(i) \quad K(x, y) := x^{-r-1} \mathbb{1}_{[0, x]}(y)$$

$$\Rightarrow T_k f(y) = \int_0^\infty K(x, y) f(x) dx =$$

$$\text{LHS} = \left(\int_0^\infty \left(x^{-\frac{r+1}{p}} \int_0^x f(y) dy \right)^p dx \right)^{\frac{1}{p}} = \|T_k f\|_p$$

$$\begin{aligned} & \left\{ \int_0^\infty x^{-\frac{r+1}{p}} \mathbb{1}_{[0, x]}(y) f(y) dy \right\} = (T_k f)(y) \\ & =: K(y, x) \end{aligned}$$

$$K(x, y) = y^{-\frac{r+1}{p}} \mathbb{1}_{[0, y]}(x)$$

$$K(\lambda x, \lambda y) = (\lambda y)^{-\frac{r+1}{p}} \mathbb{1}_{[0, \lambda y]}(\lambda x)$$

$K(\lambda x, \lambda y) = \lambda^r K(x, y)$ in a slightly broader sense.

$$\text{Let } K_\lambda(x, y) := K(\lambda x, \lambda y), \quad (\lambda^{-1} K)(x, y) = \lambda^{-r} K(x, y)$$

$$\text{then } T_{K_\lambda} = T_{\lambda^{-1} K} \Leftrightarrow (T_{K_\lambda} f)(y) = f$$

$$\begin{aligned} (T_{K_\lambda} f)(y) &= \int_0^\infty K_\lambda(x, y) f(x) dx \\ &= \int_0^\infty K(\lambda x, \lambda y) f(x) dx \\ &= \int_{x=0}^\infty (\lambda y)^{-\frac{r+1}{p}} \mathbb{1}_{[0, \lambda y]}(\lambda x) f(x) dx \end{aligned}$$

Now,

$$y \in (0, \infty)$$

$$(T_K f)(y) = \int_{x=0}^\infty K(x, y) f(x) dx = \int_{x=0}^y y^{-\frac{r+1}{p}} f(x) dx$$

$$\begin{aligned} &= y^{1-\frac{r+1}{p}} \int_{t=0}^1 f(y^{-1} t) y^{-1} dt \\ &= y^{1-\frac{r+1}{p}} \int_{t=0}^1 f(yt) dt \end{aligned}$$

$$\text{Moreover } \|T_{K_\lambda} f\|_p = \left(\int_{y=0}^\infty \left(\int_{t=0}^1 y^{1-\frac{r+1}{p}} f(yt) dt \right)^p dy \right)^{\frac{1}{p}}$$

$$\leq \int_{t=0}^1 \left(\int_{y=0}^\infty y^{1-\frac{r+1}{p}} f(yt) dy \right)^p dt$$

$$\begin{aligned} i) \text{ UBS} &= \left(\int_0^\infty \left(\int_0^y \int_{t=0}^1 f(x) dx dy \right)^p y^{-r-1} dy \right)^{\frac{1}{p}} \\ &\leq \int_{t=0}^1 y \int_{t=0}^y f(ty) dt dy \end{aligned}$$

$$= \left(\int_{y=0}^\infty \left(\int_{t=0}^y f(ty) dt \right)^p y^{p+r+1} dy \right)^{\frac{1}{p}}$$

$$\leq \int_{t=0}^1 \left(\int_{y=0}^\infty f(ty)^p t dy \right)^{\frac{1}{p}} y^{p+r+1} dy dt$$

$$= \int_{t=0}^1 \left(\int_{y=0}^\infty f(y)^p t dy \right)^{\frac{1}{p}} t^{-p-r-1} dt$$

$$= \int_{t=0}^1 \left(\int_{x=0}^t f(x)^p dx \right)^{\frac{1}{p}} t^{-p-r-1} dt = \left(\int_0^1 f(t)^p dt \right)^{\frac{1}{p}} = \|f\|_p^p$$

$$\begin{aligned}
 \text{LHS} &= \left(\int_{x=0}^{\infty} \left(\int_{y=0}^{\infty} f(y) dy \right)^p x^{-r} dx \right)^{\frac{1}{p}} \\
 &\quad \times \int_{x=0}^{\infty} F(x, y) x^{-r} dy = \int_{t=1}^{\infty} \int_{x=0}^{\infty} F(x, t x) x^{-r} dx dt \\
 &= \left(\int_{t=1}^{\infty} \left(\int_{x=0}^{\infty} F(tx) dx \right)^p x^{p+r-1} dx \right)^{\frac{1}{p}} \\
 &\leq \int_{t=1}^{\infty} \left(\int_{x=0}^{\infty} F(tx)^p (tx)^{p+r-1} dx \right)^{\frac{1}{p}} dt \\
 &\quad \underbrace{\int_{x=0}^{\infty} F(tx)^p (tx)^{p+r-1} dx}_{tx=t} = t^{-p-r} = e^{-p-r} \int_{y=0}^{\infty} F(y)^p y^{p+r-1} dy \\
 &= \int_{t=1}^{\infty} t^{1-\frac{p}{r}} \left(\int_{y=0}^{\infty} F(y)^p y^{p+r-1} dy \right)^{\frac{1}{p}} dt \\
 &= \left[-\frac{p}{r} t^{\frac{p}{r}-1} \right]_{t=1}^{\infty} = \text{RHS}
 \end{aligned}$$

Note for $p \neq 2, r=1, F(y) = |g'(y)|$ with $g(0)=0$:

$$\begin{aligned}
 \text{LHS} &= \left(\int_{x=0}^{\infty} \left(\int_{y=0}^x s(y) dy \right)^2 x^{-2} dx \right)^{\frac{1}{2}} \geq \left(\int_0^{\infty} x^2 g(x^2) dx \right)^{\frac{1}{2}} \\
 &\quad \int_0^x |g'(y)| dy \geq \int_0^x g'(y) dy = g(x) - g(0) = g(x)
 \end{aligned}$$

$$\text{RHS} = 2 \left(\int_{y=0}^{\infty} (y F(y))^2 y^2 dy \right)^{\frac{1}{2}} = 2 \left(\int_0^{\infty} |g'(y)|^2 dy \right)^{\frac{1}{2}}$$

$$\text{S6. LHS}^2 = \text{RHS}^2 \Rightarrow \int_0^{\infty} |g'(y)|^2 dy = \frac{1}{4} \text{LHS}^2 \geq \frac{1}{4} \int_0^{\infty} g(x^2) dx$$

(standard Hardy on coro)

Exercises 4 - Miscellaneous on Chapter 3

- (1) $T T^*$ -trick: Let H be a Hilbert and X , a normed vector space, $T: H \rightarrow X$ linear and continuous and T^* the adjoint of T . Show that

$$\|T\|_{H \rightarrow X} = \|T^*\|_{X^* \rightarrow H} = \|TT^*\|_{X^* \rightarrow X}^{1/2}.$$

- (2) a) Let $f \in L^1_{loc}(\mathbb{R}^d)$. $x \in \mathbb{R}^d$ is called a Lebesgue point of f provided that $\exists c \in \mathbb{C}: \lim_{r \rightarrow 0} \frac{1}{|\mathcal{B}_r(x)|} \int_{\mathcal{B}_r(x)} |f - c| = 0$. Show that a.e. $x \in \mathbb{R}^d$ is a Lebesgue point and that $c = f(x)$ for a.e. x .
- b) Fundamental theorem of calculus: Let $f \in L^1_{loc}(\mathbb{R})$, $F(x) := \int_0^x f(y) dy$. Show that F is differentiable at every Lebesgue point of f and that $F' = f$ a.e.

- (3) Let (X, \mathcal{B}, μ) a measure space and \mathcal{B} a σ -finite σ -subalgebra of \mathcal{F} . The orthogonal projection from $L^2(X, \mathcal{F}, \mu)$ to its closed subspace $L^2(X, \mathcal{B}, \mu)$ is denoted by $E(\cdot; \mathcal{B})$.

Show that

a) $\int_X f \overline{E(g; \mathcal{B})} d\mu = \int_X E(f; \mathcal{B}) \overline{g} d\mu = \int_X E(f; \mathcal{B}) \overline{E(g; \mathcal{B})} d\mu$
 for all $f, g \in L^2(X, \mathcal{F}, \mu)$

- b) $E(\cdot; \mathcal{B})$ is the unique map $: L^2(X, \mathcal{F}, \mu) \rightarrow L^2(X, \mathcal{B}, \mu)$ s.t.
 $\int_X E(f; \mathcal{B}) g d\mu = \int_X fg d\mu \quad \forall f \in L^2(X, \mathcal{F}, \mu)$
 $\forall g \in L^2(X, \mathcal{B}, \mu)$

c) Deduce that $E(f, \mathcal{B}) = \overline{E(f, \mathcal{B})}$, $f \in g \Rightarrow E(f, \mathcal{B}) \leq E(g, \mathcal{B})$

$E(hf, \mathcal{B}) = hE(f, \mathcal{B}) \quad \forall h \in L^\infty(X, \mathcal{B}, \mu)$ as well as

$$|E(f, \mathcal{B})| \leq E(|f|, \mathcal{B})$$

d) $E(\cdot, \mathcal{B}) : L^p(X, \mathcal{B}, \mu) \rightarrow L^p(X, \mathcal{B}, \mu)$ is continuous.

$$\text{and } \|E(\cdot, \mathcal{B})\|_{p \rightarrow p} \leq 1, \quad \forall 1 \leq p < \infty.$$

Prop 3.2

Hint: Show this for $p=1$ and ∞ .

e) Let $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \dots$ an increasing family of \mathcal{B} 's, and let $\bigvee_{n=1}^{\infty} \mathcal{B}_n$ be the σ -algebra generated by $\bigcup_{n=1}^{\infty} \mathcal{B}_n$.

Show $E(f, \mathcal{B}_n) \xrightarrow{n \rightarrow \infty} E(f, \bigvee_{n=1}^{\infty} \mathcal{B}_n)$ in L^p , $1 \leq p < \infty$, for all $f \in L^p(X, \mathcal{B}, \mu)$.

Hint: See Tao, Chapter 2; Prop 3.5, if necessary.

f) Let $X = \mathbb{R}$, \mathcal{B} = Borel σ -algebra, μ = Lebesgue measure,

$\mathcal{B}_n = \sigma\text{-algebra generated by } \left\{ \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right) \right\}_{k \in \mathbb{Z}}$.

Show $E(f, \mathcal{B}_n)(x) = 2^n \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} f(y) dy$, for $x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right)$

Ex 3.4.

and that $\bigvee_{n=1}^{\infty} \mathcal{B}_n = \mathcal{B}$.

④ Understand the one-dimensional Riemann - Stieltjes Lemma

(4.1 and 4.2 in Chapter 3 of Tao's notes).

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Tao

We can thus interpret D as the largest L^2 operator norm of the T_r :

$$D = \sup_r \|T_r\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)}.$$

To compute this operator norm we make the following observation:

Lemma 1.12 (TT^* identity). *Let $T : H \rightarrow X$ be a continuous map from a Hilbert space to a normed vector space, and let $T^* : X^* \rightarrow H$ be its adjoint. Then*

$$\|T\|_{H \rightarrow X} = \|T^*\|_{X^* \rightarrow H} = \|TT^*\|_{X^* \rightarrow X}^{1/2}.$$

Proof The first identity is just duality. Then we have

$$\|TT^*\|_{X^* \rightarrow X} \leq \|T\|_{H \rightarrow X} \|T^*\|_{X^* \rightarrow H} = \|T^*\|_{X^* \rightarrow H}^2$$

which gives the lower bound in the second identity. For the upper bound, observe that for any $f \in X^*$ that

$$\|T^*f\|_H^2 = \langle f, TT^*f \rangle \leq \|f\|_{X^*} \|TT^*f\|_X \leq \|f\|_{X^*}^2 \|TT^*\|_{X^* \rightarrow X};$$

taking square roots gives the upper bound as desired. \blacksquare

In light of this identity we know that

$$D^2 = \sup_r \|T_r T_r^*\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)}.$$

Now let us take a look at what $T_r T_r^*$ is. Observe that T_r is an integral operator with kernel

$$K(x, y) := \frac{1}{|B(x, r(x))|} 1_{|x-y| \leq r(x)}.$$

Thus the adjoint is given by

$$T_r^* g(y) = \int_{\mathbb{R}^d} \frac{1}{|B(x, r(x))|} 1_{|x-y| \leq r(x)} g(x) dx$$

and then TT^* is given by

$$T_r T_r^* g(x') = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|B(x, r(x))| |B(x', r(x'))|} 1_{|x-y| \leq r(x)} 1_{|x'-y'| \leq r(x')} g(x) dy dx.$$

Note that the y integral can be computed fairly easily. First we observe that the y integral vanishes unless $|x-x'| \leq r(x)+r(x')$, and in the latter case it enjoys a bound of $O_d(\min(r(x), r(x'))^d)$. Also, $|B(x, r(x))| \sim_d r(x)^d$ and $|B(x', r(x'))| \sim_d r(x')^d$. Putting this together we see that

$$|T_r T_r^* g(x')| \lesssim_d \int_{\mathbb{R}^d} 1_{|x-x'| \leq r(x)+r(x')} \frac{1}{\max(r(x), r(x'))^d} |g(x)| dx.$$

It is natural to split this integral into the regions $r(x) \leq r(x')$ and $r(x) \geq r(x')$, leading to the bound

$$|T_r T_r^* g(x')| \lesssim_d \int_{\mathbb{R}^d} 1_{|x-x'| \leq 2r(x)} \frac{1}{r(x)^d} |g(x)| dx + \int_{\mathbb{R}^d} 1_{|x-x'| \leq 2r(x')} \frac{1}{r(x')^d} |g(x)| dx.$$

Comparing this with the formulae for T_r and $T_{r'}$, we obtain the interesting pointwise inequality

$$|T_r T_r^* g(x')| \lesssim_d T_{2r} |g|(x') + T_{2r'}^* |g|(x')$$

Bzg

1. Let H Hilbert space and \mathcal{X} normed vector space

$T: H \rightarrow \mathcal{X}$ linear & cont. $T^*: \mathcal{X}^* \rightarrow H$ adjoint of T ,

$$\text{Show } \|T\|_{H \rightarrow \mathcal{X}} \stackrel{(1)}{=} \|T^*\|_{\mathcal{X}^* \rightarrow H} \stackrel{(2)}{=} \|TT^*\|_{\mathcal{X}^* \rightarrow \mathcal{X}}^{\frac{1}{2}}$$

pf (1): $\|T\|_{H \rightarrow \mathcal{X}} = \sup_{\|f\|_{\mathcal{X}^*}=1} \|Tf\|_H = \sup_{\|g\|_{\mathcal{X}}=1} \sup_{\|f\|_{\mathcal{X}^*}=1} |(g, f)| =$

dual: $\mathcal{X}^* \times \mathcal{X} \rightarrow \mathbb{C}$

$$\text{isom on } \mathcal{X}^*: \|g\|_{\mathcal{X}} = \sup_{\|f\|_{\mathcal{X}^*}=1} |(g, f)|$$

$$(g, f) \mapsto \langle g, f \rangle = g(f)$$

$$= \sup_{\|f\|_{\mathcal{X}^*}=1} |\langle g, f \rangle|$$

$$\|T^*\|_{\mathcal{X}^* \rightarrow H} = \sup_{\|g\|_{\mathcal{X}}=1} \|T^*g\|_H = \sup_{\|g\|_{\mathcal{X}}=1} \sup_{\|f\|_{\mathcal{X}^*}=1} |\langle T^*g, f \rangle_H| = \sup_{\|g\|_{\mathcal{X}}=1} |\langle g, Tf \rangle| = \|T\|_{H \rightarrow \mathcal{X}}$$

(2): $\|T^*\|_{\mathcal{X}^* \rightarrow H}^2 = \sup_{\|f\|_{\mathcal{X}^*}=1} \|T^*f\|_H^2$

$$\|T^*f\|_H^2 = \langle T^*f, T^*f \rangle_H = \langle f, TT^*f \rangle$$

$$\leq \|f\|_{\mathcal{X}^*} \|TT^*f\|_{\mathcal{X}} \quad \text{or} \quad \leq \sup_{\|g\|_{\mathcal{X}}=1} |\langle g, TT^*f \rangle|$$
$$= \|TT^*f\|_{\mathcal{X}} \leq \|TT^*\|_{\mathcal{X}^* \rightarrow \mathcal{X}} \|f\|_{\mathcal{X}^*}$$

$$\Rightarrow \|T^*\|_{\mathcal{X}^* \rightarrow H}^2 \leq \|TT^*\|_{\mathcal{X}^* \rightarrow \mathcal{X}}$$

(2): $\|TT^*\|_{\mathcal{X}^* \rightarrow \mathcal{X}} \leq \|T\|_{H \rightarrow \mathcal{X}} \|T^*\|_{\mathcal{X}^* \rightarrow H} = \|T\|_{H \rightarrow \mathcal{X}}^2$

2.

a) Let $f \in L^1_{loc}(\mathbb{R}^d)$ $\Rightarrow x \in \mathbb{R}^d$ is called a Lebesgue point of f if

$$\forall c \in \mathbb{C} : \lim_{r \rightarrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f - c| = 0$$

Show that for a.e. $x \in \mathbb{R}^d$, x is a Leb. pt. and $c = f(x)$.Lebesgue differentiation thm.: $f \in L^1_{loc}(\mathbb{R}^d)$

$$\Rightarrow \lim_{r \rightarrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)| dy = 0 \text{ for a.e. } x \in \mathbb{R}^d$$

$$\Rightarrow \text{For a.e. } x \in \mathbb{R}^d \quad \exists \quad c = f(x) :$$

 $\Rightarrow x$ Leb. pt. with $c = f(x)$

b)

Let $F \in L^1_{loc}(\mathbb{R})$, $F(x) := \int_0^x f(y) dy$ Show that F is differentiable at every Leb. pt. of f and that $F' = f$ a.e.Let $x \in \mathbb{R}$ be a Leb. pt.

$$\text{Then by a)} \quad \lim_{r \rightarrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |F(y) - F(x)| dy = 0$$

$$\text{and by } |A_r f - f(x)| \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} |F(y) - F(x)| dy$$

$$F(x) = \lim_{r \rightarrow 0} A_r F = \lim_{r \rightarrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) dy$$

$$\begin{aligned} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) dy &= \frac{1}{2r} \int_{x-r}^{x+r} f(y) dy \\ &\geq \frac{1}{2r} \left(\int_{x-r}^x f(y) dy + \int_x^{x+r} f(y) dy \right) \quad [x-r, x] \cup [x, x+r] \\ &= \frac{1}{2r} \left(\int_0^x f(y) dy - \int_{0-x}^0 f(y) dy + \int_0^x f(y) dy - \int_0^x f(y) dy \right) \\ &= \frac{1}{2r} (F(x) - F(x-r)) + P(x+r) - P(x) \\ &= \frac{1}{2r} (F(x) - F(x-r)) + \frac{1}{2r} (P(x+r) - P(x)) \end{aligned}$$

$$\lim_{r \rightarrow 0} \frac{1}{2r} (P(x+r) - P(x-r)) = f(x)$$

To get the usual limit

$$\text{Note: } |\{x, x+r\}| = \frac{1}{2} |\{x-r, x+r\}|$$

$$\Rightarrow \frac{1}{|\{x, x+r\}|} \int_{\{x, x+r\}}^{\{x+r\}} |f(y) - f(x)| dy \leq \frac{2}{|\{x-r, x+r\}|} \int_{\{x-r, x+r\}}^{\{x+r\}} |P(y) - P(x)| dy \xrightarrow[r \rightarrow 0]{} 0$$

$$\Rightarrow \left| \frac{1}{r} \int_x^{x+r} (f(x) dy - f(x)) \right| = \left| \frac{1}{r} \int_x^{x+r} (P(x) - f(x)) dy \right| \leq \frac{1}{r} \int_x^{x+r} |f(x) - P(x)| dy \xrightarrow[r \rightarrow 0]{} 0$$

$$\Rightarrow P(x) := \lim_{r \rightarrow 0} \frac{1}{r} \int_x^{x+r} (P(x+r) - P(x)) = \lim_{r \rightarrow 0} \frac{1}{r} \int_x^{x+r} f(y) dy = f(x)$$

3. Let (X, \mathcal{B}, μ) be a measure space and \mathcal{B} a σ -algebra
be a subalgebra of \mathcal{B} . The orthogonal proj. $L^2(X, \mathcal{B}, \mu) \rightarrow L^2(X, \mathcal{B}, \mu)$
denoted $E(\cdot; \mathcal{B})$

Show

$$a) \int_X f E(g, \mathcal{B}) d\mu = \int_X E(f, \mathcal{B}) g d\mu = \int_X E(f, \mathcal{B}) E(g, \mathcal{B}) d\mu$$

$$\forall f, g \in L^2(X, \mathcal{B}, \mu)$$

$$P = E(\cdot, \mathcal{B}) : L^2(X, \mathcal{B}, \mu) \ni \eta \mapsto \eta_1 \in L^2(X, \mathcal{B}, \mu) \subseteq \eta$$

projection op. on closed subspace

$\Rightarrow P$ self-adjoint & idempotent ($P^2 = P$)

$$\text{i.e. } \langle f, Pg \rangle_n = \langle Pf, g \rangle_n = \langle P^2 f, g \rangle_n = \langle Pf, Pg \rangle_n$$

b) Show $E(\cdot, \mathcal{B})$ is the unique map $: L^2(X, \mathcal{B}, \mu) \rightarrow L^2(X, \mathcal{B}, \mu)$ as

$$\int_X E(f, \mathcal{B}) g d\mu = \int_X fg d\mu \quad \forall f \in L^2(X, \mathcal{B}, \mu), g \in L^2(X, \mathcal{B}, \mu)$$

By the property of orthogonal proj. $\langle (f - P\varphi), g \rangle_n = 0 \quad \forall \varphi \in \mathcal{K}$

$$\Leftrightarrow \langle S_\varphi g \rangle_n = \langle PS_\varphi g \rangle_n \quad \forall \varphi \in \mathcal{K}, g \in X$$

Uniqueness: $\langle f, g \rangle_n = \langle Qf, g \rangle_n \quad \forall f \in \mathcal{K}, g \in X$. $Q : \mathcal{K} \rightarrow X$

$$\Rightarrow \langle (P - Q)f, g \rangle_n = 0 \quad \forall f \in \mathcal{K}, g \in X$$

$$g \in (P - Q)f \in X \Rightarrow \| (P - Q)f \|_n^2 = 0 \Rightarrow Pf = Qf \quad \Rightarrow P = Q$$

c) Deduce that $E(\bar{f}, \mathcal{B}) = E(f, \mathcal{B})$

$$f \leq g \Rightarrow E(f, \mathcal{B}) \leq E(g, \mathcal{B})$$

$$|E(f, \mathcal{B})| \leq |E(\bar{f}, \mathcal{B})|$$

$$\text{i.e., } P, \bar{S} \vdash \bar{P}\bar{f}$$

$$f \leq g \Rightarrow Pf \leq Pg$$

$$|Pg| \leq |Pf|$$

$$\text{Hence, get } \int_{\mathbb{X}} E(\bar{f}, \mathcal{B}) g d\mu = \int_{\mathbb{X}} Pg d\mu = \overline{\left(\int f g d\mu \right)} = \int_{\mathbb{X}} E(f, \mathcal{B}) g d\mu$$

$$f = u + iv, \quad (g \in \mathfrak{a} + i\mathfrak{b})$$

$$E(\bar{f}, \mathcal{B}) = E(u, \mathcal{B}) + E(v, \mathcal{B})$$

$$\begin{aligned} \int_{\mathbb{X}} E(\bar{f}, \mathcal{B}) g d\mu &= \int_{\mathbb{X}} E(u, \mathcal{B}) g d\mu + \int_{\mathbb{X}} E(v, \mathcal{B}) g d\mu \\ &= \int_{\mathbb{X}} ug d\mu + \int_{\mathbb{X}} vg d\mu = \int_{\mathbb{X}} (u+iv)g d\mu. \end{aligned}$$

$$\begin{aligned} \int_{\mathbb{X}} E(\bar{f}, \mathcal{B}) g d\mu &= \int_{\mathbb{X}} E(u, \mathcal{B}) g d\mu + \int_{\mathbb{X}} E(-iv, \mathcal{B}) g d\mu \\ &= \int_{\mathbb{X}} ug d\mu + \int_{\mathbb{X}} -ivg d\mu = \int_{\mathbb{X}} (u-iv)g d\mu \end{aligned}$$

$$h \in L^{\infty}(\mathbb{X}, \mathcal{B}, \mu) \Rightarrow h\bar{f} \in L^2(\mathbb{X}, \mathcal{B}, \mu)$$

$$hg \in L^2(\mathbb{X}, \mathcal{B}, \mu)$$

$$\int_{\mathbb{X}} E(h\bar{f}, \mathcal{B}) g d\mu = \int_{\mathbb{X}} h\bar{f} g d\mu = \int_{\mathbb{X}} fg d\mu$$

$$= \int_{\mathbb{X}} E(f, \mathcal{B}) hg d\mu = \int_{\mathbb{X}} h E(f, \mathcal{B}) g d\mu$$

$$\Rightarrow E(h\bar{f}, \mathcal{B}) = h E(f, \mathcal{B}) + \int_{\mathbb{X}} hg d\mu \quad \forall g \in L^2(\mathbb{X}, \mathcal{B}, \mu)$$

$$f \leq g \text{ pointwise a.e.} \Rightarrow (g-f) \geq 0 \text{ a.e.}$$

$$\Rightarrow \int_{\mathbb{X}} (g-f) \psi d\mu \geq 0 \quad \forall \psi \in L^2(\mathbb{X}, \mathcal{B}, \mu)$$

$$\int_{\mathbb{X}} E(g-f, \mathcal{B}) \psi d\mu \Rightarrow E(g, \mathcal{B}) - E(f, \mathcal{B}) = E(g-f, \mathcal{B}) \geq 0$$

$$\text{or, } f \geq 0 \Rightarrow E(f, \mathcal{B}) \geq 0 \quad \forall$$

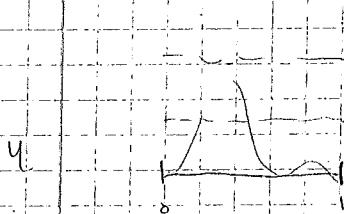
a.e.

$$h(x) = \begin{cases} |E(S, B)(x)| & |E(S, B)(x)| > 0 \\ 0 & \text{otherwise} \end{cases}, \quad E(S, B)(x) > 0$$

$$\Rightarrow |E(T, B)| = h(E(T, B)) = E(h(T, B)) > 0$$

$$\Rightarrow |E(T, B)| = \operatorname{Re} E(h(T, B)) \leq E(\operatorname{Re} h(T, B))$$

$$\leq E(1_{\mathbb{R}^2}, B) \leq E(1_{\mathbb{R}}, B)$$



and the sign

Uncertainty principles

Nam 11/05/09

position

position x

momentum $p = h/x$

$$\text{Heisenberg unc. princ. } \Delta x \cdot \Delta p \geq \frac{\hbar}{4}$$

$f \in L^2(\mathbb{R}^d)$ $f(x)$ density

$\hat{f}(p)$ momentum dens.

Heisenberg: (complexity 1D)

$$\left(\int_{\mathbb{R}} x^2 |f(x)|^2 dx \right) \left(\int_{\mathbb{R}} p^2 |\hat{f}(p)|^2 dp \right) \geq \frac{\hbar^2}{4}, \quad \|f\|_2^2 = 1$$

convention $f(x) = \int f(x) e^{i k x} dk$
 $(\text{Tao: } e^{-2\pi i x k})$

$$\int x^2 |f(x)|^2 = \langle \hat{x}, x^2 \hat{x} \rangle$$

$$\int p^2 |\hat{f}(p)|^2 = \langle \hat{p}, p^2 \hat{p} \rangle = \int |\hat{p}'(p)|^2 = \int |\hat{p}'|^2$$

$$(x) \Leftrightarrow \left(\int x^2 |f(x)|^2 \right) \left(\int |\hat{p}'(p)|^2 dp \right) \geq \frac{\hbar^2}{4}$$

$$\begin{aligned} \operatorname{Re} \langle \hat{x}, x \hat{p} \rangle &= \operatorname{Re} \int \hat{x}(x) x \hat{p}(x) dx = \int x \frac{1}{2} \int x (\hat{p}(x))^2 dx \\ &= -\frac{1}{2} \int (\hat{p}(x))^2 dx = -\frac{1}{2} \end{aligned}$$

$$\Rightarrow \frac{1}{4} = |\operatorname{Re} \langle \hat{x}, x \hat{p} \rangle|^2 \leq \|\hat{p}'\|_2^2 \|\hat{x}\|_2^2$$

HC

$$\text{Hardy: } -\Delta \geq \frac{1}{4x^2} \Leftrightarrow \int |\hat{p}'(p)|^2 \geq \frac{1}{4} \int \frac{1}{x^2} p^2 dp$$

$$\left(\int x^2 |f(x)|^2 \right) \left(\int |\hat{p}'(x)|^2 \right) \geq c$$

sharp: $\frac{n}{n+1}$

$$\geq \left(\int x^2 |\hat{f}(x)|^2 \right) \left(\int \frac{1}{x^2} p^2 dp \right) \geq \underbrace{\left(\int |\hat{p}'|^2 \right)^2}_{\frac{n}{n+1}}$$

Hardy uncertainty principle

" $f(x)$ ", " $\hat{f}(z)$ " cannot decay too fast"

R: If $|f(x)| \leq C e^{-\alpha x^2}$, $|\hat{f}(z)| \leq C e^{-\beta z^2}$ $\alpha, \beta > 0$

then

1) If $\alpha \beta > \frac{1}{4}$ then $f = 0$

2) If $\alpha \beta = \frac{1}{4}$ then $f = C e^{-\alpha x^2}$

Ex: $F = e^{-\alpha x^2} \Rightarrow \hat{F}(z) = \text{const. } e^{-\frac{\alpha z^2}{4}}$

If $f(x) = e^{-x^2/2}$ then $\hat{F}(z) = \text{const. } e^{-z^2/2}$

Proof: By scaling, $a = b = \frac{1}{2}$

($\hat{f}_c(x) = f(cx)$, $\hat{f}_c(z) = \text{const. } \hat{f}\left(\frac{z}{c}\right)$)

Lemma: If \hat{f} decays exp. then f is analytic in a strip

$\overbrace{\dots}^{\text{analytic}} \subset [a, b]$

consequence: If \hat{f} decays faster than any exp. then

f is entire

Returning to the proof we have f, F entire.

$$f(z) = \int_{\mathbb{R}} f(x) e^{-ixz} dx$$

$$|\hat{f}(z)| \leq \int_{\mathbb{R}} |f(x)| |e^{-ixz}| dx \leq C \int_{\mathbb{R}} e^{-\frac{1}{2}x^2} e^{-2\pi|Im z|x} dx$$

$$= C \int_{\mathbb{R}} e^{-\frac{1}{2}(x - 2\pi|Im z|)^2} dx \cdot e^{2\pi|Im z|^2} = C e^{2\pi|Im z|^2}$$

$$C e^{-\frac{1}{2}z^2} = c$$

$$F(z) := e^{\frac{1}{2}z^2} f(z)$$

$$|F(z)| \leq \text{const. when } z \in \mathbb{R}$$

$$|F(z)| \leq \text{const. when } z \in \mathbb{C} \quad \text{since } |\hat{f}(z)| \leq C e^{-\frac{1}{2}z^2} \quad z \in \mathbb{R}$$

Q: Is it true that $|F(z)|$ is const. in $z \in \mathbb{C}$?

Ans: If $|F(z)| \leq \text{const.}$ then Liouville $\Rightarrow F(z) = \text{const.}$

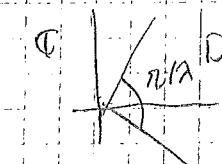
Lecture 6, Sec. 6.1

Thus, if $D = \text{angle with opening } \frac{\pi}{2}$

$f(z)$ is an entire function on D

and $|f(z)| \leq M$ on ∂D

and $|f'(z)| \leq C e^{\frac{|z|}{2}}$ for $z \in D$



then $|f(z)| \leq M$ for $z \in D$

→ Answer: Let $s > 0$ and $\theta \leq \frac{\pi}{2}$. $0 \leq \frac{\pi}{2} - \theta$ (dep on s)

$$\text{Def. } g(z) = e^{isz^2} f(z) = e^{isz^2} e^{z^2/2} f(z)$$

$$|g(z)| \leq C e^{-2sRez + Imz} e^{(Rez)^2 - (Imz)^2/2} |f(z)| e^{z^2/2}$$

$$= C e^{-2sRez + Imz + Rez^2/2}$$

$$\text{Diagram: } \begin{array}{c} \theta \\ \text{Im } z \\ \text{Re } z \end{array} \quad 0 < \arg z < \theta \quad |g(z)| \leq C \text{ on } \partial D_\theta$$

$\text{Im } z \geq \frac{1}{\cos \theta}$ → $\text{Im } z$ part works over rest part
for G large enough

Applying PL with $\lambda > 2$ and $|g(z)| \leq C e^z$

Another v. of H.U.P.

Then (Beurling 1960's)

$$\text{If } f \in L^1(\mathbb{R}) \text{ and } \int_{-\infty}^{\infty} |f(x)| |\hat{f}(y)| e^{ixy} dy dx < \infty$$

then $\hat{f} = 0$.

Consequence: If $|f| \leq C e^{-\lambda x^2}$, $|\hat{f}| \leq C e^{-\lambda y^2}$, $x, y \geq 0$, $C, \lambda > 0$
then $\hat{f} = 0$. (1st statement of previous H.U.P.)

mod:

$$\int_{\text{first}} M(x) dx = \int_{\text{last}} |F(y)| M(y) dy$$

$$\text{with } M(x) := \int_{\text{last}} |F(y)| e^{xy} dy$$

$$M(y) := \int_{\text{first}} |F(x)| e^{xy} dx$$

Now note if $F \neq 0$ then M grows at least exponentially

$\Rightarrow |F|$ decays at least exponentially

ii) If F has cpt. supp.

$$P(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} F(\omega) e^{ix\omega} d\omega$$

$M \rightarrow$ at least exp

$$y_0 \mapsto \int_{y_0/2}^{\infty} |F(y)| dy \rightarrow \text{at least expo}$$

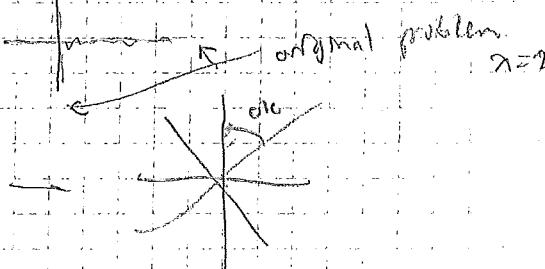
\int_{last} entire \checkmark faster exp. on IR

$M/M \nearrow$ faster exp.

Def. $P(z) = \int_{-\infty}^{\infty} F(\omega) S(i\omega) dz'$, $z \in \mathbb{C}$

$$|P(z)| \leq \int_{-\infty}^{\infty} |F(\omega)| |S(i\omega)| dz' \leq \int_{\text{IR}} |F(\omega)| M(\omega) dz'$$

and $|P(\text{IR})| \leq \text{const.}$



$$|P(z)| \leq C e^{|Im z|^2}$$

use one more symmetry

$$\lambda_{\text{avg}}(w) = \frac{1}{|E|} \sum_{e \in E}$$

$$\leq \lambda_0(\frac{d}{2}) + \lambda_6(\frac{d}{2})$$

$$\text{Ex: } Hf = CPV \frac{1}{x} * f$$

$$K(x,y) = \frac{1}{\pi} \frac{1}{x-y}$$

$$\text{Ex: } \| \cdot \| \sim \| K(\cdot, \cdot) \| \approx b$$

CZ operator: $\pi: C(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ bdd if

$$Tf(y) = \int K(x,y) f(x) dx, \quad \forall f \in L^2 \text{ compact}, y \in \text{supp } f$$

$$K: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d} \text{ s.t.}$$

$$1) |K(x,y)| \leq C|x-y|^{-d}$$

$$2) \text{"H\"older cont." in } x \Rightarrow y$$

$$\text{sufficient: } \left| \frac{\partial_x K(x,y)}{\partial_y K(x,y)} \right| \leq C|x-y|^{-d-1}$$

Show: $T \in CZ_0 \Rightarrow T: L^1 \rightarrow L^1$ bdd (Cx)

Then

Proof:

$$\text{To show (Cx): } f \in L^1 \cap L^2, \lambda > 0 \text{ show } \| Tf \|_{L^1} \leq C_d \| f \|_1$$

$$\sup_{x > 0} \lambda \left| \left\{ x : |Tf(x)| \geq \lambda \right\} \right|$$

$$\text{Replace } f \text{ by } \lambda f \Rightarrow \lambda = 1$$

$$\text{Replace } f \text{ by } f(x) = f(x) \rightarrow \text{Assume } \| f \|_{L^1} = 1$$

$$\Rightarrow \text{Show } \left| \left\{ x : |Tf(x)| \geq 1 \right\} \right| \leq C_d \quad \forall f \in L^2 \cap L^1, \| f \|_{L^1} = 1$$

$$CZ_0 \Rightarrow f = g + \sum_Q b_Q$$

$$\lambda_{Tf}(1) \leq \lambda_{Tg}(1) + \lambda_{\sum_Q b_Q}$$

$$\text{Chebychev: } \lambda_{Tg}(1) \leq (1)^p \| Tg \|_p^p \quad \text{choose } p=2$$

$$T \text{ is } L^1 \text{ on } U \rightarrow \mathbb{R}^n \Rightarrow \|\lambda_T g\|_2 \leq C_d \|g\|_2$$

$$\begin{aligned} \text{Lemma 2, ch 1} \Rightarrow & \|u_C\|_0 \|g\|_{L^\infty} \\ & \leq \|u_C\|_{W^{1,2}} \|g\|_2 = C_d \end{aligned}$$

$$\lambda \sum_Q b_Q \stackrel{(1)}{\leq}$$

$$\begin{aligned} \text{Lemma 1, f} \Rightarrow & \|Tb_Q\|_{L^1(Q \cap VQ)} \leq C_d \|b_Q\|_1 \\ & \leq C_d 2^{d+1} |Q| \end{aligned}$$

$$\Rightarrow \left\| \sum_Q Tb_Q \right\|_{L^1(Q \cap VQ)} \leq \sum_Q \|Tb_Q\|_{L^1(Q)} \leq C_d 2^{d+1} |VQ|$$

Chetyshov

$$\sum_{Q \in \mathcal{D}_k} \frac{1}{2} \left| \left\{ x \in \bigcup Q : \left| \sum_Q Tb_Q \right| \geq \frac{1}{2} \right\} \right|$$

$$\left| \left\{ x \in \bigcup Q : \left| \sum_Q Tb_Q \right| \geq \frac{1}{2} \right\} \right| \leq |VQ| \leq C_d |VQ|$$

$$\Rightarrow \lambda \sum_Q b_Q \stackrel{(1)}{\leq} C_d 2^{d+2} + C_d$$

$$\Rightarrow \lambda_T f(1) \leq C_d + \frac{C_d}{2} = \frac{3}{2} C_d$$

X

$$\underline{\text{C202: 200}} \Rightarrow f = g + \sum_Q b_Q \text{ s.t.}$$

$$\|g\|_2 \leq \|f\|_2$$

$$\|g\|_\infty \leq 2^d$$

$$\sup_Q b_Q \leq 1, \quad \sum_Q b_Q = 0, \quad |VQ| \leq \frac{\|f\|_2}{\lambda}$$

$$\|b_Q\|_1 \leq 2^{d+1} |Q|$$

App:

$$\text{VM} \quad \lim_{N \rightarrow \infty} \int_{-N}^N e^{ixz} \hat{f}(z) dz = f(x) ? \quad \text{even if } \hat{f} \notin L^1$$

Thm 2.12: $1 < p < \infty$; $N > 0$, $S_N f(x) := \text{conv} \sum_{n=-N}^N e^{inx} \hat{f}(z) h_z$

$$f \in S(\mathbb{R}) \quad = \mathcal{F}^{-1} \mathbb{I}_{[-N, N]} \mathcal{F} f$$

$\Rightarrow S_N$ extends to a bdl op. on $L^p(\mathbb{R}^2)$

$$\text{and } \|S_N f - f\|_{L^p} \xrightarrow{N \rightarrow \infty} 0 \quad \forall f \in L^p(\mathbb{R}^2)$$

prof: $S_N f = \mathcal{F}^{-1} \mathbb{I}_{[-N, N]} \mathcal{F} f$

$$= \frac{1}{2} \mathcal{F}^{-1} T_N \operatorname{sgn}(z) T_N \mathcal{F} f$$

$$+ \frac{1}{2} (\operatorname{sgn}(z-N) - \operatorname{sgn}(z+N)) \underbrace{\mathcal{F}^{-1} T_N \operatorname{sgn}(z) T_N \mathcal{F} f}_{\mathcal{F} e^{ixN} f}$$

$$T_\alpha g(x) := g(x-\alpha)$$

$$= \frac{1}{2} e^{ixN} \underbrace{\mathcal{F}^{-1} \operatorname{sgn}(z)}_{M_N \text{ bdl}} \mathcal{F} e^{-ixN} f + \frac{1}{2} e^{ixN} \underbrace{\mathcal{F}^{-1} \operatorname{sgn}(z) \mathcal{F} e^{ixN} f}_{M_H f}$$

$$= \frac{1}{2} \left(M_N H M_N - M_{N-H} M_N \right) f$$

bdl on L^p if $1 < p < \infty$ uniformly in N

$$\Rightarrow \|S_N\|_{L^p \rightarrow L^p} \leq C \quad \text{indep. of } N$$

Know: $\|S_N f - f\|_{L^p} \xrightarrow{N \rightarrow \infty} 0 \quad \forall f \in S(\mathbb{R}^2)$

$$\Rightarrow \forall f \in L^p \text{ let } f_n \in S(\mathbb{R}) \quad \|S_n f - f\|_{L^p} \xrightarrow{N \rightarrow \infty} 0$$

$$\begin{aligned} \Rightarrow \|S_N f - f\|_{L^p} &\leq \|S_N(f-f_n)\|_{L^p} + \|f-f_n\|_{L^p} \\ &\quad + \|S_N f_n - f_n\|_{L^p} \\ &\leq \|S_N\| \|f-f_n\| + \dots \xrightarrow{n \rightarrow \infty} 0, \quad N \rightarrow \infty \end{aligned}$$

Fefferman: On \mathbb{R}^2 : $\mathcal{F}^{-1} \mathbb{I}_{B(0)} \mathcal{F} : L^p \rightarrow L^p$
unbounded if $p \neq 2$

Fourier multipliers

$m \in S'(\mathbb{R}^d) \rightsquigarrow$

1. spatial multiplier $S \ni f \mapsto m * f \in S'$
2. Fourier mult. $S \ni f \mapsto \hat{f}(m\hat{f}) \in S'$

$$m(D)f$$

Hölder indices 1, simple to analyse

2. on L^2 easy to understand (isometry + spatial mult.)

Q: Are Fourier mult. and PDOs C* algs? Answer: Yes!

Fourier mult. \leftrightarrow convolution op's, $m(D)f = (\mathcal{F}^{-1}m) * f$

$$es'$$

Then (Hörmander 60's) L^p and $A: L^p(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d)$ cont.

and translation inv. i.e. $A T_{x_0} f = T_{x_0} A f \forall x_0$

Then $\exists m \in S'(\mathbb{R}^d)$: $A f = m * f$

\Rightarrow different ways to think of the same type of op.

Def: $M^p(\mathbb{R}^d) := \{m \in S': m(D): L^p \rightarrow L^p \text{ bdd}, f \in \mathcal{A}(L^p)\}$
with op. norm $\|m\|_{M^p} \mapsto$ Banach algebra

$$p \geq 2: M^2 = L^\infty(\mathbb{R}^d), \|m\|_{M^2} = \|m\|_\infty$$

adjoint: $m(D)^* = \overline{m(D)}: L^{p'} \rightarrow L^{p'} \Rightarrow \overline{m} \in M^{p'}, p \neq 2$
 $m \in M^p \Rightarrow$

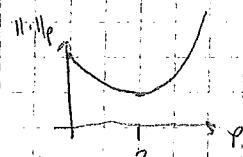
$$\|m\|_{M^p} = \|\overline{m}\|_{M^{p'}} = \|m\|_{M^{p'}}$$

Riesz-Thorin $\Rightarrow m(D): L^2 \rightarrow L^2$ cont.

$$\|m\|_{L^\infty} = \|m\|_\infty \leq \|m\|_{M^p}^{1-p} \|m\|_{M^{p'}}^{p} \neq \|m\|_{M^p}$$

$$\Rightarrow M^p \subseteq L^\infty$$

$$M^\infty = M^1$$



Fact: $M^\infty = \mathcal{F} L^1(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d)$

$$\|u\|_{M^0} = \|u\|_m = \|\mathcal{F}^{-1} u\|_1$$

Summary:

$$M^2 \circ M^p = M^p \Rightarrow M^{\infty} = M$$

$$L^\infty(\mathbb{R}^d)$$

$$\mathcal{F} L^1(\mathbb{R}^d)$$

Bx (Fefferman): $\mathcal{B}_{B_1(C_0)}(D) \subset M^p(\mathbb{R}^d)$ if $p \neq 2$, $d \geq 1$. but it is for $d \geq 1$

2 ways to proceed \rightarrow smooth m , possibly more general (PDOs)
 \rightarrow restrict to simple singularities of m
 (Hörmander-Mikhlin mult. thm.
 $m \in C^k(\mathbb{R}^d \setminus \{0\})$)

1st task: Def. PDOs, show that they have C^∞ -kernels

2nd: apply to the more general cases

$$a(x, D) f(x) = \int_{\mathbb{R}^d} e^{ix \cdot \xi} a(x, \xi) \hat{f}(\xi) d\xi = (x \cdot) \underset{\xi \rightarrow x}{\mathcal{F}}^{-1}(a(x, \xi)) \hat{f}(\xi)$$

a should be \approx polynomial in ξ

$$\text{def: } S^m := \left\{ a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) : \forall \alpha, \beta \in \mathbb{N}_0^d \exists C_{\alpha, \beta}, \forall (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \right. \\ \left. |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{m+|\beta|} \right\}$$

$$\text{Ex: } a(x, \xi) = a_0(x) + a_1(x) \xi + \dots + a_3(x) \xi^3 \in S^3(\mathbb{R}) \text{ if } a_k \in C_{\text{loc}}^\infty(\mathbb{R})$$

$$\Rightarrow a(D) f = a_0(x) f + \dots + a_3(x) D^3 \hat{f}(x)$$

$$\text{Ex: } S^{-\infty} := \bigcap_{m \in \mathbb{R}} S^m \supseteq S(\mathbb{R}^d) \\ \supseteq C_{\text{loc}}^\infty(\mathbb{R}^d; S(\mathbb{R}^d))$$

Q: What is the kernel of such op?

$$\text{Ex. 5.2} \quad a(x, D) f(x) = \int_{\mathbb{R}^d} k(x, x-y) f(y) dy$$

$k(x, \cdot) := \int_{\mathbb{R}^d} a(x, z) \in S'(\mathbb{R}^d)$ dep. on point x

$k(x, z)$ is fin for $|z| \neq 0$

$$|k(x, z)| \leq C_{\epsilon} |z|^{-N} + |z|^{\beta} \geq \epsilon.$$

$$a(x, 0) = \text{op}(a)$$

Then: $a \in S^m(\mathbb{R}^d) \Rightarrow k(x, y) = k(x, x-y)$ is a singular kernel

$$k(x, z) \in C^\infty(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})) \quad \text{and}$$

$$\text{If } d+m+1+\alpha+N > 0 \quad \forall z \neq 0 \quad |\partial_x^\alpha \partial_z^\beta k(x, z)| \leq C_{\alpha, \beta, N} |z|^{-(N+d-m)}$$

$$|z| \approx |z|^{-d-m} \text{ close to } z=0$$

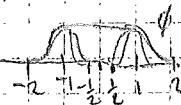
\hookrightarrow faster than $|z|^{-N}$ at $|z|=0$

(exponential decay $\sim e^{-r}$ at infinity, controlled singularity at 0)

For the proof: Decompose \mathbb{R}^d into spherical harmonic shells

"Littlewood-Paley decom"

$$0 \leq \phi \in C_0^\infty(\mathbb{R}^d)$$



$$\delta(z) := \phi(z) - \phi(\frac{z}{2})$$

$$\delta(2^{-j}z) \text{ supp. in } |z| \in (2^{j-1}, 2^j]$$

$$\text{Partitions of 1: } ① \quad 1 = \sum_{j=-\infty}^{\infty} \delta(2^{-j}z) \quad (= \sum \phi(\frac{z}{2^j}) - \phi(\frac{z}{2^{j+1}})) \quad \forall z \neq 0$$

$$② \quad 1 = \phi(z) + \sum_{j=1}^{\infty} \delta(2^{-j}z) \quad \forall z \in \mathbb{R}^d$$

$$\text{Operator decomps: } \mathcal{I} = \sum_{j=-\infty}^{\infty} \delta(2^{-j}D) \quad \text{for every } \chi \text{ at most 2 times to}$$

$$1 = \phi(D) + \sum_{j=1}^{\infty} \delta(2^{-j}D)$$

Exercises 5 — Integral operators from PDE

- ① a) Let $\psi : [0, \infty) \rightarrow [0, \infty]$ nonincreasing and $\Psi : \mathbb{R}^n \rightarrow [0, \infty]$ defined by $\Psi(x) := \psi(|x|)$. Show $\forall f \in L^p(\mathbb{R}^n)$:

$$|f * \Psi(x)| \leq (\int_{\mathbb{R}^n} \Psi) \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$$

$$\text{and } |f * \Psi_t(x)| \leq (\int_{\mathbb{R}^n} \Psi) \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy \quad \forall t > 0.$$

Here, $\Psi_t(x) = t^{-n} \Psi(x/t)$.

Hint: First take $\nu = \sum_{j=1}^N a_j \mathbb{1}_{(-r_j, r_j)}$, $a_j, r_j \in (0, \infty)$.
An arbitrary ψ can be approximated by such sums.

- b) Let $\phi \in L^1(\mathbb{R}^n)$, $\int \phi = 1$, $|\phi(x)| \leq \psi(|x|)$ with $\psi : [0, \infty) \rightarrow [0, \infty)$ as in a) and bounded. Show

$$\sup_{t>0} |f * \phi_t(x)| \leq C_\phi \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy. \quad \text{a.e. } x$$

- c) Check that the proof of Lebesgue's differentiation theorem yields $\lim_{t \rightarrow 0^+} f * \phi_t(x) = f(x)$ for a.e. $x \in \mathbb{R}^n$.

- d) Let $\phi_1(x) = c_1 (1 + |x|^2)^{-\frac{n+1}{2}}$, $\phi_2(x) = c_2 e^{-\frac{|x|^2}{4}}$, c_1, c_2 s.t. $\int \phi_{1/2} = 1$. Given $f \in L^p(\mathbb{R}^n)$, let $u_1(t, x)$ be $f * (\phi_1)_t(x)$ and $u_2(t, x) = f * (\phi_2)_{\sqrt{t}}(x)$. Then

$$\Delta_{t,x} u_1(t, x) = 0 \quad \text{and} \quad (\partial_t - \Delta_x) u_2(t, x) = 0 \quad \text{for } t > 0, x \in \mathbb{R}^n,$$

$$\text{and } \lim_{t \rightarrow 0^+} u_{1/2}(t, x) = f(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

(2) Let $S^m := \{ a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) : \forall \alpha, \beta \in \mathbb{N}_0^n, \exists C_{\alpha\beta}, |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} (1+|x|)^{m-|\alpha|} \}$

The operator $L_\xi := (1+|x|^2)^{-1}(I - \Delta_\xi)$ satisfies $L_\xi^N e^{ix\xi} = e^{ix\xi} \forall N$.

$$\begin{aligned} a) \text{ Check that for } a \in S^m, \quad \text{op}(a)f(x) &:= \int d\xi \, a(x, \xi) \hat{f}(\xi) e^{ix\xi} \\ &= \int d\xi \, L_\xi^N [a(x, \xi) \hat{f}(\xi)] e^{ix\xi} \end{aligned}$$

defines an operator $\text{op}(a) : S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^n)$, if a is a polynomial in ξ , $\text{op}(a)$ is a differential operator.

b) Write $\text{op}(a)f(x) = (k(x, \cdot) * f)(x)$, i.e.

$$k(x, \cdot) = \int \frac{d\xi}{\xi} a(x, \xi), \quad \text{a distribution } \in S'(\mathbb{R}^n) \quad \forall x \in \mathbb{R}^n.$$

Show that $k(x, \cdot)$ is a function away from the origin ($\xi = 0$) and satisfies $|k(x, z)| \leq C_N |z|^{-N}$ for $|z| \geq 1$, $N \in \mathbb{N}$, uniformly in x .

Hint: $\partial_\xi^\alpha a(x, \xi)$ is integrable in ξ for $|\alpha| \geq m+n+1$

$$\text{and } \left| \int \frac{d\xi}{\xi} \partial_\xi^\alpha a(x, \xi) \right| = |z^\alpha k(x, z)|.$$

c) Using the notation from b), and $n=1$, what is

$$k(x, \cdot) \text{ for } a(x, \xi) = \sum_{j=0}^m a_j \xi^j, \quad a_j \in \mathbb{C}^2$$

Hint: What is $\delta^{(j)} * f$?

Remark: Operators of the form $\text{op}(a)$, $a \in S^m$, are called "pseudo differential operators". In DifFun1, we only considered x -independent symbols.

nation of (9) and (12).^f If we examine the argument above, we see that the bound A' in (12), and therefore the bounds A_p in (11), do not depend on the regularity of K , but are a function of only the bound A in (9) and (10), the exponent p , and the constant c appearing in the decomposition theorem of §4. This concludes the proof of Theorem 3.

5.2 Since $L^p \cap L^q$ is a dense linear subspace of L^p (when $p < \infty$), we can use Theorem 3 to extend T to all of $L^p(\mathbf{R}^n)$, $1 < p < q$; this extension also satisfies the inequality (11), but now for all of L^p . Similarly we can extend T to L^1 , and there it satisfies the weak-type inequality (12). In fact for $p > 1$, whenever $\{f_n\}$ is a sequence in $L^p \cap L^q$ that converges in L^p norm, then by (11) the sequence $\{T(f_n)\}$ is Cauchy in the L^p norm. Likewise, when $p = 1$, $T(f_n)$ converges in measure by (12). Observe that the extension of T so obtained is unique. We summarize our discussion as a corollary:

COROLLARY. *The operator T in Theorem 3 has a unique extension to all of L^p , $1 \leq p < q$, that satisfies the inequalities (11) and (12).*

In what follows we shall also use the symbol T to denote the extension given by the corollary.

5.3 It is worthwhile to point out where the key assumption (10) is used in the proof of Theorem 3. It enters only via the inequality

$$\int_{B(\bar{y},\delta)} |Tf| d\mu \leq A \int_{B(\bar{y},\delta)} |f| d\mu,$$

which holds whenever f is a function supported in the ball $B(\bar{y},\delta)$ and satisfies the cancellation property

$$\int_{B(\bar{y},\delta)} f(x) d\mu(x) = 0.$$

This inequality will be a crucial fact in the extension of results of this kind to Hardy spaces, as we will see Chapter 3, §3.1.

5.4 We return to the interpolation theorem used in §5.1 above. The case for $q = \infty$ is already subsumed in the proof of Theorem 1

$$\mu\{|(Tf)(x)| > \alpha\} \leq A'' \left(\alpha^{-1} \int_{|f|>\alpha} |f| dx + \alpha^{-q} \int_{|f|\leq\alpha} |f|^q dx \right),$$

where A'' depends only on the constants A and A' appearing in (9) and (12). This results by splitting $f = f^\alpha + f_\alpha$, where $f^\alpha(x) = f(x)$ when $|f(x)| > \alpha$, and $f_\alpha(x) = f(x)$ when $|f(x)| \leq \alpha$. If we then apply (9) to Tf^α , and apply (12) to Tf_α , we get the desired distribution inequality for $Tf = Tf^\alpha + Tf_\alpha$. Now we can complete the argument by noting that

$$\int |(Tf)(x)|^p dx = p \int_0^\infty \mu\{x : |(Tf)(x)| > \alpha\} \alpha^{p-1} d\alpha.$$

6. Examples of the general theory

Having presented the basic ideas concerning maximal functions and singular integrals in their general abstract setting, we intend now to illustrate the theory by briefly indicating several examples that will be the subject of later study. Our aim in this section will be only to state in simplest form their salient facts; we postpone the actual proofs of most of the asserted properties to later chapters where these matters are taken up in detail. In keeping with our wish to quickly get to the point, we have chosen most of our examples from the context of the “classical” setting of the theory: here \mathbf{R}^n is given its usual Euclidean structure; i.e., the balls $B(x,\delta)$ are $\{y : |y-x| < \delta\}$, where $|\cdot|$ is the standard Euclidean norm; also $d\mu(x)$ is the usual Lebesgue measure dx .

6.1 Approximations of the identity. Suppose Φ is a fixed function on \mathbf{R}^n that is appropriately small at infinity; for example, take $|\Phi(x)| \leq A(1+|x|)^{-n-\varepsilon}$. We also assume here that Φ is normalized by the condition $\int \Phi dx = 1$. From such a Φ , we fashion an “approximation to the identity” via convolutions as follows. For each $t > 0$, we set

$$\lim_{t \rightarrow 0} (f * \Phi_t)(x) = f(x), \quad \text{for a.e. } x$$

whenever $f \in L^p(\mathbf{R}^n)$ for some p , $1 \leq p \leq \infty$.

Indeed, this result (like the corollary in §3.1 above) is a consequence of the fact that we have the following pointwise estimate.

^f We prove this particular case of the interpolation theorem in §5.4 below. See *Fourier Analysis*, Chapter 5, §2, and *Singular Integrals*, Chapter 1, §4 (and Appendix B of *Singular Integrals*) for more complete versions.

PROPOSITION.

$$\sup_{t>0} |f * \Phi_t(x)| \leq c_\Phi Mf(x).$$

The proof of this majorization (and its variants) can be found in Chapter 2, §2.1. For a more precise form of (16), see §8.16 below. We give two of the original and most important examples. First, if

$$\Phi(x) = c_n(1 + |x|^2)^{-(n+1)/2},$$

where

$$c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}},$$

then $\Phi_t(x)$ is the *Poisson kernel*, and

$$u(x, t) = (f * \Phi_t)(x)$$

gives the solution of the Dirichlet problem for the upper half space

$$\mathbf{R}_+^{n+1} = \{(x, t) : x \in \mathbf{R}^n, t > 0\},$$

namely

$$\Delta u = \left(\frac{\partial^2}{\partial t^2} + \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} \right) u(x, t) = 0, \quad u(x, 0) \equiv f(x).$$

The second example is the *Gaussian kernel*

$$\Phi(x) = (4\pi)^{-n/2} e^{-|x|^2/4}.$$

This time, if $u(x, t) = (f * \Phi_{t^{1/2}})(x)$, then u is the solution of the heat equation

$$\left(\frac{\partial}{\partial t} - \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} \right) u(x, t) = 0, \quad u(x, 0) \equiv f(x).$$

6.2 Singular Integrals. The main result proved in Theorem 3

for singular integrals is a conditional one, guaranteeing the boundedness on L^p for a range $1 < p \leq q$, on the assumption that the boundedness on L^q is already known; the most important instance of this occurs when $q = 2$. In keeping with this, we consider bounded linear transfor-

bounded function m on \mathbf{R}^n (the “multiplier”), so that T can be realized as

$$\widehat{Tf}(\xi) = m(\xi) \widehat{f}(\xi),$$

where $\widehat{\cdot}$ denotes the Fourier transform. Alternatively, at least on test functions $f \in \mathcal{S}$, T can be realized in terms of convolution with a kernel K ,

$$Tf = f * K, \quad (17)$$

where K is the distribution given by $\widehat{K} = m^\dagger$.

We shall now examine how Theorem 3 (for $q = 2$) applies to this class of operators. To proceed further, we assume that the distribution K agrees away from the origin with a function that is locally integrable away from the origin; we denote this function by $K(x)$. Then (17) implies that

$$Tf(x) = \int K(x-y) f(y) dy, \quad \text{for a.e. } x \notin \text{supp } f$$

whenever f is in L^2 and f has compact support. This is the representation (8) in the present context. Next, the basic condition (10) is then equivalent with

$$\int_{|x| \geq c|y|} |K(x-y) - K(x)| dx \leq A, \quad (18)$$

for all $y \neq 0$, where $c > 1$.

6.2.1 Let us consider the condition (18) further. First, as is easily seen, it is a consequence of the differential inequalities

$$\left| \left(\frac{\partial}{\partial x} \right)^\alpha K(x) \right| \leq A_\alpha |x|^{-n-|\alpha|} \quad \text{for all } \alpha, \quad (18_\alpha)$$

or its weaker form, (here $\gamma > 0$ is fixed)

$$|K(x-y) - K(x)| \leq A \frac{|y|^\gamma}{|x|^{n+\gamma}}, \quad \text{whenever } |x| \geq c|y|. \quad (18_\gamma)$$

How do K , satisfying such conditions, come about? It turns out that, roughly speaking, such conditions on K have equivalent versions when stated in terms of the Fourier transform of K , namely the multiplier m . In Chapter 6, we shall prove the following proposition:

[†] For these two realizations of translation-invariant operators on $L^2(\mathbf{R}^n)$, see

and $\overline{M}_\infty f(x) = \sup_j |Mf_j(x)|$. Then, in analogy with the case $q = 2$ handled by Theorem 1, one can show that

$$\begin{aligned} |\{\overline{M}_q f(x) > \alpha\}| &\leq \frac{A_q}{\alpha} \int |f|_q dx \quad \text{and} \\ \|\overline{M}_q f\|_p &\leq A_{p,q} \|f\|_q, \quad 1 < p < \infty. \end{aligned} \quad (13)$$

The proof for $1 < q < \infty$ is nearly identical to that for $q = 2$, except one begins with the easy case $p = q$, instead of $p = 2$. Observe also that $q = \infty$ reduces to the scalar-valued case, because then $\overline{M}_\infty f \leq M(|f|_\infty)$. Finally, it should be observed that analogues of (13) fail when $q = 1$ (see §5.1 below).

(ii) The weighted inequalities (8) and (9) naturally raise the question as to what happens when the weights on both sides of the inequality are the same. In view of the proposition, an obvious sufficient condition is that $M\omega \leq c\omega$. This is the A_1 condition; its variants, which give necessary and sufficient conditions for $q > 1$, are the subject of Chapter 5. One can also observe that (8) is sharp in the sense that, if $\omega\{Mf > \alpha\} \leq A/\alpha \int |f|_\omega dx$, for all f and α , then $\omega \geq cM\omega$; this follows by choosing for f a sequence converging weakly to a point mass at an arbitrary $x \in \mathbb{R}^n$.

2. Nontangential behavior and Carleson measures

In many situations, the study of a function f defined on \mathbb{R}^n can be closely connected with related properties of a corresponding function F defined on the (open) upper half-space \mathbb{R}_{+}^{n+1} , with F constructed from f by some averaging process. The simplest example arises when the average is carried out by a suitable “approximation of the identity” as follows.

Fix an integrable function Φ on \mathbb{R}^n with $\int_{\mathbb{R}^n} \Phi dx = 1$. For $t > 0$, set $\Phi_t(x) = t^{-n} \Phi(x/t)$, so that $\int \Phi_t = 1$ for all t . Now let

$$F(x, t) = (f * \Phi_t)(x). \quad (14)$$

Then, as is well known, for appropriate f and Φ , $F(x, t) \rightarrow f(x)$ as $t \rightarrow 0$ in a variety of senses. Some are quite easy to see, such as convergence in the L^p norm for $f \in L^p$, $1 \leq p < \infty$. Others, such as the one we shall consider here, are deeper and have to do with the possibility of pointwise convergence in the nontangential sense; this involves a corresponding maximal function

$$F^*(x) = \sup_{|x-y| < t} |F(y, t)|. \quad (15)$$

For the systematic study of such functions F , it will be important to free ourselves from the restriction that F be given as an average of functions in (14), and we will therefore turn our attention to a general class of functions in \mathbb{R}_{+}^{n+1} for which we have nontangential control. However, before proceeding in this more general setting, it will be helpful to review briefly some of the properties of functions F that do arise in the form (14).

2.1 Nontangential maximal functions and averages. To begin with, we recall the relation of the averages (14) with the maximal function Mf given by

$$(Mf)(x) = \sup_{r>0} c_n r^{-n} \int_{|y| < r} |f(x-y)| dy. \quad (16)$$

Whenever Φ is a nonnegative function on \mathbb{R}^n that is radial and (radially) decreasing, then

$$\sup_{t>0} |f * \Phi_t(x)| \leq Mf(x) * \int_{\mathbb{R}^n} \Phi dy.$$

To see this, it suffices to verify that the inequality

$$|f * \tilde{\Phi}(x)| \leq Mf(x) \quad (17)$$

holds for functions Φ of the above type that are normalized by the condition that $\int \Phi = 1$.

First take $\tilde{\Phi}$ to be of the form $\sum_{j=1}^N a_j \chi_{B_j}$, where each a_j is a positive constant and each χ_{B_j} is the characteristic function of a ball B_j that is centered at the origin. Then, since $\sum a_j |B_j| = 1$ and $(f * \chi_{B_j})(x) \leq |B_j| Mf(x)$, the inequality (17) follows immediately. In general, any nonnegative, integrable, radial, and radially decreasing Φ can be approximated by such finite sums; so the inequalities (17) and (16) hold as claimed.

The implication of the above for nontangential control is then a simple consequence.

PROPOSITION. Assume that Φ has a radial majorant that is non-increasing, bounded, and integrable. Then, with $F(x, t) = f * \Phi_t(x)$, we have

$$F^*(x) = \sup_{|y-x| < t} |F(y, t)| \leq c Mf(x).$$

Proof. If the radial nonincreasing majorant of $\tilde{\Phi}(x)$ is $\tilde{\Phi}(|x|)$, then $\sup_{|u| \leq 1} |\tilde{\Phi}(x-u)|$ has a radial majorant $\tilde{\Psi}(|x|)$, where $\tilde{\Psi}(r) = \tilde{\Phi}(0)$ for $0 \leq r \leq 1$, and $\tilde{\Psi}(r) = \tilde{\Phi}(r-1)$ for $r > 1$. The integrability of $\tilde{\Psi}$ then follows from that of $\tilde{\Phi}$ and the finiteness of $\tilde{\Phi}(0)$. We can therefore apply (16) to obtain our conclusion.

1.3 To pass from merely formal statements we need to make precise the definition of the pseudo-differential operator (1) and the class of symbols that is used. We shall consider in this chapter the standard symbol class, denoted by S^m , which is the most common and useful of the general symbol classes. A function a belongs to S^m (and is said to be of *order* m) if $a(x, \xi)$ is a C^∞ function of $(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n$ and satisfies the differential inequalities

$$|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \leq A_{\alpha, \beta}(1 + |\xi|)^{m - |\alpha|}, \quad (2)$$

for all multi-indices α and β . For instance, (2) holds when a is a polynomial in ξ (independent of x) of degree m .

Given a symbol $a \in S^m$, the operator T_a will initially be defined on the Schwartz class of testing functions \mathcal{S} . In fact, it is immediate that the integral (1) converges absolutely and is infinitely differentiable. An integration by parts argument shows that $T_a(f)$ is a rapidly decreasing function. Indeed, observe that

$$(I - \Delta_\xi)^N e^{2\pi i x \cdot \xi} = (1 + 4\pi^2 |x|^2)^{-1} e^{2\pi i x \cdot \xi},$$

and define the operator

$$L_\xi = (1 + 4\pi^2 |x|^2)^{-1} (I - \Delta_\xi);$$

thus $(L_\xi)^N e^{2\pi i x \cdot \xi} = e^{2\pi i x \cdot \xi}$. Inserting this in (1) and carrying out the repeated integrations by parts gives us

$$(T_a f)(x) = \int (L_\xi)^N [a(x, \xi) \hat{f}(\xi)] e^{2\pi i x \cdot \xi} d\xi,$$

from which the rapid decrease of $T_a(f)$ is evident. Since a similar argument works for any partial derivative of $T_a(f)$, we see that T_a maps \mathcal{S} to \mathcal{S} , and that this mapping is continuous. It is also useful to note that if $\{a_k\}$ is a pointwise convergent sequence of symbols in S^m that satisfy the inequalities (2) uniformly in k , then $T_{a_k}(f) \rightarrow T_a(f)$ in \mathcal{S} whenever $f \in \mathcal{S}$; we shall make use of this fact later.

1.4 An alternative way of writing the definition (1) is as a repeated integral

$$(T_a f)(x) = \int \int a(x, \xi) e^{2\pi i \xi \cdot (x-y)} f(y) dy d\xi. \quad (3)$$

However, the integral (3) does not necessarily converge absolutely, even when $f \in \mathcal{S}$. To deal with this, and also to facilitate other manipulations with pseudo-differential operators, it is convenient to approximate a given symbol by symbols of compact support. For this purpose, fix a function $\gamma \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ with $\gamma(0, 0) = 1$. Set $a_\varepsilon(x, \xi) = a(x, \xi) \gamma(\varepsilon x, \varepsilon \xi)$. Notice that if a belongs to the symbol class S^m , then so do the a_ε , and

Next observe that, as was remarked at the end of §1.3, $T_{a_\varepsilon} \rightarrow T_a$, in the sense that $T_{a_\varepsilon}(f) \rightarrow T_a(f)$ in \mathcal{S} , whenever $f \in \mathcal{S}$, as $\varepsilon \rightarrow 0$. Moreover, since the alternate definition (3) clearly holds when a has compact support, we get that

$$(T_a f)(x) = \lim_{\varepsilon \rightarrow 0} \int \int a_\varepsilon(x, \xi) e^{2\pi i \xi \cdot (x-y)} f(y) dy d\xi.$$

In addition, one can verify directly the duality relation

$$\langle T_a f, g \rangle = \langle f, T_a^* g \rangle, \quad (4)$$

whenever $f, g \in \mathcal{S}$; here

$$(T_a^* g)(y) = \lim_{\varepsilon \rightarrow 0} \int \int \bar{a}_\varepsilon(x, \xi) e^{2\pi i \xi \cdot (y-x)} g(x) dx d\xi, \quad (5)$$

and $\langle f, g \rangle$ denotes $\int_{\mathbf{R}^n} f(x) \bar{g}(x) dx$. Imitating our proof that T_a maps \mathcal{S} to \mathcal{S} , we set

$$L_x = (1 + 4\pi^2 |x|^2)^{-1} (I - \Delta_x),$$

and note that the right side of (5) equals

$$\lim_{\varepsilon \rightarrow 0} \int \int (L_x)^N [\bar{a}_\varepsilon(x, \xi) g(x)] e^{2\pi i \xi \cdot (y-x)} dx d\xi,$$

from which it is clear that T_a^* also maps \mathcal{S} to \mathcal{S} .

Thus the pseudo-differential operator T_a , initially defined as a mapping from \mathcal{S} to \mathcal{S} , extends via the duality (4) to a mapping from the space of tempered distributions to itself. Notice also that T_a is automatically continuous on this space.

1.5 We note that our symbols (at least when the order $m = 0$) satisfy conditions for large ξ of the kind enjoyed by the standard multipliers arising for the translation-invariant singular integrals described in Chapter 1, §6.2. However, our symbols are, in addition, always assumed to be smooth for all ξ .† This restriction has the advantage of retaining the local behavior of such operators, while eliminating a variety of problems for large x that are not always relevant in applications such as partial differential equations. An additional restriction of this type which is sometimes made is that the symbol $a(x, \xi)$ have compact support in x . While this assumption often simplifies an argument, we shall see below that it is not essential, and we shall not be bound by it.

2. An L^2 theorem

After these preliminaries, we state the first main result.

THEOREM 1. Suppose a is a symbol of order 0, i.e., that $a \in S^0$. Then the operator T_a , initially defined on \mathcal{S} , extends to a bounded operator from $L^2(\mathbb{R}^n)$ to itself.

To prove the theorem, it suffices to show that

$$\|T_a(f)\|_{L^2} \leq A \|f\|_{L^2}, \quad \text{whenever } f \in \mathcal{S}, \quad (6)$$

with A independent of f . In fact, suppose that $f \in L^2$, and let $\{f_n\}$ be a sequence in \mathcal{S} so that $f_n \rightarrow f$ in L^2 . Then because of (6), $T_a(f_n)$ converges in L^2 norm, while, in view of the remarks in §1.4 above, $T_a(f_n)$ converges to $T_a(f)$ in the sense of distributions.

2.1 Turning to the proof of (6), we establish it first under the assumption that the symbol $a(x, \xi)$ has compact support in x . This restriction makes it possible to use the Fourier transform in the x variable, and expand the symbol in a simple way using products of symbols depending on x or ξ only.

We write

$$a(x, \xi) = \int_{\mathbb{R}^n} \hat{a}(\lambda, \xi) e^{2\pi i x \cdot \lambda} d\lambda,$$

where

$$\hat{a}(\lambda, \xi) = \int_{\mathbb{R}^n} a(x, \xi) e^{-2\pi i x \cdot \lambda} dx.$$

Since a has compact x -support, the integral defining $\hat{a}(\lambda, \xi)$ converges. An integration by parts shows that, for each multi-index α ,

$$(2\pi i)^{\alpha} \hat{a}(\lambda, \xi) = \int_{\mathbb{R}^n} [\partial_x^\alpha a(x, \xi)] e^{-2\pi i x \cdot \lambda} dx$$

and that $|(2\pi i)^{\alpha} \hat{a}(\lambda, \xi)| \leq c_\alpha$, uniformly in ξ . As a result we have

$$\sup_{\xi} |\hat{a}(\lambda, \xi)| \leq A_N (1 + |\lambda|)^{-N} \quad (7)$$

for arbitrary $N \geq 0$. Now

$$\begin{aligned} (T_a f)(x) &= \int a(x, \xi) \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi \\ &= \int \int \hat{a}(\lambda, \xi) \widehat{f}(\xi) e^{2\pi i x \cdot \lambda} (\widehat{T}_{a(\lambda, \xi)} f)(x) d\xi d\lambda, \end{aligned}$$

where

$$(T^\lambda f)(x) = e^{2\pi i x \cdot \lambda} (T_{a(\lambda, \xi)} f)(x).$$

Since for each λ , $T_{a(\lambda, \xi)}$ is a multiplier operator on the Fourier transform side, by Plancherel's theorem we have that

$$\|T_{a(\lambda, \xi)} f\|_{L^2} \leq \sup_{\xi} |\hat{a}(\lambda, \xi)| \cdot \|\widehat{f}\|_{L^2} = \sup_{\xi} |\hat{a}(\lambda, \xi)| \cdot \|f\|_{L^2}.$$

§2. AN L^2 THEOREM

$$\|T_a\| \leq A_N \int (1 + |\lambda|)^{-N} d\lambda < \infty,$$

if we choose $N > n$. Thus (6) is proved when a has compact support in x .

2.2 The proof for general symbols requires that we also use the singular integral realization of the operator T_a . That is, we shall write

$$(T_a f)(x) = \int_{\mathbb{R}^n} k(x, z) f(x - z) dz \quad (8)$$

with the following understanding. First, for each x , $k(x, \cdot)$ is the distribution whose Fourier transform is the function $a(x, \cdot)$; written formally this is the identity

$$a(x, \xi) = \int_{\mathbb{R}^n} k(x, z) e^{-2\pi i x \cdot \xi} dz.$$

Thus (8) can be interpreted as the convolution of the distribution $k(x, \cdot)$ with the function $f \in \mathcal{S}$, evaluated at the point x .

We are about to prove that, away from the origin, the distribution $k(x, \cdot)$ agrees with a function that is rapidly decreasing at infinity. Therefore, (8) can be interpreted as a convergent integral for x lying outside the support of f , if we assume that the complement of that support is nonempty. We will also use the more standard notation

$$(T_a f)(x) = \int K(x, y) f(y) dy,$$

again for $x \notin \text{supp}(f)$, where we have set $K(x, y) = k(x - x - y)$.

The precise study of the singular integral kernel $k(x, \cdot)$ will be taken up in §4 below. For our immediate purposes a very crude estimate will suffice, namely that, away from the origin, $k(x, \cdot)$ is a function that satisfies the inequality

$$|k(x, z)| \leq A_N |z|^{-N}, \quad \text{for all } |z| \geq 1 \text{ and all } N > 0, \quad (9)$$

uniformly in x . To see this, write

$$(T_a f)(x) = \int a(x, \xi) \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

By the standard properties of convolutions of test functions and distributions, this integral equals $[k(x, \cdot) * f](x)$, where $k(x, \cdot)$ is the distribution whose Fourier transform is the function $a(x, \cdot)$, as we have written in (8). Next, $(-2\pi i z)^\alpha k(x, \cdot)$, with the distribution $k(x, \cdot)$ thought of as acting on functions of z , equals the inverse Fourier transform of $\partial_x^\alpha a(x, \xi)$; by (2), $\partial_x^\alpha a(x, \xi)$ is integrable in ξ whenever $|\alpha| \geq n+1$. This shows that $k(x, \cdot)$ equals a function $k(x, z)$ away from the origin ...

Bx6

1. a) Let $\varphi: [0, \infty) \rightarrow [0, \infty]$ non-increasing

and $\Psi: \mathbb{R}^n \rightarrow [0, \infty]$, $\Psi(x) := \varphi(|x|)$

Show $\forall f \in L^p(\mathbb{R}^n)$:

$$|\mathcal{F}_x \Psi(x)| \leq \left(\int_{\mathbb{R}^n} \Psi \right) \sup_{r>0} \int_{B(x,r)} |f(y)| dy$$

$$|\mathcal{F}_x \Psi_\epsilon(x)| \leq \left(\int_{\mathbb{R}^n} \Psi \right) \quad \text{for } \epsilon > 0$$

$$\Psi_\epsilon(x) := \epsilon^{-1} \Psi(\frac{x}{\epsilon})$$

PF:

$$\text{Let } \Psi = \sum_{j=1}^N a_j \mathbb{1}_{B(x,j)}, \quad a_j \text{ as } j \in (0, \infty) \text{ (non-increasing)}$$

$$(\mathcal{F}_x \Psi)(x) = \int_{\mathbb{R}^n} |\mathcal{F}(x+y)| \Psi(y) dy$$

$$= \sum_{j=1}^N \int_{\mathbb{R}^n} |\mathcal{F}(x+y)| a_j \mathbb{1}_{B(x,j)}(y) dy$$

$$\Rightarrow \Psi(x) = \sum_{j=1}^N a_j \mathbb{1}_{B(x,j)}$$

$$|\mathcal{F}_x \Psi(x)| = \left| \int_{\mathbb{R}^n} \Psi(x-y) f(y) dy \right|$$

$$\leq \sum_{j=1}^N \int_{\mathbb{R}^n} a_j \mathbb{1}_{B(x,j)}(x-y) |\mathcal{F}(y)| dy$$

$$= \sum_{j=1}^N a_j \int_{B(x,j)} |\mathcal{F}(y)| dy$$

$$= \sum_{j=1}^N a_j |B_{x,j}(x)| \int_{B_{x,j}(x)} |\mathcal{F}(y)| dy$$

$$= \int_{\mathbb{R}^n} \sum_{j=1}^N a_j \mathbb{1}_{B_{x,j}(x)}(y) dy \leq \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\mathcal{F}(y)| dy$$

$$= \int_{\mathbb{R}^n} \sum_{j=1}^N a_j \mathbb{1}_{B_{x,j}(x)} = \int_{\mathbb{R}^n} \Psi(x) dx$$

For arbitrary $\Psi: [0, \infty) \rightarrow [0, \infty]$ non-increasing s.t. $\int_0^\infty \Psi < \infty$

$$\text{We can approx. } \Psi \leftarrow \Psi = \sum_{j=1}^N c_j \mathbb{1}_{B_{j-1}} \text{ with } \sum_{j=1}^N c_j = \int_0^\infty \Psi \mathbb{1}_{B_{j-1}} dx$$

$$+ \sum_{j=1}^N c_j' \mathbb{1}_{B_j} \leftrightarrow \Psi$$

$\Psi_N \in \Psi$

$$|\mathcal{F} * \Psi(x) - \mathcal{F} * \Psi_N(x)| = |\mathcal{F} * (\Psi - \Psi_N)(x)|$$

$T: L^p \rightarrow \mathcal{F} * \Psi$ is bounded $L^p \rightarrow L^p$

$$\Rightarrow \left(\int |\mathcal{F} * \Psi(x) - \mathcal{F} * \Psi_N(x)|^p dx \right)^{1/p} = \|T(\Psi - \Psi_N)\|_{L^p} \leq \|T\| \|\Psi - \Psi_N\|_p$$

$$\Rightarrow \mathcal{F} * \Psi_N \rightarrow \mathcal{F} * \Psi \text{ in } L^p \quad \left(\mathcal{F} * (\Psi - \Psi_N)(x) \leftarrow \int \mathcal{F}(x-y) (\Psi - \Psi_N)(y) dy \right)$$

$\Rightarrow \mathcal{F} * \Psi_N(x) \rightarrow \mathcal{F} * \Psi(x) \text{ a.e. } x \in \mathbb{R}^n$

using

$$|\mathcal{F} * \Psi(x)| = \lim_{N \rightarrow \infty} |\mathcal{F} * \Psi_N(x)|$$

$$\leq (\int |\Psi_N|^p) \sup_{B_r(x)} \frac{1}{|B_r(x)|} \int_{B_r(x)} |\mathcal{F}(y)| dy$$

$$\leq \int_{\mathbb{R}^n} |\Psi|$$

$$\Psi_\epsilon(x) := \epsilon^{-n} \Psi(\frac{x}{\epsilon}), \quad \epsilon > 0$$

$$\Rightarrow \int_{\mathbb{R}^n} \Psi_\epsilon = \int_{\mathbb{R}^n} \epsilon^{-n} \Psi(\frac{x}{\epsilon}) dx = \int_{\mathbb{R}^n} \Psi(y) dy = \int_{\mathbb{R}^n} \Psi$$

↗

b) Let $\Phi \in L^1(\mathbb{R}^n)$, $\|\Phi\|_1 = 1$, $|\mathcal{F}\Phi(x)| \leq \Psi(|x|)$

with $\Psi: [0, \infty) \rightarrow [0, \infty)$ as in a) and bounded, $\Psi \in L^1$

$$\text{Show } \sup_{\epsilon > 0} |\mathcal{F} * \Phi_\epsilon(x)| \leq C_\Phi \sup_{r > 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |\mathcal{F}(y)| dy$$

~~$$\Phi = \Phi_+ + \Phi_-$$

$$|\mathcal{F} * \Phi(x)| \leq |\mathcal{F} * \Phi_+(x)| + |\mathcal{F} * \Phi_-(x)|$$

$$\leq (\int_{\mathbb{R}^n} |\Phi_+|) \sup_{B_r(x)} + (\int_{\mathbb{R}^n} |\Phi_-|) \sup_{B_r(x)}$$

$$= (\int_{\mathbb{R}^n} |\Phi|) \sup_{B_r(x)}$$~~

$$\begin{aligned} |\mathcal{F} * \Phi_\epsilon(x)| &= \left| \int_{\mathbb{R}^n} \Phi_\epsilon(x-y) \mathcal{F}(y) dy \right| \leq \int_{\mathbb{R}^n} |\Phi_\epsilon(x-y)| |\mathcal{F}(y)| dy \\ &= \epsilon^{-n} |\Phi(\frac{x}{\epsilon})| \leq \epsilon^{-n} \Psi(\frac{|x|}{\epsilon}) \\ &= \Psi_\epsilon(|x|) \end{aligned}$$

$$\leq \int_{\mathbb{R}^n} \Psi_\epsilon(x-y) |\mathcal{F}(y)| dy = |\mathcal{F}| * \Psi_\epsilon(x)$$

$$\leq (\int_{\mathbb{R}^n} \Psi) \sup_{r > 0} \frac{1}{|B_r(x)|} \underbrace{\int_{B_r(x)} |\mathcal{F}(y)| dy}_{C_\Phi}$$

↗

c) Show $\lim_{\epsilon \rightarrow 0^+} f * \Phi_\epsilon(x) = f(x)$ a.e. $x \in \mathbb{R}^n$

$$\begin{aligned} f * \Phi_\epsilon(x) - f(x) &= \int f(x-y) \Phi_\epsilon(y) dy - \int f(x) \Phi_\epsilon(y) dy \\ &= \int (f(x-y) - f(x)) \Phi_\epsilon(y) dy \end{aligned}$$

• For $f \in C_c(\mathbb{R}^n)$:

$$\text{take } \epsilon > 0, \quad |f(x) - f * \Phi_\epsilon(x)| \leq \underbrace{\int |f(x-y) - f(x)| |\Phi_\epsilon(y)| dy}_{\leq C}$$

$$\leq \int_{|y| < R} |f(x-y) - f(x)| |\Phi_\epsilon(y)| dy + C \int_{|y| \geq R} |\Phi_\epsilon(y)| dy$$

$$\int_{|z| \geq R} |\Phi_\epsilon(z)| dz$$

$$\leq \sup_{|y| < R} |f(x-y) - f(x)| \cdot \int_{|z| \geq R} |\Phi_\epsilon(z)| dz \stackrel{\epsilon \text{ small}}{\leq} \frac{\epsilon}{2} \text{ for } R \text{ big}$$

$$\leq \frac{\epsilon}{2} \text{ for } \epsilon \text{ small}$$

(f uniformly cont.)

• Approx $f \in L^p$ by $f_k \in C_c$, i.e. $\|f_k - f\|_p \rightarrow 0$

$$\Rightarrow f_k(x) \rightarrow f(x) \text{ a.e. } x \in \mathbb{R}^n \quad \|f_k - f\|_p < 2^{-k}$$

$$\lim_{\epsilon \rightarrow 0^+} f_k * \Phi_\epsilon(x)$$

$$f * \Phi_\epsilon(x) + (f_k - f) * \Phi_\epsilon(x)$$

$$|(f_k - f) * \Phi_\epsilon(x)| \leq \sup_{\epsilon > 0} |(f_k - f) * \Phi_\epsilon(x)| \leq C_p \sup_{r>0} \frac{1}{B_r(x)} \int_{B_r(x)} |f_k - f|$$

$$= C_p \sup_{r>0} A_r |f_k - f|(x)$$

$$\text{Hardy-Littlewood} \Rightarrow \left\| \sup_{r>0} A_r |f_k - f| \right\|_{L^p(\mathbb{R}^n)}, 1/p \leq \infty \rightarrow 0$$

$$\Rightarrow \sup_{r>0} A_r |f_k - f|(x) \xrightarrow{k \rightarrow \infty} 0, \text{ a.e. } x \in \mathbb{R}^n$$

$$\lim_{\epsilon \rightarrow 0^+} f * \Phi_\epsilon(x) = f(x) \text{ a.e. } x \in \mathbb{R}^n$$

d) Let $\Phi_1(x) := C_1(1+x^2)^{-\frac{n}{2}}$, C_1, C_2 s.t. $\int \Phi_{1,2} = 1$

$$\Phi_2(x) := C_2 e^{-\frac{|x|^2}{4}}$$

$\mathcal{F} \in L^p(\mathbb{R}^n)$, $u_1(t,x) := \mathcal{F} * (\Phi_1)_{t^{\frac{1}{2}}}(x)$

$$t > 0 \quad u_2(t,x) := \mathcal{F} * (\Phi_2)_{t^{\frac{1}{2}}}(x)$$

$$\begin{aligned} u_1(t,x) &= \mathcal{F} * (\Phi_1)_{t^{\frac{1}{2}}}(x) = \int (\Phi_1)_{t^{\frac{1}{2}}}(x-y) \mathcal{F}(y) dy \\ &= C_1 \int t^{-\frac{n}{2}} \Phi_1(t^{\frac{1}{2}}(x-y)) \mathcal{F}(y) dy \end{aligned}$$

$$\partial_t u_1 = -\frac{n}{2} t^{-\frac{1}{2}} u_1 + C_1 \int t^{-\frac{n+2}{2}} \Phi_1'(t^{\frac{1}{2}}(x-y)) \mathcal{F}(y) dy$$

$$(\Phi_{1,2})_{t^{\frac{1}{2}}} \in C^\infty, \quad \mathcal{F} \in L^p \Rightarrow u_{1,2} \in C_x^\infty$$

$$u_2(t,x) = \int (\Phi_2)_{t^{\frac{1}{2}}}(x-y) \mathcal{F}(y) dy = C_2 \int t^{\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}} \mathcal{F}(y) dy$$

$$\partial_t u_2 = C_2 \int \left(-\frac{n}{2} t^{-\frac{1}{2}} + \frac{|x-y|^2}{4t} t^{-\frac{3}{2}} \right) t^{\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}} \mathcal{F}(y) dy$$

$$\Delta_x u_2 = C_2 \int t^{-\frac{n}{2}} (\Delta_y e^{-\frac{|x-y|^2}{4t}}) \mathcal{F}(y) dy$$

$$\nabla_x e^{-\frac{|x-y|^2}{4t}} = -\nabla_x \frac{|x-y|^2}{4t} e^{-\frac{|x-y|^2}{4t}} = -\frac{(x-y)}{2t} e^{-\frac{|x-y|^2}{4t}}$$

$$\nabla_x \left(\dots \right) = +\nabla_x (x-y) \frac{1}{2t} e^{-\frac{|x-y|^2}{4t}} - (x-y) \cdot \nabla_x e^{-\frac{|x-y|^2}{4t}}$$

$$\Rightarrow e^{\frac{|x-y|^2}{4t}} \left(\dots \right) = -\frac{\nabla_x x}{2t} + \frac{(x-y) \cdot (x-y)}{2t} = -\frac{n}{2} t^{-\frac{1}{2}} + \frac{|x-y|^2}{4t^2}$$

$$\Rightarrow (\partial_t - \Delta_x) u_2 = 0$$

similarly $(\partial_t^2 - \Delta_x) u_1 = 0$

$\Phi_{1,2} \geq 0$, spher. symmetric & radially nonincreasing

$$\int \Phi_1 dr = C \int_0^\infty (1+r^2)^{-\frac{n+1}{2}} r^{n-1} dr < \infty$$

$\sim r^{-2}, r \rightarrow \infty$

$$\int \Phi_2 dx = C \int_0^\infty e^{-\frac{|x|^2}{4}} r^{n-1} dr < \infty$$

$$\Rightarrow \Phi_{1,2} \in L^1$$

$$(a) \rightarrow (c) \rightarrow \lim_{t \rightarrow 0^+} u_{k,1,2}(x) = \lim_{t \rightarrow 0^+} \mathcal{F} * (\Phi_k)_{t^{\frac{1}{2}}}(x) = \mathcal{F}(x)$$

a.e. x

(2) Let $\mathcal{S}^m := \left\{ a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) : \text{there } \alpha \in \mathbb{N}_0^n \text{ s.t. } C_{\alpha, m} := \left| \partial_x^\alpha \partial_z^\alpha a(x, z) \right| \leq C_{\alpha, m} (1+|z|)^{m-|\alpha|} \right\}$

$$L_3 := (1+|x|^2)^{-1} (1-\Delta_z)$$

$$\begin{aligned} L_3 e^{ix \cdot z} &= (1+|x|^2)^{-1} (e^{ix \cdot z} - \Delta_z e^{ix \cdot z}) \\ &\Rightarrow L_3^N e^{ix \cdot z} = e^{ix \cdot z} \quad \forall N \geq 0 \end{aligned}$$

a) Check that for $a \in \mathcal{S}^m$, $\text{op}(a)f(x) := \int a(x, z) \hat{f}(z) e^{ix \cdot z} dz$
 $\in \int L_3^N (a(x, z) \hat{f}(z)) e^{ix \cdot z} dz$
defines an op. $\text{op}(a) : S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$

If a is a polynomial in z , then $\text{op}(a)$ is a diff. op.

Show $\text{op}(a) \in S'(\mathbb{R}^n)$

For $f \in S(\mathbb{R}^n)$, $\partial_x^\alpha a(x, \cdot), \hat{f}(\cdot) \in C^\infty(\mathbb{R}^n)$

$$\begin{aligned} \left| \int \partial_x^\alpha (\partial_z^\beta (a(x, z) \hat{f}(z))) dz \right| &\leq \sum_{\substack{\beta_1 + \beta_2 = \beta \\ \beta_1 + \beta_2 = \beta}} \left| \int \partial_z^{\beta_1} (\partial_z^{\beta_2} a(x, z)) \partial_z^{\beta_2} \hat{f}(z) dz \right| \\ &\leq \sum_{\substack{\beta_1 + \beta_2 = \beta \\ \beta_1 + \beta_2 = \beta}} |z|^\alpha C_{\beta, \beta} (1+|z|)^{m-|\beta|} |\partial_z^{\beta_2} \hat{f}(z)| \\ &\leq C_{\alpha, \beta, \gamma} \quad \forall \alpha, \beta, \gamma \\ \Rightarrow \partial_x^\alpha a(x, \cdot) \hat{f}(\cdot) &\in S(\mathbb{R}^n) \quad \forall \gamma \quad \Rightarrow L_3^N [a(x, z) \hat{f}(z)] \in S'(\mathbb{R}^n) \quad \forall N \end{aligned}$$

$$\text{PB} \Rightarrow \int \left[\int \partial_x^\alpha a(x, z) \hat{f}(z) dz \right] e^{ix \cdot z} dz$$

$$= \int L_3^N \partial_x^\alpha a(x, z) \hat{f}(z) dz e^{ix \cdot z} \quad \forall N \geq 0$$

$$\begin{aligned} \partial_x^\alpha \text{op}(a) f(x) &= \sum_{\beta_1 + \beta_2 = \beta} \int \partial_x^{\beta_1} a(x, z) \hat{f}(z) \partial_z^{\beta_2} e^{ix \cdot z} dz \\ &= \sum_{\beta_1 + \beta_2 = \beta} \int L_3^N \left[(\partial_z^{\beta_2} \partial_x^{\beta_1} a(x, z)) \hat{f}(z) \right] e^{ix \cdot z} dz \end{aligned}$$

$$\begin{aligned} \Rightarrow \left| x^\alpha \partial_x^\alpha \text{op}(a) f(x) \right| &\leq \sum_{\beta_1 + \beta_2 = \beta} \int |x^\alpha (1+|x|^2)^{-N} (1-\Delta_z)^N | \left| (\partial_z^{\beta_2} \partial_x^{\beta_1} a(x, z)) \hat{f}(z) \right| dz \\ &\leq C_{\alpha, \beta} \int (1+|z|)^{-M} dz \leq C_{\alpha, \beta} \quad \text{indep. of } x \end{aligned}$$

$$\Rightarrow \text{op}(a) f \in S(\mathbb{R}^n)$$

Let $a(x, z) = \sum_{|\alpha| \leq m} a_\alpha(x) z^\alpha$, $a_\alpha \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$
 $\partial_x^\alpha a_\alpha \in L^\infty(\mathbb{R}^n)$ HKB
 $\Rightarrow a \in S^m$

$$\begin{aligned} f \in S &\Rightarrow \text{op}(a)f(x) = \int a(x, z) \hat{f}(z) e^{izx} dz \\ &= \sum_{|\alpha| \leq m} (a_\alpha(x) \int z^\alpha \hat{f}(z) e^{izx} dz) \\ &\quad (2\pi)^n \int_{\mathbb{R}^n} z^{\alpha+1} \hat{f}(z) dz = (2\pi)^n D^\alpha f(x) \\ &= \sum_{|\alpha| \leq m} (2\pi)^n a_\alpha(x) D^\alpha f(x) \quad \text{differential op.} \end{aligned}$$

b) Write $\text{op}(a)f(x) = (k(x, \cdot) * f)(x)$, $k(x, \cdot) = \int_{\mathbb{R}^n} a(x, z) \hat{f}(z) dz \in S'(\mathbb{R}^n)$
 $a \in S'$ $\hat{f} \in S'$
 $\hat{f} = \mathcal{F}^{-1} f$
 $\hat{f} = \mathcal{F}^{-1} a(x, \cdot) \hat{f}(z) = a(x, z) \hat{f}(z)$

$$\begin{aligned} \text{op}(a)f(x) &= \int a(x, z) \hat{f}(z) e^{izx} dz \\ &= (2\pi)^n \int_{\mathbb{R}^n} a(x, z) \hat{f}(z) dz \\ &= (2\pi)^n \mathcal{F}^{-1} (\mathcal{F} a(x, \cdot) * \hat{f}) \\ &= (2\pi)^n \mathcal{F}^{-1} (\mathcal{F} \mathcal{F}^{-1} a(x, \cdot) * \hat{f}) \\ &= (2\pi)^n \mathcal{F}^{-1} (\mathcal{F} a(x, \cdot) * f) \\ \text{if } \phi \in C_0^\infty : g(\phi) &= (\mathcal{F}^{-1} a(x, \cdot)) * \mathcal{F} * \phi(0) = \mathcal{F}^{-1} a(x, \cdot) (\mathcal{F} * \phi) \\ g(\phi) &= (\mathcal{F}^{-1} a(x, \cdot)) \left(\sum_n \hat{\phi}_n \right) \\ (\mathcal{F}^{-1} \hat{\phi})^n * \phi &= (2\pi)^n (\mathcal{F} \hat{\phi})^n = (\mathcal{F} \phi)^n \end{aligned}$$

Show that, $k(x, \cdot) \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$

and $|k(x, z)| \lesssim C_N |z|^{1-N}$, $1 \leq N \leq n$, uniformly in x

$$\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \Rightarrow \langle k(x, \cdot), \phi \rangle = \langle \mathcal{F}^{-1} a(x, \cdot), \phi \rangle$$

$$\phi = \hat{\psi} \Rightarrow \langle k(x, \cdot), \phi \rangle = \langle \mathcal{F}^{-1} a(x, \cdot), \psi \rangle$$

$$= \int a(x, z) \psi(z) dz$$

$$\begin{aligned}
\phi \in C_0^\infty & \quad \langle \text{op}(a) f, \phi \rangle = \iint \iint a(x, z) \hat{f}(z) e^{izx} dz dx \phi(x) dx \\
&= \iint \iint a(x, z) e^{ixz} \hat{f}(z) \phi(x) dx dz \\
&\quad \left\langle [k(x, \cdot) * f](x), \phi(x) \right\rangle \stackrel{\text{def}}{=} ([k(x, \cdot) * f](x) * \phi(x))(\phi) \\
&= \iint \iint \underbrace{[k(x, x-y) * f(y)] dy}_{\mathcal{F}_y^{-1} a(x, y)} \phi(x) dx \\
&\quad \mathcal{F}_y^{-1} a(x, y) = (2\pi)^n \int a(x, z) e^{i(x-y)z} dz \\
&= (2\pi)^n \iint \iint a(x, z) e^{ixz} f(y) e^{-iyz} \phi(x) dx dy dz
\end{aligned}$$

$k(x, \cdot) \in S'(\mathbb{R}^n)$

$$\begin{aligned}
(-i\tau)^\alpha k(x, \cdot) &= (-i\tau)^\alpha \mathcal{F}_{z \rightarrow z}^{-1} a(x, z) = \mathcal{F}_{z \rightarrow z}^{-1} \partial_z^\alpha a(x, z) \\
\left(\Leftrightarrow \mathcal{F}_{z \rightarrow z}^{-1} (-i\tau)^\alpha k(x, z) = \partial_z^\alpha \underbrace{\mathcal{F}_{z \rightarrow z}^{-1} k(x, z)}_{a(x, z)} \right)
\end{aligned}$$

For $|\alpha| \geq m+1$:

$$\begin{aligned}
|\partial_z^\alpha a(x, z)| &\leq C_{\alpha, 0} (1+|z|)^{m-|\alpha|} = C_{\alpha, 0} (1+|z|)^{-n+1} \in L^1 \\
\Rightarrow \mathcal{F}_{z \rightarrow z}^{-1} \partial_z^\alpha a(x, z) &\in L^\infty \\
\Rightarrow z \mapsto (-i\tau)^\alpha k(x, z) &\in L^\infty \\
\text{and } |z|^\alpha |k(x, z)| &= |(-i\tau)^\alpha k(x, z)| \\
&= \left| (2\pi)^n \int \partial_z^\alpha a(x, z) e^{izx} dz \right| \\
&\leq C \int |\partial_z^\alpha a(x, z)| dz \leq C C_{\alpha, 0} (1+|z|)^{-n+1} dz \\
&\leq c_\alpha \text{ uniformly in } x
\end{aligned}$$

\Rightarrow For N large and corresp. x 's:

$$|z|^N k(x, z) = \sum_{|\alpha| \leq N} (-i\tau)^\alpha k(x, z) \in L^\infty(\mathbb{R}^n)$$

$$\Rightarrow k(x, \cdot) \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$$

c) bzl. what is $\kappa(x_1)$ for $\kappa(x_1) = \sum_{j=0}^m a_j s^{(j)}$, $y \in \mathcal{A}$?

Notes: $s^{(0)} * f = f$

since if $\phi \in C_0(\mathbb{C}^n)$, $s^{(0)} * \ell * \phi = \ell * s^{(0)} * \phi$

$$= f * (s_0 * \delta^0 \phi) = f * \delta^0 \phi = \underbrace{\delta^0 f * \phi}_{f(s)}$$

$$\Rightarrow \sum_j a_j s^{(j)} * f = \sum_j a_j P(f)$$

$$\text{From a: } \kappa(x_1) f = \sum_j (x_1)_j a_j D^j f = \sum_j 2x_1 j a_j (i)^j f^{(j)}$$

$$= \sum_j 2x_1 j a_j (i)^j s^{(j)} * f = k(x_1) * f$$

$$\Rightarrow \kappa(x_1) = \sum_j 2x_1 j a_j (i)^j s^{(j)}$$

$$\phi \in \langle [k(x_1) * f](x), \phi \rangle = ((\kappa(x_1) * f)(x) * \phi(x))(\phi)$$

S^m in $C^0(\mathbb{R}^d)$ conv. iff

$$\text{op}(a) f = \int_{\mathbb{R}^d} (a(x, \xi) \hat{f}(\xi))$$

$$S^m = \{a \in C^0(\mathbb{R}^d \times \mathbb{R}^d); \|a_x\|_a^m\}$$

$$S^\infty = \cap_{m \in \mathbb{N}} S^m$$

$$\text{Thus } a \in S^m \Leftrightarrow K(x, y) := k(x, x-y) := \int_{\mathbb{R}^d} a(x, \xi)$$

is a singular kernel.

i.e.

$$k(x, y) \in C^0(\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x)\})$$

$$\text{and initial } N > 0 \quad \forall z \neq 0 \quad |\partial_x^\alpha \partial_z^\beta k(x, z)| \leq C_{\alpha, \beta, N} |z|^{-d-N}$$

tool: Littlewood-Paley decomposition $\mathbb{R}^d =$

$$j\text{th ring} = \{z : |z| \in [2^{j-1}, 2^{j+1})\}$$

$$1 = \sum_{j=-\infty}^{\infty} \delta(2^j z) = \sum_j (\phi(\frac{z}{2^j}) - \phi(\frac{z}{2^{j+1}})) \quad \forall z \neq 0$$

$$1 = \phi(z) + \sum_{j=1}^{\infty} \delta(2^j z) \quad \forall z$$

$$\Rightarrow 1 = \sum_{j=-\infty}^{\infty} \delta(2^j 0)$$

$$1 = \phi(0) + \sum_{j=1}^{\infty} \delta(2^j 0) \quad \sum_{j=1}^{\infty} \Delta_j = \delta_0 - \delta_{2^j 0}$$

Convergence of operator decompositions:

$$f \in S'(\mathbb{R}^d) \Rightarrow S_0 f + \sum_{j=1}^l \Delta_j f = S_l f = f \phi(2^{-l} z) \xrightarrow{l \rightarrow \infty} f \quad \begin{matrix} \xrightarrow{l \rightarrow \infty} \\ \xrightarrow{m \rightarrow l} \\ \xrightarrow{1 \text{ unit on cpt.}} \end{matrix}$$

The other one:

$$\sum_{j=-\infty}^k \Delta_j f = S_k f - \underbrace{S_{-1} f}_{\xrightarrow{k \rightarrow \infty} 0} \quad \xrightarrow{k \rightarrow \infty} f \in S'$$

$$\text{No: } f \geq 1 \Rightarrow \int \sim \infty \Rightarrow S_{-1} f \sim f^{-1} \delta_0 \notin S$$

$$\text{But: } f \in S \quad (\text{or } f \in L^1) \Rightarrow S_{-1} f \xrightarrow{l \rightarrow \infty} 0$$

\Rightarrow conv. holds in $S'(\mathbb{R}^d)$

$$\text{proof: } \text{op}(a) = \text{op}(a_0) \amalg \sum_{j=1}^{\infty} \text{op}(a_j) D_j$$

$$= \text{op}(a_0) + \sum_{j=1}^{\infty} \text{op}(a_j)$$

$$a_0 = a(x, z) \delta(z) \in S^{-\infty}$$

$$a_j = a(x, z) S(2^j z) \in S^{-\infty}$$

$\in S^m$ uniformly in x

$$\text{kernel of } a_j: f \mapsto \int_{-\infty}^{\infty} a_j(y) e^{iyx} dy \in S_x(\mathbb{R}^n)$$

$$\in S_x(\mathbb{R}^n) \forall x$$

$$\Rightarrow \text{op}(a_j)f(x) = \int k_j(x-y) f(y) dy \quad \text{Hilbert}$$

$$k(x, \cdot) = \sum_{j=0}^{\infty} k_j(x, \cdot) \text{ converges in } S, \quad \forall x$$

\rightarrow conv. in S'

$$\text{Show } \sum_{j=0}^{\infty} |\partial_x^\alpha \partial_z^\beta k_j(x, z)| \leq C_{\alpha, \beta, N} |z|^{-1-n-|\alpha|-|\beta|}$$

Ex: The Hilbert transform has symbol $\text{sgn } z \notin C^\infty$

\Rightarrow not $\Psi(D)$ but still we have:

Thm 4.4: (Hörmander-Mikhlin multiplikator schm.)

$$m \in L^\infty(\mathbb{R}^d) \text{ s.t. } |\partial_z^\alpha m(z)| \leq C_K |z|^{-d-\alpha} \text{ for suff. many } \alpha$$

$$(0 < \alpha \leq d+2)$$

Then $m(D) \in C_0$ and therefore $m(D): L^p \rightarrow L^p$

continuously

$$\text{Rem: } |\partial_z^\alpha \partial_z^\beta m(z)| \leq C \quad \forall z \quad \forall \alpha, \beta \in \{0, 1\}^d \text{ suff. const.}$$

proof: $m \in L^\infty \Rightarrow m(D): L^2 \rightarrow L^2$ bounded

\Rightarrow have to show $K(xy) = \langle b(x-y), \int_{-\infty}^{\infty} m(s) \rangle, x \neq y$

is a singular kernel

$$\text{use } 1 = \sum_{j=-\infty}^{\infty} S(2^j z) \Rightarrow m(D) = \sum_{j=-\infty}^{\infty} (m, S(2^j \cdot))(D)$$

$$= m(D)$$

$\sum_{j=-\infty}^{\infty} m_j(z) \sin(2^{-j}z)$ conv. Vfes

$$k_0(z) = \sum_{j=-\infty}^{j=1} m_j(z)$$

$$k \in \sum_{j=-\infty}^{\infty} k_j \text{ conv. in } S'$$

We show estimates for $\sum |k_j| z^j k_j(z)|$ for $|z| > 0$

Hence $\sum k_j$ conv. uniformly by Weierstrass test

$$\sum |\partial_k k_j|$$

$\Rightarrow k \in C(\mathbb{R}, L^\infty)$ and satisfies $|k(z)| \leq C|z|^{-d}$

$$|zk(z)| \leq C|z|^{-d-1}$$

Exercises 6 - Calculus of pseudo differential operators

Recall that $S^m = \{a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) : \exists C_{\alpha\beta} : |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} (1+|\xi|)^{m-|\beta|}\}$.

and set $S^{-\infty} := \bigcap_{m \in \mathbb{R}} S^m$, $S^\infty := \bigcup_{m \in \mathbb{R}} S^m$.

① a) (Borel's Lemma) Let $(a_j)_{j \in \mathbb{N}_0} \subseteq \mathbb{C}$, $(\varepsilon_j)_{j \in \mathbb{N}_0} \subseteq (0, \infty)$, $\forall k \in \mathbb{N}$ sufficiently fast and $\eta \in C_c^\infty(\mathbb{R})$, $\eta(x) = 1$ for $|x| \leq 1$ and $\eta(x) = 0$ for $|x| \geq 2$.

Show that $f(x) := \sum_{j=0}^{\infty} \frac{a_j}{j!} x^j \eta(x/\varepsilon_j) \in C^\infty(\mathbb{R})$ and $\partial_x^j f(0) = a_j \forall j$.

b) (Asymptotic summation) Let $a_j \in S^{m_j}$, $m_j \downarrow -\infty$, ε_j as above and $\varphi \in C^\infty(\mathbb{R}^d)$, $\varphi(x) = 1$ for $|x| \geq 2$, $\varphi(x) = 0$ for $|x| \leq 1$.

Show that $a(x, \xi) := \sum_{j=0}^{\infty} a_j(x, \xi) \varphi(\varepsilon_j \xi) \in S^{m_0}$ and

$$(*) \quad a = \sum_{j=0}^k a_j \in S^{m_{k+1}} \quad \forall k \in \mathbb{N}_0.$$

Notation: We write $a \sim \sum_{j=0}^{\infty} a_j$ if $(*)$ holds. Observe that $a \sim 0$ if and only if $a \in S^{-\infty}$. Using 1_b , we define an associative product $\# : S/S^{-\infty} \times S/S^{-\infty} \rightarrow S/S^{-\infty}$ via

$$a \# b \sim \sum_{|\alpha| \geq 0} \frac{i^\alpha}{\alpha!} D_\xi^\alpha a(x, \xi) D_x^\alpha b(x, \xi).$$

② a) Let $a \in S^m$, $b \in S^{\tilde{m}}$, $m, \tilde{m} \in \mathbb{R} \cup \{-\infty\}$. Show that $a+b \in S^{\max\{m, \tilde{m}\}}$, $\partial_x^\alpha \partial_\xi^\beta a \in S^{m-|\beta|}$, $ab \in S^{m+\tilde{m}}$, $a \# b \in S^{\max\{m, \tilde{m}\}}/S^{-\infty}$.
 If $|a(x, \xi)| \geq C<\xi>^m$, show that $a^{-1} \in S^{-m}$ and
 $n := a \# a^{-1} - 1$, $\tilde{n} := a^{-1} \# a - 1 \in S^{-1}/S^{-\infty}$.

b) (Neumann series) Let $b \in S^{-1}$, $b^{\#n} = \underbrace{b \# \dots \# b}_{n \text{ factors}}$.
 Show that $(1+b + b^{\#2} + b^{\#3} + \dots) \# (1-b) \sim 1$.

c) Continuing a), let $s = \sum_{j=1}^{\infty} r^{#j}$, $\tilde{s} = \sum_{j=1}^{\infty} \tilde{r}^{#j}$, $b = a^{-1} \# (1+s)$, $\tilde{b} = (1+\tilde{s}) \# a^{-1}$. Show that $\tilde{b} \# a \sim a \# b \sim 1$ and therefore $\tilde{b} \sim \tilde{b} \# a \# b \sim b$, or $a \# b - 1 \sim b \# a - 1 \sim 0$.

Interpretation: Given $a \in S^m$ elliptic ($|a| \geq C\langle \cdot \rangle^m$), we have found an inverse $b \in S^{-m}$ of a up to a negligible remainder $\in S^{-\infty}$.

③ a) Given $a \in S^m$, define the adjoint of $a(x,D)$ with respect to the L^2 -scalar product: $\forall f, g \in S(\mathbb{R}^d): \langle a(x,D)^* f, g \rangle_{L^2} := \langle f, a(x,D)g \rangle_{L^2}$. Integrate by parts as in Ex. 5.2 to show that $a(x,D)^*: S \rightarrow S$ and conclude that $a(x,D)$ extends by duality, $\forall f \in S' \forall g \in S'$ $\langle f, a(x,D)g \rangle := \langle a(x,D)^* f, g \rangle$, to an operator on S' .

b) Let $a \in S^{-\infty}$. Show that $a, a^*: \mathcal{E}' \xrightarrow{\text{opc}} S$ and therefore, by duality, also $a^*, a: S' \rightarrow \mathbb{C}^\infty$.

Hint: Integrate by parts yet again.

$$BSC. 6. \quad f^m = \sum_{\alpha \in C_0^\infty(\mathbb{R}^d)} |a_\alpha| \epsilon^{|\alpha|} |\partial_x^\alpha g(x)| \leq C_{\alpha, \beta} (1 + |x|)^{m - |\alpha|}$$

$$S^{-\infty} = \bigcap_{m \in \mathbb{N}_0} S^m, \quad S^{\infty} = \bigcup_{m \in \mathbb{N}} S^m$$

b) Let $(a_j)_{j \in \mathbb{N}_0} \subseteq \mathbb{C}$, $(\varepsilon_j)_{j \in \mathbb{N}_0} \subseteq (0, \infty)$, $a_j \rightarrow 0$ suff. fast,
 $n \in C_0^\infty(\mathbb{R})$, $n_{\text{new}} = 1$ for $|x| \leq 1$, $n(x) = 0$ for $|x| \geq 2$

$$\text{Show that } f(x) := \sum_{j=0}^{\infty} a_j x^j n\left(\frac{x}{\varepsilon_j}\right) \in C_c^\infty(\mathbb{R})$$

$$\text{and } f^{(k)}(0) = a_k, \quad k \geq 0$$

$$f_N(x) := \sum_{j=0}^N a_j x^j n\left(\frac{x}{\varepsilon_j}\right) \in C_c^\infty(\mathbb{R})$$

$$\text{Supp } a_j \subseteq \text{Supp } n\left(\frac{x}{\varepsilon_j}\right) \subseteq \text{Supp } n \subseteq [-2\varepsilon_j, 2\varepsilon_j] \subseteq [-R, R]$$

$\forall j \in \mathbb{N} \text{ some } \varepsilon_j$

$$\Rightarrow \text{Supp } f_N \subseteq [-R, R]$$

$$\|g_j\|_{L^\infty} = \|a_j x^j n\left(\frac{x}{\varepsilon_j}\right)\|_{L^\infty} \leq \frac{|a_j|}{j!} |2\varepsilon_j|^j \|n\|_{L^\infty}$$

$$\|f_N\|_{L^\infty} \leq \sum_{j=0}^N \frac{|a_j|}{j!} 2^j \varepsilon_j^j \|n\|_{L^\infty}$$

$|a_{j,\varepsilon}| < \infty$ for $\varepsilon_j \downarrow 0$ suff. fast

$$\|f_N - f_M\|_{L^\infty} \leq \sum_{N \leq j \leq M} \frac{|a_j|}{j!} 2^j \varepsilon_j^j \|n\|_{L^\infty} \rightarrow 0, \quad N, M \rightarrow \infty$$

More generally:

$$\begin{aligned} (\alpha, \varepsilon_j) \cdot \partial^\alpha g_j(x) &= \sum_{\alpha_1 + \alpha_2 = \alpha} a_j \underbrace{\partial^{\alpha_1} x^j}_{j!} \underbrace{\partial^{\alpha_2} n\left(\frac{x}{\varepsilon_j}\right)}_{\substack{s!/(j-s)! \\ \varepsilon_j^{-s} n^{(s)}\left(\frac{x}{\varepsilon_j}\right)}} \\ &= \sum_{\alpha_1 + \alpha_2 = \alpha} \frac{a_j}{(j-s)!} \left(\frac{x}{\varepsilon_j}\right)^{j-s} \varepsilon_j^{-s} n^{(s)}\left(\frac{x}{\varepsilon_j}\right) \\ &\leq 2^{j-\alpha} \end{aligned}$$

$$\Rightarrow \|\partial^\alpha g_j\|_{L^\infty} \leq C_{\alpha, s, j} \varepsilon_j^{j-\alpha}$$

\Rightarrow For suff. small ε_j , $\|\partial^\alpha g_j\|_{L^\infty} \leq 2^{-j}$ for $j > \alpha$

$$\begin{aligned} \Rightarrow \|\partial^\alpha f_N\|_{L^\infty} &\leq \sum_{j=0}^N \|\partial^\alpha g_j\|_{L^\infty} \leq \sum_{j=0}^N \|\partial^\alpha g_j\|_{L^\infty} + \sum_{j=N+1}^N 2^{-j} \\ &\leq C_\alpha \end{aligned}$$

uniform convergence uniformly for each x and $f \in C_c^\infty$

Note: $n^{(\alpha)}(0) = 0 \quad \forall \alpha > 0$

$$\Rightarrow \partial^* g_0(d) = \sum_{\substack{d_1+d_2=d \\ d_1=j}} a_j \cdot \varepsilon_j^{j-d} \cdot n^{(d_2)}(d) = a_j \cdot \delta_{j,d}$$

$$\Rightarrow \partial^\alpha f = \sum_{\beta \geq 0}^N \partial^\alpha \partial_\beta f = a_\alpha$$

b) Let $a_j \in S^{(j)}$, $\lim a_j = \infty$, $\epsilon_j > 0$ as above.

$$\varrho \in C^\infty(\mathbb{R}^d), \quad \varrho(x) = 1 \quad \text{for } |x| \geq 2 \\ \varrho(x) = 0 \quad \text{for } |x| \leq 1$$

Show that $a(x_1, \xi) := \sum_{j=0}^{\infty} a_j(x_1, \xi) e(\xi_j)$ is S^{no}

$$\text{and } a = \sum_{j>0} q_j \in S^{\text{left}} \quad \forall k \in \mathbb{N}_0$$

PF: Obviously, $A_N^{(\infty)} = \sum_{j=0}^N a_j(\infty, \varepsilon) e(\varepsilon, z) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$

$$\partial_x^{\alpha} \partial_z^{\beta} (a_j(\lambda z) \psi(\varepsilon_j z)) = \sum_{B_1 + B_2 = B} (\partial_x^{\alpha} \partial_z^{B_1} a_j(\lambda z)) \underbrace{\partial_z^{B_2} \psi(\varepsilon_j z)}_{\lesssim \varepsilon_j^{|B_2|}} \partial_z^{B_2} \psi(\varepsilon_j z),$$

$$\sum_{B_1+B_2=B} C_{B_1, B_2} (1+|z|^2)^{m_0 - |B_1|} E_j^{|B_2|} C_{e, B_2} x^{|\lambda|_B z}$$

$$\leq \left\{ C_{\alpha, \beta_1} (1+|\beta_1|)^{m-|\beta_1|}, \quad \beta_2 = 0 \right\} \quad \text{for } \beta_2 > 0.$$

$$C_{\alpha(B)}^{(1)} (1 + |z|)^{B_1} \sim (1 + \frac{2}{|z|})^{B_2}, \quad B_2 > 0.$$

$$\Rightarrow \left| \partial_x^\alpha \partial_z^\beta (a_j(\varepsilon z) b_j(\varepsilon z)) \right| \leq \sum_{\alpha, \beta} (1/31)^{m_j - 176} \chi$$

$$\Rightarrow \psi_1(x_1, y_1) \psi_2(x_2, y_2) \in S^{(m)}$$

$(m_j \rightarrow +\infty \Rightarrow A_N \in S^{no} \quad \forall N \in N)$

$$\partial_x^\alpha \partial_y^\beta A_N = \sum_{j+m_j \geq |B|} \partial_x^\alpha \partial_y^\beta (a_j(\log y) g(\log y)) + \sum_{j+m_j < |B|} -$$

↓
or $\frac{1+\frac{1}{\delta}}{\delta} \leq 2$

$\leq C(1+|y|)^{n-10}$

$\partial_x^2 \partial_z^6 u$: nonlocality (in compactness).

$$\Rightarrow \text{as } C^{\infty}(\mathbb{R}^d \times \mathbb{R}^d) \text{ and } |\partial_x^\alpha \partial_y^\beta g| \leq \sum_{j=1}^n C_{\alpha_j \beta_j} (1+|y|)^{n-j}$$

as is ^{me}

Obviously, $a - \sum_{j=0}^k a_j \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$

$$\text{and } a - \sum_{j=0}^k a_j = \sum_{j=0}^k a_j(x, z) (-1 + \varphi(\varepsilon_j z)) + \sum_{j=0}^{\infty} a_j(x, z) \varphi(\varepsilon_j z)$$

$\underbrace{\text{supp}(1)}_{\subseteq B_{2\varepsilon_j}(0)} \subseteq B_{2\varepsilon_j}(0) \quad \underbrace{\varepsilon_j}_{\in S^{m-k+1}}$

$$\Rightarrow |\partial_x^\alpha \partial_z^\beta (a - \sum_{j=0}^k a_j)| \leq \sum_{j=0}^k C_{\alpha j} (1 + |z|)^{m-j} + C_{\alpha k} (1 + |z|)^{m-k}$$

$$\Rightarrow a - \sum_{j=0}^k a_j \in S^{m-k+1}$$

Note that $a \in \sum_{j=0}^{\infty} a_j \Leftrightarrow a - \sum_{j=0}^k a_j \in S^{m-k+1} \text{ if } k \in \mathbb{N}_0$

$$S^{m-k+1} / S^{-\infty} \times S^{-\infty} / S^{-\infty} \rightarrow S^{-\infty} / S^{-\infty}$$

$$(a, b) \mapsto a \# b := \left[\sum_{|\alpha| \geq 0, |\alpha|} D_x^\alpha a D_x^\alpha b \right] / S^{-\infty}$$

2.9) Let $a \in S^m$, $b \in S^m$, $m, m' \in \mathbb{N} \cup \{-\infty\}$,

Show that 1) $a+b \in S^{m \wedge m'}$

$$2) \partial_x^\alpha \partial_z^\beta a \in S^{m-|\alpha|}$$

$$3) ab \in S^{m+m'}$$

$$4) a \# b \in S^{m+m'} / S^{-\infty}$$

If $|a(z)| \geq C(|z|)^m = C(1 + |z|^2)^{\frac{m}{2}}$ show that

$$5) a^{-1} \in S^{-m}$$

$$6) r := a \# a^{-1} - 1 \in S^{-1} / S^{-\infty}$$

$$7) \tilde{r} := a^{-1} \# a - 1 \in S^{-1} / S^{-\infty}$$

$$1) |\partial_x^\alpha \partial_z^\beta (a_m)| \leq C_{\alpha \beta} a (1 + |z|)^{m-|\alpha|} + C_{\alpha \beta} b (1 + |z|)^{m-|\alpha|}$$

$$2) |\partial_x^\alpha \partial_z^\beta (\partial_x^\gamma \partial_z^\delta a)| \leq C_{\alpha \beta, \gamma \delta} (1 + |z|)^{m-|\alpha|+|\beta|}$$

$$3) |\partial_x^\alpha \partial_z^\beta (ab)| \leq \sum_{\alpha_1 + \alpha_2 = \alpha} \sum_{\beta_1 + \beta_2 = \beta} |\partial_x^{\alpha_1} \partial_z^{\beta_1} a| \cdot |\partial_x^{\alpha_2} \partial_z^{\beta_2} b|$$

$$\leq C_{\alpha \beta} (1 + |z|)^{m-|\alpha|} C_{\alpha_2 \beta_2} (1 + |z|)^{m-|\alpha_2|}$$

$$4) \partial_x^\alpha \partial_z^\beta ($$

$$\begin{aligned}
 & D_x^\alpha D_3^\beta \left(\sum_{|\alpha|>0} \frac{i^\alpha}{\alpha!} D_3^\alpha a D_x^\beta b \right) = \sum_{\alpha_1, \alpha_2 \geq 0} \sum_{\substack{\alpha_1 + \alpha_2 = \alpha \\ \beta_1 + \beta_2 = \beta}} \sum_{|\alpha_1|>0} \\
 & \frac{(-i)^\alpha}{\alpha!} \left(D_x^{\alpha_1} D_3^{\beta_1} a \right) \left(D_x^{\alpha_2} D_3^{\beta_2} b \right) \\
 & \stackrel{(1)}{\leq} \sum_{|\alpha|>0} C_{\alpha, \beta} (1+|z|^3)^{m-|\beta|+|\alpha|} C_{\alpha+\beta, \beta_2, b} (1+|z|^3)^{m-|\beta_2|} \\
 & \stackrel{(2)}{\leq} \sum_{|\alpha|>0} C_{\alpha, \beta} (1+|z|^3)^{m+|\alpha|-|\beta|+|\beta_1|} \\
 & \stackrel{(3)}{\in} S^{m+|\alpha|}/S^{-\infty}
 \end{aligned}$$

5) $|a(x, z)| \geq c |z|^m \Rightarrow a^{-1}(x, z) = \frac{1}{a(x, z)} \in C^\infty(C^k \times \mathbb{R}^d)$ and

$$\begin{aligned}
 |D_x^\alpha D_3^\beta a^{-1}| &\leq |a|^{-1+|\beta|-1} \prod_{j=1}^n |a_j| \prod_{j=1}^m |D_{\beta_j} a_j| \\
 &+ \dots + |a|^{-2} |D_x^\alpha D_3^\beta a| \\
 &\leq (\langle z \rangle^m)^{|\beta|+1} (\langle z \rangle^{m-1})^{|\alpha|} (\langle z \rangle^{m-1})^{|\beta|} \\
 &+ \dots + (\langle z \rangle^m)^3 \langle z \rangle^{m-|\beta|} \\
 &\leq \langle z \rangle^{m-|\beta|} \underbrace{\langle z \rangle^{-|\beta|}}_{S^{-|\beta|} \subseteq S^{-1}}
 \end{aligned}$$

$$6) a \# a^{-1} = 1 + \sum_{|\alpha|>0} \underbrace{\frac{i^\alpha}{\alpha!} D_3^\alpha a}_{\in S^{m-|\alpha|}} \underbrace{D_x^\alpha a^{-1}}_{\in S^{-m}} \Rightarrow a \# a^{-1} \in S^1/S^{-\infty}$$

$$7) a' \# a = 1 + \sum_{|\alpha|>0} \underbrace{\frac{i^\alpha}{\alpha!} D_3^\alpha a'}_{\in S^{m-|\alpha|}} \underbrace{D_x^\alpha a}_{\in S^m} \Rightarrow a' \# a - 1 \in S^1/S^{-\infty}$$

$$b) \text{ Let } b \in S^1, b^{\# n} := b \# b \# \dots \# b, b^{\# 0} := 1$$

$$\text{Show that } \left(\sum_{k=0}^{\infty} b^{\# k} \right) \# (1-b) \sim 1$$

8) Note: $b^{\# k} \in S^k/S^{-\infty}$ (so the series converges)

$$z_N := \sum_{k=0}^N b^{\# k} \in S^N$$

Note: $a \# 1 = 1 \# a = a$

$$z_N \# (1-b) = z_N - z_N \# b = \sum_{k=0}^N b^{\# k} - \sum_{k=1}^{N+1} b^{\# k}$$

$$= b^{\# \sigma} + b^{\#(\pi\tau)} = 1 - b^{\#(\pi\tau)}$$

Brave

210

$$\sum_{k=1}^{\infty} b_k$$

144

$$\Sigma_n + R_n = \Sigma$$

For any water levels in

$$\sum_{k=1}^{\infty} b_k$$

$\mathbb{P} \in \{\mathbb{N}, \mathbb{H}\}$

—Sincere

$$M > N : b^{\# M} \cdot (1-b) \leq b^{\# M} - b^{\#(M+1)} \in S^{-M} / f_{-\infty}$$

Hence these do not produce any new terms. In $S^{\otimes k}/S^{<\otimes k}$

Taking $N \rightarrow \infty$ we hence have

$$\sum_n \#(1-b) = 1 \in S^{-(N+1)} / \text{span}$$

$$a \leftarrow s_0 (-1)^{t+1} \in \mathbb{Z}_q$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n = 1 \in S$$

$$c) \quad 14 \sin \sum_{n=1}^{10} n^{\frac{1}{2}} = 14 (1+5)$$

$$b^2 = (1+s^2) + a^2$$

$$v = \alpha + \alpha^{-1} - 1$$

$$f = g + h$$

Show Bananab

an h b n b Ha t t b ~ k

$$(a+b-1) \sim b+a-1 \sim 0$$

1

By: a

$$\text{By a) } \bar{x} = a + a^{-1} - 1 \in S^+ / S^{ab} \Rightarrow x^{\#}, x^{\circ}, x^{\circ\circ} \in S^+ / S^{ab}$$

$$x = 5^k + 1$$

$$\Rightarrow b \# a = \left(\sum_{j=0}^n r^{A_j} \right) \# a^{-1} \# a$$

$$a \# b = a \# a^{-1} \# \left(\sum_{j=0}^{\infty} r^{\#} \right)$$

$$a \# b = a \# a^{-1} \# \left(\sum_{r \geq 0} r^{\#} \right)$$

Hence, since $\frac{S^{-\infty}}{S^{-\infty}} = \frac{S^m}{S^{-\infty}} = S^{m-\infty}$ i.e. $S^{\infty} \approx 0$

$$\tilde{b} \text{ plus } b = \tilde{b} + 1 + b + c \sim \tilde{b} + 1 \sim \tilde{b}$$

$$\text{and } b \# a \# b \sim 1 \# b + 2 \# b \sim 1 \# b \sim b$$

a) Given $a \in S^m$ we define

$$\forall f, g \in S(\mathbb{R}^d) \quad \langle g, a(x, \cdot)^* f \rangle := \langle a(x, \cdot) g, f \rangle_{L^2}$$

$$= \int_{\mathbb{R}^d} \exp(a) g(x) f(x) dx$$

$$= \int_{\mathbb{R}^d} \left(\left(\int_{\mathbb{R}^d} a(x, z) g(z) e^{izx} dz \right) f(x) \right) dx$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d} g(z) \underbrace{a(x, z)}_{\text{decays rapidly}} f(x) e^{-ixz} dx dz$$

$$= \langle \hat{g}, \mathcal{T}_{x \rightarrow z} (a(x, \cdot) f(x)) \rangle_{L^2}$$

motivation $\in S(\mathbb{R}^d)$ as seen in Ex. 5.2, cp. later

Physical

$$= (\text{inv } \mathcal{F}_{\text{ray}}) \hat{g}(y), \mathcal{F}_{\text{ray}} \mathcal{T}_{x \rightarrow z} (a(x, \cdot) f(x))_2$$

$$= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} g(y) e^{iyx} dy \right) a(x, z) f(x) e^{-izx} dx dz$$

$$= (\text{inv } \mathcal{F}_{\text{ray}}) \int_{\mathbb{R}^d} g(y) \text{ inv } \mathcal{F}_{\text{ray}} \int_{\mathbb{R}^d} a(x, z) f(x) e^{-izx} dx e^{iyx} dz dy$$

$$= \int_{\mathbb{R}^d} g(y) (\mathcal{F}f)(y) dy = \langle g, Tf \rangle$$

$$\text{where } (\mathcal{F}f)(y) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x, z) f(x) e^{-izx} dx e^{iyx} dz$$

$$= (\text{inv } \mathcal{F})^{-1} \left(\text{inv } \int_{\mathbb{R}^d} a(x, z) f(x) e^{-izx} dx \right)$$

\Rightarrow suffices to show that $\mathcal{F} \in S(\mathbb{R}^d)$

$$\text{cp. } \exp(a) f(z) := \int_{\mathbb{R}^d} a(x, z) f(x) e^{ixz} dx \in S(\mathbb{R}^d)$$

As in Ex. 5.2

$$\left| \frac{\partial}{\partial z} \exp(a) f(z) \right| \leq C_{a,b} \cdot \text{lip}_p(f)$$

$\Rightarrow \mathcal{F}: S(\mathbb{R}^d) \rightarrow S(\mathbb{R}^d)$

RHS persons

$$\text{Now } \langle g, (a(x,D) - Tg) \rangle_2 = 0 \quad \forall g \in S^{\perp}$$

$S(\mathbb{R}^d) \rightarrow S^{\perp}$

\Rightarrow For every $f \in S(\mathbb{R}^d)$, $a(x,D)^* f$ is given by Tf

Note! Well-def. since $g \in S(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$ dense:

$$\langle g, Tf \rangle_2 = \langle g, T'f \rangle_2 \quad \forall g$$

$$\Leftrightarrow \langle g, (Tf - T'f) \rangle_2 \quad \forall g \in S$$

$$\Leftrightarrow (Tf - T'f) \in S^\perp = S^\perp = (L^2)^\perp = 0$$

$$\Leftrightarrow Tf = T'f$$

For $f \in S'$ we therefore have a cont. linear form on S :

$$g \mapsto \langle f, a(x,D)^* g \rangle =: \langle a(x,D)^* f, g \rangle \quad g \in S$$

i.e. on S : ~~symmetric~~ symmetric

$$f \in S \iff \langle f, a(x,D)^* g \rangle = \langle a(x,D)f, g \rangle \quad \forall g \in S$$

and hence $a(x,D)|_{S'}|_S = a(x,D)$

i.e. it extends $a(x,D)$ from S to S'

b) Let $a \in S^{-\infty}$ i.e. $|D_x^\alpha D_y^\beta a(x,y)| \leq C_\alpha (1+|x|)^{-N} \quad \forall N$

then as in Ex 5.2: $\text{op}(a)f(x) := \int a(x,y) \hat{f}(y) e^{ix-y} dy$

now: $f \in S' \iff \hat{f} \in S'^\perp$ i.e. $|\hat{f}(y)| \leq C(1+|y|)^N$ for some N

$\Rightarrow \text{op}(a)f \in S$

similarly for a^*

(using $\hat{a}^*(y) = \overline{a(-y)}$)

$$\langle a^* f, g \rangle = \langle f, a g \rangle$$

By duality / $a: \mathcal{E}' \rightarrow S \Rightarrow a^*: S' \rightarrow (\mathcal{E}')' = C^\infty$

$$\langle af, g \rangle = \langle f, a^* g \rangle$$

$$a^*: \mathcal{E}' \rightarrow S \Rightarrow a: S' \rightarrow C^\infty$$

Recall, Thm: $k(x, z) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d \setminus \{z\})$ and

$$\forall d+m+|\alpha|+N > 0 \quad \forall z \neq 0 \quad |\partial_x^\alpha \partial_z^\beta k(x, z)| \leq C_{\alpha, \beta, N} |z|^{-d-m-|\alpha|-|\beta|}$$

Fourier multipliers: $m \in L^\infty(\mathbb{R}^d)$

$$|\partial_\xi^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|} \quad \forall \xi \neq 0 \quad \forall 1 \leq |\alpha| \leq d+2$$

$$\Rightarrow m(0) \in L^p \text{ bounded}, \quad p \in (1, \infty)$$

$$m(0)f = kx \cdot f, \quad \|k(x)\| \leq C|x|^{d-1}$$

$$|\partial_x k(x)| \leq C|x|^{d-1}$$

exponential decay for $|z| \rightarrow \infty$

$\sim |z|^{-d-m}$ for $z \rightarrow 0$

behaves like $m \in \mathcal{S}$

Ex: Ulbert wave $H = m(0)$, $m(\xi) = -i \operatorname{sgn}(\xi)$

Fractional derivatives: $|D|^\alpha$, $m(\xi) = |\xi|^\alpha$

$$|D|^\alpha, \quad m(\xi) = (1 + |\xi|^2)^{\frac{\alpha}{2}}, \quad \text{even } \alpha$$

$$\text{Riesz trans: } \operatorname{sgn}(D) = \frac{D}{|D|}, \quad m(\xi) = \frac{i\xi}{|D|}$$

$$= \frac{(D_1, \dots, D_d)}{\sqrt{|D|}} = (R_1, \dots, R_d)$$

Elliptic regularity on L^p

$$W^{2,p}(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d) : \partial^\alpha f \in L^p(\mathbb{R}^d) \quad \forall |\alpha| \leq 2\}$$

$$\text{V. norm: } \sum_{|\alpha|=2} \|\partial^\alpha f\|_p \quad \text{or} \quad \|f\|_p + \sum_{|\alpha|=2} \|\partial^\alpha f\|_p$$

Prop: $f \in L^p, |Df| \in L^p \Rightarrow f \in W^{2,p} \quad p \in (1, \infty)$

Proof: $\partial_{x_j} \partial_{x_k} f = -P_j P_k \Delta f$ by taking $\mathcal{F}|_{S^1}$ on both sides

$$\Rightarrow \sum_{j, k} \|\partial_{x_j} \partial_{x_k} f\|_p = \sum_{j, k} \|P_j P_k \Delta f\|_p \leq C \|Af\|_p$$

$$\text{More general: } \sum_{|\alpha|=k} \|\partial^\alpha f\|_p \leq C \underbrace{\|D^k f\|_p}_{\|f\|_p}$$

Main thm for PDO's: $\text{op}(ab) = \text{op}(a) \circ \text{op}(b)$

- Corollaries:
- $a \in S^0 \Rightarrow \text{op}(a) \in C_0 \Rightarrow \text{op}(a): L^p \rightarrow L^p$ bounded ($V_{p, \text{loc}}$)
 - $a \in S^m, \operatorname{Re} a \geq C \|g\|^m \wedge \|g\|_2 \Rightarrow \operatorname{Re} \langle \text{op}(a)f, f \rangle \geq \frac{C}{2} \|f\|_2^2 - C \|f\|_2$
(ellipticity)
(Garding's thm.)
 - L^p regularity (local, "macro-local" variants)

"Domination": $\forall c \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d), \forall \delta > 0, \exists m, \epsilon \in (0, 1)$

$$\forall f \in \text{op}(a(m)) \gamma(\epsilon \eta, \epsilon g) f \xrightarrow{\epsilon \rightarrow 0} \text{op}(a)f \text{ in } S'(\mathbb{R}^d)$$

\Rightarrow uniformly in m

Calculus: $\text{op}(a) \circ \text{op}(b) = \text{op}(ab)$ via explicit formula (for \mathcal{C})

$$\langle a(b, D)^* f, g \rangle_{L^2} := \langle f, a(b, D)g \rangle_{L^2}$$

$$\Rightarrow \exists b \in S^m; a(b, D)^* = \text{op}(b)$$

$$\forall N > 0 \quad b = \sum_{|\alpha|=N} \frac{b_\alpha}{\alpha!} D^\alpha \quad a(b, D) = \sum_{|\alpha|=N} a_\alpha D^\alpha$$

$$\text{Cor: } (\sum_{|\alpha|=m} a_\alpha D^\alpha) \circ (\sum_{|\beta|=n} b_\beta D^\beta) = \sum_{|\alpha|=m, |\beta|=n} (a_\alpha b_\beta) D^{\alpha+\beta}$$

$$(\sum_{|\alpha|=m} a_\alpha D^\alpha)^* f = \sum_{|\alpha|=m} D^\alpha (\bar{a}_\alpha f) = \sum_{|\alpha|=m} \bar{a}_\alpha D^\alpha f = \sum_{|\alpha|=m} \bar{a}_\alpha D^\alpha f + \text{lower order}$$

$$a(D) \circ b(D) = (ab)(D) \quad (+ \text{ lower order})$$

$$a(D)^* = \bar{a}(D) \quad (+ \text{ lower order})$$

Complex Microlocal

B^F Banach, $F \subset B$ cont.

$$\mathcal{L} = \{ \text{OpRes} \in \mathbb{B} \}$$

$$\mathcal{N}_{B^F}(\mathcal{L}) = \left\{ u \in L^1(B) : \exists C, \text{ s.t. } \|u(x+iy)\|_p \leq C \right\}$$

Ex: $L^2(\mathbb{R}^n)$, $p=2$

$$[E, F]_\theta := \sum_{\theta} \{ u \}, u \in \mathcal{N}_{B^F}(\mathcal{L}) \quad \text{for } \theta \in [0, 1]$$

also
Banach
space

(norm = $\sup_{x \in B, y \neq 0}$)

$$\text{Ex: } E = \mathbb{A}, F = \mathbb{S}^1$$

$$\mathcal{N}_{E, F}(\mathcal{L}) = \{ \text{bdd. Hilb.} : u(x+iy) \geq 0 \}$$

By Schwartz Reflection Principle, we extend to $\tilde{\mathcal{L}} = \{ u \in L^1(\mathbb{R}) : u(x+iy) = -u(x-iy) \}$

$$u \in \mathcal{N}_{E, F}(\mathcal{L}) \quad \& \quad u(x+iy) \geq 0 \Rightarrow u \geq 0 \text{ on } \mathbb{R}$$

$$\Rightarrow \mathcal{N}_{E, \mathbb{S}^1}(\mathcal{L}) = \mathbb{S}^1 \Rightarrow [E, \mathbb{S}^1]_\theta = 0 \quad \forall \theta \in [0, 1]$$

$$\text{sim. } [E, \mathbb{S}^1]_\theta = 0, \forall \theta \in [0, 1]$$

Prop 1: If E, F as above, \tilde{E}, \tilde{F} Banach, $\tilde{F} \subset \tilde{B}$ cont.

Suppose $T: \tilde{E} \rightarrow \tilde{E}$ cont. lin.
 $F \rightarrow \tilde{F}$

Then $\forall \theta \in [0, 1], T: [E, F]_\theta \rightarrow [\tilde{E}, \tilde{F}]_\theta$.

(cont.)

Def: $[H, \mathcal{D}(A)]_\theta, H$ Hilb. space, A op. self-adj. op on H

From spec-thm \Rightarrow unitary $A = U^{-1} B U$

$$B u(x) = M_b u(x) = b(x) u(x), \quad \mathcal{D}(B) = \{u \in L^2 : b(x)u \in L^2\}$$

assume $b(x) \geq 0$

Twice

$$\text{Def. } A^\theta = U^{-1} B^\theta U, \quad \mathcal{D}(B^\theta) = \{u \in L^2 : b^\theta(x)u \in L^2\}$$

$$\mathcal{D}(A^\theta) = U^{-1} \mathcal{D}(B^\theta)$$

Prop 2: For all $\theta \in [0, 1]: [H, \mathcal{D}(A)]_\theta = \mathcal{D}(A^\theta)$

Applications:

Recall: $H^s(\mathbb{R}^n) = \{u \in S' : \langle \zeta \rangle^s u(\zeta) \in L^2\}, s \in \mathbb{R}$

$$= \{u \in S' : \exists \alpha \in \mathbb{N} + \mathbb{Z}^n \text{ s.t. } \langle \zeta \rangle^{-\alpha} u(\zeta) \in L^2\}$$

Diagram 1: $\langle \zeta \rangle^{-1} \langle \zeta \rangle^s \zeta = \zeta^s, s = (1-\alpha)_+^{1/2}$

$\mathcal{A}^s : D(\mathcal{A}^s) \rightarrow L^2$ self-adj. on L^2

Swes': $\mathcal{A}^s \in \mathcal{C}^{\infty}$

so: $\mathcal{A}^s = \frac{D(\mathcal{A}^s)}{\mathcal{A}^s \text{ on } \mathcal{M}} \in D(\mathcal{A}^s)$

Prop 2 \Rightarrow w.r.t. $H = L^2(\mathbb{R}^n), A = \mathcal{A}^s$

$$[L^2(\mathbb{R}^n), H^s(\mathbb{R}^n)]_0 = H^{s_0}(\mathbb{R}), 0 \in [0, n]$$

For all $\theta, t \in \mathbb{R}$ $[H^\theta(\mathbb{R}^n), H^t(\mathbb{R}^n)]_0 = H^{\theta t + (1-\theta)s}, 0 \in [0, n]$

\mathcal{A}^s boundary form $H^t \rightarrow H^{t-s}$ $t \in \mathbb{R}$, $s \in \mathbb{R}$

$M_{\mathcal{A}} : H^k(\mathbb{R}^n) \rightarrow H^k(\mathbb{R}^n) \quad \forall k \in \mathbb{N} \cup \{0\}$

then extend to $t \in \mathbb{R}$ by Prop 1

$$k \in \mathbb{Z}, k \neq 0 \quad \exists t \in \mathbb{R} \text{ s.t. } -k > t > -(k+1)$$

$$[H^{-k}, H^{-(k+1)}]_0 = H^{-\theta t + (1-\theta)(k+1)} = H^{-(k+1)\theta} = H^t \quad \text{for some } \theta \in [0, 1]$$

Case $F \neq E$, $E \subset F$: If $E \hookrightarrow V, F \hookrightarrow V$, V some Banach space
(or Hilbert space)

$$G := E + F \subseteq V \text{ Banach sp.}$$

def. $M_{E+F}(V) = \{u(x) \text{ bdd. cont. on } G, u|_E \in M_E(V)\}$

$$\|u(x)\|_G \leq \|u(x)\|_E + \|u(x)\|_F \quad \forall x \in G$$

Prop 3: $0 < \theta < 1, E = L^p(\mathbb{R}, \mu), F = L^{p_2}(\mathbb{R}, \mu)$

$$[L^p(\mathbb{R}, \mu), L^{p_2}(\mathbb{R}, \mu)]_0 = L^q(\mathbb{R}, \mu) \quad \frac{1}{q} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$$

$$T: L^{p_1} \xrightarrow{L^{p_2}} L^q \text{ cont.} \Rightarrow T: [L^{p_1}, L^{p_2}]_0 \rightarrow \{f_i, L^q\}_0$$

Exercises 7 - Useful facts

① a) Let H be a Hilbert space and $[\cdot, \cdot]: H \times H \rightarrow \mathbb{C}$ sesquilinear and such that $[h, h] \geq 0 \ \forall h \in H$. Show the Cauchy-Schwarz inequality $|[f, g]| \leq [f, f]^{\frac{1}{2}} [g, g]^{\frac{1}{2}}$ $\forall f, g \in H$.

b) Let M be a subset of a Banach space X and $\text{span } M := \left\{ \sum a_i m_i : a_i \in \mathbb{C}, m_i \in M \right\}$. Show that $\text{span } M$ is dense in $X \iff$ Every continuous linear functional on X , which vanishes on M , is 0.

$\hookleftarrow H-B$

c) Let A be a Banach algebra, $\varphi: A \rightarrow \mathbb{C}$ linear and multiplicative ($\varphi(ab) = \varphi(a)\varphi(b)$). Show that φ is continuous and $\|\varphi\| \leq 1$. If A is unital $\Rightarrow \|\varphi\| \in \{0, 1\}$.

Hint: If $\exists a \in A$ s.t. $\|a\| < 1$, but $\varphi(a) = 1 \Rightarrow \varphi\left(\sum_{n=1}^{\infty} a^n\right) = 1 + \varphi\left(\sum_{n=1}^{\infty} a^n\right) = 1 + \varphi(a)(1 + \sum_{n=2}^{\infty} a^n)$

Banach algebra
homomorph.

Reformulation of $r(a) = \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}}$

Gelfand
transform
of $L'(G) = T_{L^1(G)}$

Let G be a locally compact abelian group (say $G = \mathbb{R}$) and $h \in L'(G)$. Show that $\|\underbrace{h * \dots * h}_{n \text{ factors}}\|_{L'(G)}^{\frac{1}{n}} \xrightarrow{n \rightarrow \infty} \|\hat{h}\|_{L^\infty(\mathbb{R})}$.

Hint: This is a (complete) Banach algebra $\lim_{n \rightarrow \infty} \|\hat{h}^n\|^{\frac{1}{n}} = \sup |\hat{h}(a)|$.

② a) Prove that \mathbb{R} is a locally compact abelian group.

b) Prove that if G_1, \dots, G_N are locally compact abelian groups, so is $G_1 \times \dots \times G_N$ (with the product topology and component-wise multiplication).

1.b) \Rightarrow

c) Prove that an infinite-dimensional Banach space is not locally compact. Take a net $a_n \in \mathbb{R}^n$. If a_n is cpt then bounded sequences contain conv. subseq. But $B_E(a) \subseteq \mathbb{R}^n$ for some E , and \mathbb{R}^n has an orthonormal basis $\{e_j\} \Rightarrow E\{e_j\} \subseteq B_E(a)$.

Remark: A vector space with a Hausdorff topology, such that multiplication with scalars and vector addition are continuous, is locally compact \iff it is homeomorphic to \mathbb{R}^n for some n .

(3) Let G be a locally compact abelian group and m the Haar measure on G . Show:

- Rudin p. 101*
- a) If $\emptyset \neq V \subset G$ open $\Rightarrow m(V) > 0$.
- Rudin p. 101*
- b) $E \subset G$ measurable $\Rightarrow m(E) = m(\bar{E})$.

(4) a) Let $\gamma: \mathbb{R} \rightarrow \mathbb{C}$ continuous; $\gamma(x+y) = \gamma(x)\gamma(y)$ $\forall x, y \in \mathbb{R}$ and γ not identically 0. By continuity $\exists \delta > 0: \int_0^\delta \gamma(x) dx = \alpha \neq 0$. Show $\alpha \gamma(x) = \int_x^{x+\delta} \gamma(t) dt$. Conclude that γ is differentiable and $\gamma'(x) = \gamma'(0) \gamma(x)$. Show that the only bounded solutions are $\gamma_y(x) := e^{iyx}$ for some $y \in \mathbb{R}$.

- Rudin 1.2.6*
- b) Endow the space of all such γ with the compact-open topology (as described in Rudin, 1.2.6). Show that $\gamma \mapsto \gamma_y$ is a homeomorphism.
- c) Show that all bounded γ as in a), with \mathbb{R} replaced by \mathbb{R}/\mathbb{Z} are of the form $\gamma_n(x) = e^{2\pi i n x}$, $n \in \mathbb{Z}$.

(5) Let G be a locally compact abelian group. $\phi: G \rightarrow \mathbb{C}$ is said to be positive definite, if $\sum_{n,m=1}^N c_n \bar{c}_m \phi(x_n - x_m) \geq 0$ for all $x_1, \dots, x_N \in G$, $c_1, \dots, c_N \in \mathbb{C}$.

Rudin 1.2.7 \Rightarrow Let $f \in L^2(G)$, $\tilde{f}(x) = \overline{f(-x)}$. Show that $f * \tilde{f}$ is positive definite and $\in C_0(G)$.

for 5 Hint: Use Young's inequality to deduce the last assertion.

$$(3) \quad Tf = \int_G f dm \quad (f \in C_c(G)).$$

1.1.2. If V is a non-empty open subset of G , then $m(V) > 0$. For if $m(V) = 0$ and K is compact, finitely many translates of V cover K , and hence $m(K) = 0$. The regularity of m then implies that $m(E) = 0$ for all Borel sets E in G , a contradiction.

1.1.3. We have spoken of the Haar measure of G . This is justified by the following uniqueness theorem:

If m and m' are two Haar measures on G , then $m' = \lambda m$, where λ is a positive constant.

Proof: Fix $g \in C_c(G)$ so that $\int_G g dm = 1$. Define λ by

$$\int_G g(-x) dm'(x) = \lambda.$$

For any $f \in C_c(G)$ we then have

$$\begin{aligned} \int_G f dm' &= \int_G g(y) dm(y) \int_G f(x) dm'(x) \\ &= \int_G g(y) dm(y) \int_G f(x+y) dm'(x) \\ &= \int_G dm'(x) \int_G g(y)f(x+y) dm(y) \\ &= \int_G dm'(x) \int_G g(y-x)f(y) dm(y) \\ &= \int_G f(y) dm(y) \int_G g(y-x) dm'(x) = \lambda \int_G f dm. \end{aligned}$$

Hence $m' = \lambda m$. Note that the use of Fubini's theorem was legitimate in the preceding calculation, since the integrands $g(y)f(x+y)$ and $g(y-x)f(y)$ are in $C_c(G \times G)$.

Thus Haar measure is unique, up to a multiplicative positive constant. If G is compact, it is customary to normalize m so that $m(G) = 1$. If G is discrete, any set consisting of a single point is assigned the measure 1. These requirements are of course contradictory if G is a finite group, but this will cause us no difficulty.

Having established the uniqueness of m , we shall now change our notation, and write $\int_G f(x) dx$ in place of $\int_G f dm$. Thus dx, dy, \dots will always denote integration with respect to Haar measure.

1.1.4. For any Borel set E in G , $m(-E) = m(E)$. For if we set $m'(E) = m(-E)$, m' is a Haar measure on G , and so there is a constant λ such that $m(-E) = \lambda m(E)$ for all Borel sets E . Taking E so that $-E = E$, we see that $\lambda = 1$.

1.1.5. Translation in $L^p(G)$. If G is a LCA group and $1 \leq p \leq \infty$, we shall write $L^p(G)$ in place of $L^p(m)$ (see Appendix E7). It is clear that the L^p -norms are translation invariant, i.e., that

$$(1) \quad \|f_x\|_p = \|f\|_p \quad (x \in G),$$

where, we recall, f_x is the translate of f defined by

$$(2) \quad f_x(y) = f(y - x) \quad (y \in G).$$

THEOREM. Suppose $1 \leq p < \infty$ and $f \in L^p(G)$. The map

$$(3) \quad x \rightarrow f_x$$

is a uniformly continuous map of G into $L^p(G)$.

Proof: Let $\varepsilon > 0$ be given. Since $C_c(G)$ is dense in $L^p(G)$ (Appendix E8) there exists $g \in C_c(G)$, with compact support K , such that $\|g - f\|_p < \varepsilon/3$, and the uniform continuity of g (Appendix B9) implies that there is a neighborhood V of 0 in G such that

$$(4) \quad \|g - g_x\|_\infty < \frac{\varepsilon}{3} [m(K)]^{-1/p}$$

for all $x \in V$. Hence $\|g - g_x\|_p < \varepsilon/3$, and so

$$\|f - f_x\|_p \leq \|f - g\|_p + \|g - g_x\|_p + \|g_x - f_x\|_p < \varepsilon$$

if $x \in V$. Finally, $f_x - f_y = (f - f_{y-x})_x$, so that $\|f_x - f_y\|_p < \varepsilon$ if $y - x \in V$, and the proof is complete.

Note that the same theorem (with the same proof) is true with $C_0(G)$ in place of $L^p(G)$, but that it is false for $L^\infty(G)$, unless G is discrete.

1.1.6. Convolutions. For any pair of Borel functions f and g on the LCA group G we define their convolution $f * g$ by the formula

$$(1) \quad (f * g)(x) = \int_G f(x - y)g(y)dy$$

If we continue in this way, we see that

$$V \subset \bigcap_{n=1}^{\infty} (Y + 2^{-n}V).$$

Since $\{2^{-n}V\}$ is a local base, it now follows from (a) of Theorem 1.13 that $V \subset \bar{Y}$. But $\bar{Y} = Y$. Thus $V \subset Y$, which implies that $kV \subset Y$ for $k = 1, 2, 3, \dots$. Hence $Y = X$, by (a) of Theorem 1.15, and consequently $\dim X \leq m$. $////$

1.23 Theorem *If X is a locally bounded topological vector space with the Heine-Borel property, then X has finite dimension.*

PROOF. By assumption, the origin of X has a bounded neighborhood V . Statement (f) of Theorem 1.13 shows that \bar{V} is also bounded. Thus \bar{V} is compact, by the Heine-Borel property. This says that X is locally compact, hence finite-dimensional, by Theorem 1.22.

Metrization

We recall that a topology τ on a set X is said to be *metrizable* if there is a metric d on X which is compatible with τ . In that case, the balls with radius $1/n$ centered at x form a local base at x . This gives a necessary condition for metrizability which, for topological vector spaces, turns out to be also sufficient.

1.24 Theorem *If X is a topological vector space with a countable local base, then there is a metric d on X such that*

- (a) *d is compatible with the topology of X ,*
- (b) *the open balls centered at 0 are balanced, and*
- (c) *d is invariant: $d(x + z, y + z) = d(x, y)$ for $x, y, z \in X$.*

If, in addition, X is locally convex, then d can be chosen so as to satisfy (a), (b), (c), and also

- (d) *all open balls are convex.*

PROOF. By Theorem 1.14, X has a balanced local base $\{V_n\}$ such that

$$(1) \quad V_{n+1} + V_{n+1} + V_{n+1} + V_{n+1} \subset V_n \quad (n = 1, 2, 3, \dots);$$

when X is locally convex, this local base can be chosen so that each V_n is also convex.

PROOF. Let S be the sphere which bounds the open unit ball B of \mathbb{C}^n . Thus $z \in S$ if and only if $\sum |z_i|^2 = 1$, and $z \in B$ if and only if $\sum |z_i|^2 < 1$.

Suppose $f: \mathbb{C}^n \rightarrow Y$ is an isomorphism. This means that f is linear, one-to-one, and $f(\mathbb{C}^n) = Y$. Put $K = f(S)$. Since f is continuous (Lemma 1.20), K is compact. Since $f(0) = 0$ and f is one-to-one, $0 \notin K$, and therefore there is a balanced neighborhood V of 0 in X which does not intersect K . The set

$$E = f^{-1}(V) = f^{-1}(V \cap Y)$$

is therefore disjoint from S . Since f is linear, E is balanced, and hence connected. Thus $E \subset B$, because $0 \in E$, and this implies that the linear map f^{-1} takes $V \cap Y$ into B . Since f^{-1} is an n -tuple of linear functionals on Y , the implication (d) \rightarrow (a) in Theorem 1.18 shows that f^{-1} is continuous. Thus f is a homeomorphism.

To prove (b), choose $p \in \bar{Y}$, and let f and V be as above. For some $t > 0$, $p \in tV$, so that p lies in the closure of

$$Y \cap (tV) \subset f(tB) \subset f(t\bar{B}).$$

Being compact, $f(t\bar{B})$ is closed in X . Hence $p \in f(t\bar{B}) \subset Y$, and this proves that $\bar{Y} = Y$. ////

1.22 Theorem Every locally compact topological vector space X has finite dimension.

PROOF. The origin of X has a neighborhood V whose closure is compact. By Theorem 1.15, V is bounded, and the sets $2^{-n}V$ ($n = 1, 2, 3, \dots$) form a local base for X .

The compactness of \bar{V} shows that there exist x_1, \dots, x_m in X such that

$$\bar{V} \subset (x_1 + \frac{1}{2}V) \cup \dots \cup (x_m + \frac{1}{2}V).$$

Let Y be the vector space spanned by x_1, \dots, x_m . Then $\dim Y \leq m$. By Theorem 1.21, Y is a closed subspace of X .

Since $V \subset Y + \frac{1}{2}V$ and since $\lambda Y = Y$ for every scalar $\lambda \neq 0$, it follows that

$$\frac{1}{2}V \subset Y + \frac{1}{4}V$$

so that

$$V \subset Y + \frac{1}{2}V \subset Y + Y + \frac{1}{4}V = Y + \frac{1}{4}V.$$

2011-06-06

Gmail - exercise



Douglas Lundholm <douglas.lundholm@gmail.com>

exercise

Heiko Gimperlein <gimperlein@math.ku.dk>
To: Douglas Lundholm <lundholm@math.ku.dk>

Wed, Jun 1, 2011 at 7:25 PM

Dear Douglas,

I still owe you the solution of exercise 7.1c:

Let x be as in the hint and set $y = \sum_{\{n \geq 1\}} x^n$. Then $\phi(y) = \phi(x) + \phi(x)\phi(y) = 1 + \phi(y)$, or $0 = 1$. Contradiction.

If ϕ is not identically 0, there's an x such that $\phi(x)=1$. If A has a unit (called e), $1 = \phi(x) = \phi(ex) = \phi(e)\phi(x) = \phi(e)$. Since $|e|=1$, $|\phi| \geq 1$. But from the first part, $|\phi| \leq 1$, so $|\phi|=1$.

By the way: It's always a good idea to bring your own sponge and chalk to the biology department. Chalk is available in the mail room, and you're welcome to take the sponge from my office (there's also some chalk there). Usually, kursussal 4 is locked, and one has to ask one of the biologists on the floor to open it.

Have a good weekend,
Heiko

b)

$$x \in \overline{X/M} < X$$

$$\ell \in (X/\overline{M})^*$$

$\Rightarrow \ell \in X^*$ which vanishes on \overline{M}

$$\Rightarrow \ell = 0.$$

$$\Rightarrow A(X/\overline{M})^* = \{0\}$$

Hahn-Banach \Rightarrow ~~X/\overline{M}~~ $X/\overline{M} = \{0\}$

$$\Rightarrow X = \overline{M}$$

Bsp. 7

1. a) Let H be a complex vector space with

$$[\cdot, \cdot] : H \times H \rightarrow \mathbb{C} \text{ sesquilinear \& s.t. } [\bar{h}, h] \geq 0 \quad \forall h \in H.$$

Show Cauchy-Schwarz: $|[f, g]| \leq \|f\|_H^2 \|g\|_H$.

Pf: Sez. qv linear \Rightarrow (x) holds if $f=0$ or $g=0$.

\Rightarrow WLOG: $f, g \neq 0$. Note: we could still have $[f, f] = 0$ or $[g, g] = 0$.

(If $[h, h] = 0 \quad \forall h \in H$ then) If $[f, f] = 0$:

$$0 \leq [f + \alpha g, f + \alpha g] = \underbrace{[f, f]}_0 + \underbrace{[\alpha f, g]}_{\alpha \cdot [f, g]} + \underbrace{[f, \alpha g]}_{\alpha \cdot [f, g]} + \underbrace{[\alpha g, \alpha g]}_{\alpha^2 [g, g]} = \alpha^2 [g, g]$$

$$\begin{aligned} &= \alpha [f, g] + \overline{\alpha} [g, f] = \alpha [f, g] + (\alpha^* [f, g]) = \alpha^2 [f, g] \\ &= 2 \operatorname{Re} \alpha [f, g] + |\alpha|^2 [g, g] \end{aligned}$$

Taking $\alpha > \epsilon, \alpha = \pm i\epsilon \Rightarrow [f, g] = 0 \Rightarrow$ (x) holds
 $\epsilon \text{ small} > 0$

\Rightarrow WLOG $\|f\|_H^2 \neq 0$. Set $f' := \frac{f}{\|f\|_H^2}, g' := \frac{g}{\|f\|_H^2}$

\Rightarrow WLOG (x) $\Leftrightarrow |[f, g]| \leq 1$ for $\|f\|_H^2 = \|g\|_H^2 = 1$

$$0 \leq [f + \alpha g, f + \alpha g] = [f, f] + \alpha [f, g] + \overline{\alpha} [g, f] + |\alpha|^2 [g, g]$$

$$= 1 + 2 \operatorname{Re} \alpha [f, g] + |\alpha|^2$$

If $[f, g] = 0$ then we are done \Rightarrow WLOG $[f, g] \neq 0 \Rightarrow \alpha := -\frac{[f, g]}{|[f, g]|}$

$$\Rightarrow 0 \leq 1 - 2 |[f, g]| + 1 \Leftrightarrow |[f, g]| \leq 1$$

The restriction problem

$$\text{for which } p \in \mathbb{N} \quad \|f\|_{L^p(\mathbb{S}^{n-1})} \leq C \|f\|_{L^p(\mathbb{R}^n)}$$

Restriction Conjecture (for the sphere):

$$(*) \text{ holds } \Leftrightarrow \left[p \geq \frac{n+1}{n-1} q \right] \quad \left[p < \frac{2n}{n+1} \right]$$

Thomas-Stein restriction thm: (*) holds for $q=2$

Fourier transforms & measures

Def. Let μ be a measure on \mathbb{R}^n with finite total var.

φ a Schw. fun.

the Fourier transf. of μ : $\widehat{\mu}(z) = \int_{\mathbb{R}^n} e^{-iz \cdot y} d\mu(y)$

the convolution of $\varphi \ast \mu$: $\varphi \ast \mu(x) = \int_{\mathbb{R}^n} \varphi(x-y) d\mu(y)$

both well-def & bounded

Lemma: μ, ν finite E.V. meas.

$$v \in S(\mathbb{R}^n)$$

$$1) \quad \widehat{v \ast \mu} = \widehat{v} \ast \widehat{\mu}$$

$$2) \quad \widehat{v \mu} = \widehat{v} \ast \widehat{\mu}$$

$$3) \quad \int v d\mu = \int \widehat{v} d\mu$$

can M. to prove using

Riesz Thm

$$\text{Car } \int_{\mathbb{R}^n} f \widehat{g} d\mu = \int_{\mathbb{R}^n} \widehat{g} \ast \widehat{f} dx$$

$$\langle \widehat{f}, \widehat{g} \rangle_{L^2(\mathbb{R}^n)} = \langle f, g \ast \widehat{\delta} \rangle_{L^2(\mathbb{R}^n)}$$

$$\text{then } \|\widehat{f}\|_{L^2(\mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)}$$

$$\Leftrightarrow \langle \widehat{f}, \widehat{f} \rangle_{L^2(\mathbb{R}^n)} \leq C^2 \|f\|_p^2$$

$$\langle f, f \ast \widehat{\delta} \rangle$$

Hilber

$$\text{iff } \|\widehat{f \ast \delta}\|_p' \leq C^2 \|f\|_p$$

Recall: • Principle of non-stationary phase

$$u \in C_c^\infty(\mathbb{R}^n), \quad \psi \in C^\infty(\mathbb{R}^n; \mathbb{R})$$

$\nabla \psi \neq 0$ on supp of u . Then NNEM

$$\left| \int_{\mathbb{R}^n} u(x) e^{i\psi(x)} dx \right| \lesssim \delta^{-N}$$

• Principle of stationary phase:

$$u \in C_c^\infty(\mathbb{R}^n), \quad \psi \in C^\infty(\mathbb{R}^n; \mathbb{R}) \quad \text{w. only one}$$

stationary pt. $x_0 \in \text{supp } u$ & non-degen. Then DC

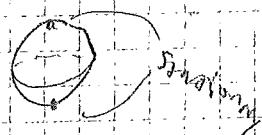
$$\left| \int_{\mathbb{R}^n} u(x) e^{i\psi(x)} dx - C u(x_0) e^{i\psi(x_0)} \delta^{-n/2} \right| \lesssim \delta^{(n+1)/2}$$

Proposition: Let σ be surface meas. on the unit sph $S^{n-1} \subset \mathbb{R}^n$.

$$\text{Then } \hat{\sigma}(x) = C \frac{e^{i2\pi|x|}}{|x|^{(n+1)/2}} + C \frac{e^{-i2\pi|x|}}{|x|^{(n+1)/2}} + O(|x|^{n/2})$$

Proof: By rotational symmetry of σ (and hence $\hat{\sigma}$) it is enough to estimate

$$\hat{\sigma}(x_{\text{ext}}) = \int_{S^{n-1}} e^{-2\pi i \langle x_{\text{ext}}, w \rangle} \sigma(dw)$$



Let Ψ_+, Ψ_- be smooth cut-off funcs supported

near the NP & SP resp.

$$\hat{\sigma}(x_{\text{ext}}) = \int \Psi_+(w) e^{-i2\pi \langle x_{\text{ext}}, w \rangle} d\sigma(w) + \int \Psi_-(w) e^{-i2\pi \langle x_{\text{ext}}, w \rangle} d\sigma(w)$$

(in coordinate basis)

$$\text{Start-phase: } \int \Psi_+(w) e^{-i2\pi \langle x_{\text{ext}}, w \rangle} d\sigma(w) = C \Psi_+(x_{\text{ext}}) e^{-i2\pi \omega_+(x_{\text{ext}})} + O(\delta^{n/2})$$

\Rightarrow condition II

Thomason: when $1 \leq p \leq \frac{2n+2}{n+3}$ then $\|f\|_{L^{p+1}} \|_{L^{\infty}} \leq \|f\|_p$

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ s.t. $\varphi = 1$ on supp of σ .

$$\Psi_k(x) = \varphi(\delta^k x) + \varphi(\delta^{-k} x) \quad \text{supp on } |x| \approx \delta^{n/2},$$

$$\text{where } |\Psi_k| \approx 1, \quad 1 = \varphi \otimes \sum_{k \geq 0} \Psi_k(x)$$

$$\|f \cdot \Psi_k\|_p = \|f \cdot \varphi \cdot \sum_{k \geq 0} \Psi_k(x)\|_p \leq \|f \cdot \varphi\|_p \|f\|_p$$

o Young's

$$\leq \|f \ast \varphi_{10}^\delta\|_p + \sum_{k>0} \|f \ast \varphi_{10}^\delta\|_p$$

We need $\|\varphi_{10}^\delta\|_p \leq 2^{-pk} \|f\|_p$

Interpolate $1 \rightarrow \infty$ bounds
 $2 \rightarrow 2$

$$\begin{aligned} & \stackrel{1 \rightarrow \infty}{\|f \ast \varphi_{10}^\delta\|_\infty} \leq \|\varphi_{10}^\delta\|_p \leq 2^{-(n-1)\delta/2} \\ (\|f\|_1) \end{aligned}$$

$$\stackrel{2 \rightarrow 2}{\|f \ast g\|_2 \leq \|g\|_\infty \|f\|_2} \text{ i.e., need estimates for}$$

$$\|\varphi_{10}^\delta\|_\infty = \|\widehat{\varphi_{10}^\delta} \ast 0^\perp\|_\infty$$

$$\widehat{\Phi_{10}}(x) = 2^{nk} \widehat{\Phi}_0(2^k x) \Rightarrow |\widehat{\Phi_{10}}(x)| \leq \frac{2^{nk}}{(1+2^k|x|)^N} \text{ N const.} \text{ Nlog. off k}$$

suff. to show $\frac{2^{nk}}{(1+2^k|x|)^N} \leq 2^k$

Just calculate integral! in M
worst possible way

$$\|T\|_{1 \rightarrow \infty} \leq 2^{\frac{n-1}{2}k}$$

$$\|T\|_{2 \rightarrow 2} \leq 2^k$$

Propagate these bounds for $\theta = \frac{p}{p'}$

$$\theta = \frac{n+2}{n+3} + p' > 0$$

proof for $p = \frac{n+2}{n+3}$

We will need Stein interpolation

$$T_\theta f = \sum_{k>0} 2^{-\frac{n+1}{2}k} (2^{-\frac{n+1}{n+3}k})^\theta f \ast (\varphi_{10}^\delta)$$

$$\text{Res 20: } \left\| \sum_{k>0} 2^{-\frac{n+1}{2}k} f \ast \varphi_{10}^\delta \right\|_\infty \leq \|f\|_1$$

$$\text{Res 21: } \left\| \sum_{k>0} 2^{(1+\theta)k} f \ast \varphi_{10}^\delta \right\|_2 \leq \|f\|_2$$

The Trace Formula - Weyl's Theorem (1959)

The trace of a trace class operator on a Hilbert space

is the sum of its eigenvalues with mults. (dim of gen. eigenspace.)

$$\text{i.e., } \operatorname{tr} T = \sum_{j=1}^{\infty} \lambda_j(T)$$

Proof: Let T be a trace class op.

$$\text{by defn. } \operatorname{tr} T = \sum_{n=1}^{\infty} \langle T e_n, e_n \rangle \quad \{e_n\} \text{ ONR}$$

$\begin{matrix} \text{Trivial} \\ \text{if } \sum_{n=1}^{\infty} \lambda_n(T) < \infty \end{matrix}$

general
case

$$T e_n = \lambda_n e_n \text{ or } T e_n = \lambda_n w_n + w_{n+1}$$

use G-S to get orthogonal set $\{f_n\}$

$$T f_n = \lambda_n f_n + \text{lin. comb. of } f_{n+1}, f_{n+2}, \dots$$

$$\langle T f_n, f_n \rangle = \lambda_n$$

problem: $\{f_n\}$ might not be a basis

let $\{f_m\}$ basis for $(\text{span } \{f_n\})^\perp$

$$\Rightarrow \operatorname{tr} T = \sum_{n=1}^{\infty} \langle T e_n, f_n \rangle + \sum_{m=1}^{\infty} \langle T e_m, f_m \rangle$$

\downarrow
 $\text{show } = 0$

Lemma 6: T cpt op on H

K ortho comp. of MS eigenvectors and gen. eigen.

Then (i) K Mv. subspace for T^*

(ii) Spec. of T^* | K consists only of $\lambda = 0$

Proof: (i) $T^* e = \lambda e + f$

suppose $\langle e, f \rangle \neq 0$, $\langle e, e \rangle \neq 0$

$$\langle e, T^* e \rangle = \langle T e, e \rangle = \langle \lambda e + f, e \rangle = 0$$

(ii) if $\lambda \neq 0$, eigen. of T^* on K .

then λ eigen. of T on H with finite mult.

(i.e. $\dim N(T-\lambda) < \infty$)

$$\text{but } \dim N(T-\lambda) = \dim N(T^* - \lambda^*) \leq \dim N(T^*)$$

$$\text{Since } u \in N((T^* - \lambda)^*) \setminus N(T^* - \lambda)^*.$$

$$(T^* - \lambda)v \neq 0 \text{ has no sol. since } (T^* - \lambda)^* v = (\lambda - \lambda)^* v = 0$$

$$(T^* - \lambda)^* v = (T^* - \lambda)u = 0$$

Fredholm alt., \exists w s.t. difference of T , $(T - \lambda)w = 0$. $\langle w, u \rangle \neq 0$

$$\sum_{n=1}^{\infty} \langle Th_n, h_n \rangle = \sum_{n=1}^{\infty} \langle h_n, T^* h_n \rangle = \sum_{n=1}^{\infty} \langle T^* h_n, h_n \rangle$$

Lidskii's Lemma: If T finite-dim op. with no non-zero eigenvalues. Then $\|T\| \geq 0$.

Recall: If compact, then so is $A := \|T\|$

$$\{s_j\} \text{ singular values of } T \subseteq \text{eva of } A$$

$$s_j(T)$$

Lemma 7: T comp. with n -th eigenvalues $\lambda_1, \lambda_2, \dots$ arranged in decreasing order of their abs. values. (Multi. mult.)

Then $\forall N \in \mathbb{N}$

$$\prod_{j=1}^N |\lambda_j(T)| \leq \prod_{j=1}^N s_j(T)$$

Proof: Let B_N be the space spanned by the first N eigenv.

$$P_N \text{ orth. proj. on } B_N \quad T_N := T|_{B_N}, \quad A_N := \|T_N\|$$

$$T_N = U_N A_N \quad \Rightarrow \quad \|\det T_N\| = \det A_N$$

$$\Rightarrow \prod_{j=1}^N |\lambda_j(T_N)| = \prod_{j=1}^N s_j(A_N) \quad (*)$$

$T P_N$ acts as T_N on E_N

$\Rightarrow 0$ on E_N^\perp

$\Rightarrow \|T P_N\| = \begin{cases} A_N & \text{on } E_N \\ 0 & \text{on } E_N^\perp \end{cases}$

$$\Rightarrow \lambda_j(A_N) = s_j(T P_N), \quad j=1, \dots, N$$

$$s_j(T P_N) = s_j(P_N^* T^*) \leq \|P_N^*\| s_j(T^*) = s_j(T)$$

Insert (*)

$$\Rightarrow \prod_{j=1}^N |\lambda_j(T_N)| \leq \prod_{j=1}^N s_j(T)$$

Lemma 8: Let $a_1 \geq a_2 \geq \dots$ and $b_1 \geq b_2 \geq \dots$

be two decreasing seq. of real numbers starting by ∞

$$\sum_{j=1}^N a_j \leq \sum_{j=1}^N b_j \quad \forall N$$

Let F be a convex fun on \mathbb{R} s.t. $F(x) \rightarrow 0$ as $|x| \rightarrow \infty$

$$\text{then } \sum_{j=1}^N F(a_j) \leq \sum_{j=1}^N F(b_j) \quad \forall N$$

$$\log \text{on } H \leq H \Rightarrow \sum_{j=1}^N \log |a_j| \leq \sum_{j=1}^N \log |b_j|$$

Choose $F(x) = \exp(x)$

$$\Rightarrow \sum_{j=1}^N |a_j| \leq \sum_{j=1}^N |b_j| \quad (\star\star)$$

Choose $F(x) = \log(1+re^x)$, $r > 0$

$$\Rightarrow \sum_{j=1}^N \log(1+r|a_j|) \leq \sum_{j=1}^N \log(1+r|b_j|)$$

$$\Rightarrow \prod_{j=1}^N (1+r|a_j|) \leq \prod_{j=1}^N (1+r|b_j|) \quad (+)$$

Let $\{\lambda_n\}$ ONB of H

P_N orth. Proj. onto $\text{span } \{\lambda_1, \dots, \lambda_N\}$, $T_N := P_N T P_N$

Lemma 9: Suppose that T is trace class with no zero eva.

$$(i) \lim_{N \rightarrow \infty} \|T_N - T\| = 0$$

$$(ii) \lim_{N \rightarrow \infty} \text{tr} T_N = \text{tr} T$$

Recall

$$r = \sup \{|a_j| : j \in \mathbb{N}\}$$

(iii) Denote the spec. radius of T_N by σ_N

Then $\sigma_N \rightarrow 0$ as $N \rightarrow \infty$

prove: (i) finite rank op's are dense among cpt. op's

(ii) by def. of trace

(iii) $(T + \lambda)$ is invertible for all $\lambda \neq 0$

Given $\delta > 0$ def. $m(\delta) = \max_{|\lambda| \geq \delta} \| (T + \lambda)^{-1} \|$

By (6), we can choose $M(\delta)$ so large that for $N > M(\delta)$,

$$\|T_N - T\| \leq \frac{1}{M(\delta)}$$

Consider $\|(T_N - T)(T - \lambda)^{-1}\| \leq \|T_N - T\| \|T - \lambda\|^{-1} < 1$, (7) \Rightarrow 8.

so we write

$$T_N - \lambda = T_N - T + T - \lambda = ((T_N - T)(T - \lambda)^{-1} + I)(T - \lambda)$$

Invisible whenever $|\lambda| \geq 8$, $N >$

$$\Rightarrow \sigma_N < 8 \text{ for } N > M(\delta)$$

\times

Denote the eigenvalues of T_N as $\lambda_j^{(N)}$, $j=1, \dots, N$.

$$\text{def. polynomial } D_N(\lambda) := \prod_{j=1}^N (1 - \lambda \lambda_j^{(N)})$$

Lemma 10: $\lim_{N \rightarrow \infty} D_N(\lambda) = e^{-\lambda \operatorname{Tr} T}$

$$\text{proof: } \log D_N(\lambda) = \sum_{j=1}^N \log (1 - \lambda \lambda_j^{(N)}) \quad \text{Note: } |\lambda_j^{(N)}| < \sigma_N$$

$$\frac{D'_N}{D_N} = \frac{\sum \lambda_j^{(N)}}{1 - \lambda \sum \lambda_j^{(N)}} \quad \text{for } |\lambda| < \frac{1}{\sigma_N} \therefore \frac{D'_N}{D_N} = \frac{1}{\lambda} \sum \frac{\lambda_j^{(N)}}{1 - \lambda \lambda_j^{(N)}}$$

$$= - \sum_{j=1}^N \sum_{k=1}^{\infty} \lambda_j^{(N)} \lambda_j^{(N)k}$$

$$S_k^{(N)} := \sum_{j=1}^N \lambda_j^{(N)k} \quad \Rightarrow \quad (\lambda) \frac{D'_N}{D_N} = - \sum_{k=1}^{\infty} S_k^{(N)} \lambda^k$$

$$\text{we have } S_1^{(N)} = \operatorname{tr} T_N$$

$$\text{For } k > 1: |S_k^{(N)}| \leq \sum_{j=1}^N |\lambda_j^{(N)}| \sigma_N^{k-1}$$

$$\text{Use (6) on } T_N \text{ for } k > 1 \quad \leq \sigma_N^{k-1} \sum_{j=1}^N \sigma_j(T_N)$$

$$\leq \sigma_N^{k-1} \|T_N\|_F \leq \|T\|_F$$

Rewrite (6)

$$\frac{D'_N}{D_N} + \operatorname{tr} T = \operatorname{tr} T + \operatorname{tr} T_N - \sum_{k=0}^{\infty} S_k^{(N)} \lambda^k$$

$$\frac{\sigma_N(\lambda)}{1 - \sigma_N(\lambda)}$$

$\rightarrow 0$, $N \rightarrow \infty$

for λ in open set

$$\lim_{N \rightarrow \infty} \int_{\mathbb{C}} \left(\frac{D'_N}{D_N} + \operatorname{tr} T \right) d\lambda' = 0$$

$\rightarrow \lambda$ for λ in bold set

$$|\rho_n(z)| \leq \prod_{j=1}^N (1+|z_j|) |\rho_j(z_j)| \leq \prod_{j=1}^N (1+|z_j|) s_j(z_j)$$

use (1) with $r=|z|$

$N \rightarrow \infty$

$$\Rightarrow |e^{-\lambda \sqrt{T}}| \leq \prod_{j=1}^{\infty} (1+|z_j|) s_j(z_j)$$

$$M r < c r \Rightarrow |(e^{-\lambda \sqrt{T}})| \leq \prod_{j=1}^M (1+|z_j|) \prod_{j=M+1}^{\infty} e^{121 s_j(z_j)}$$

$$= P_n G(\tau) \exp \left(\sum_{j=M+1}^{\infty} 121 s_j(z_j) \right)$$

Choose arg τ s.t. $-\lambda \sqrt{T} > 0$ and take $|z_j| \rightarrow 0$

$$\Rightarrow |e^{-\lambda \sqrt{T}}| \leq \sum_{m=1}^{\infty} s_j \rightarrow 0, M \rightarrow \infty$$

$$\Phi(r) = \int_{B_r(0)} B \, dx$$

$$|\Phi(r)| \leq \int_{|x| < r} |B| \, dx \leq \|B\|_q \|u\|_{L^q} \left(\frac{r}{\pi}\right)^{-1}$$

$$= \int_{B_r(0)} |B| u \, dx \sim_{\infty} (r r^\alpha)^{\frac{q-1}{q}} \sim r^{\frac{2(q-1)}{q}}$$

$$\Rightarrow \frac{|\Phi(r)|^2}{r} \leq r^{\frac{4(q-1)}{q}-2} \sim r^{(4q-2q+2)(q-1)/q-2} \sim r^{\frac{2(q-1)}{q}}$$

$$V(x) = n \left(C \int_{B^2} |\ln(1+|x|)|^{-2} \, dx \right)^{\frac{2-q}{q}} \|u\|^{q/2} (1+|x|)^{-2}$$

$$a = \frac{2}{q-2}$$

$$N \lesssim C \int_{B^2} V^{1+\alpha} (1+|x|)^{2a} \, dx = n^{1+\alpha} C^{1+(1+\alpha)\frac{2-q}{q}} \left(\int |\ln(1+|x|)|^{-2} \, dx \right)^{\frac{(1+\alpha)^2}{q}}$$

$$= n^{1+\alpha} C^{1-\frac{1}{q}} \left(\int |\ln(1+|x|)|^{-2} + \frac{4}{q-2} \, dx \right)$$

$$= n^{1+\alpha} \underbrace{\int |\ln(1+|x|)|^{-2} \, dx}_{(\text{ten rechts})} \underbrace{\left(\int |\ln(1+|x|)|^{-2} \, dx \right)^{\frac{2-q}{q}}}_{(1)^{\frac{2-q}{q}}} + 1$$

$H^1(\mathbb{R})$:

$$u_0 := H_0 e^{-x^2/2} = c e^{-x^2/2}, \quad H_{k=0,1,2, \dots}$$

$$u_{\pm, k} := \chi_{\mathbb{R}_k} H_k e^{-x^2/2}, \quad k \text{ odd} \quad H_0 \neq c$$

$$u_{\pm, k} := \chi_{\mathbb{R}_k} (H_k + c_k H_0) e^{-x^2/2} \quad \begin{matrix} H_k \in \mathbb{S} \text{ even polynomial, } k \text{ even} \\ \text{odd} \end{matrix} \quad \begin{matrix} \text{odd} \\ \text{even} \end{matrix}, \quad k \text{ odd}$$

, k even , such $c_k \in \mathbb{R}$ s.t.

$$H_k(0) + c_k H_0(0) = 0$$

$$\Rightarrow u_0 = H_0 e^{-x^2/2}$$

$$u_{+, k} + u_{-, k} = H_k e^{-x^2/2}, \quad k \text{ odd}$$

$$u_{+, k} + u_{-, k} = H_k e^{-x^2/2} + c_k H_0 e^{-x^2/2}, \quad k \text{ even}$$

$$\Rightarrow \text{Span} \{ u_0, u_{\pm, k} \}_{k \in \mathbb{N}} = \text{Span} \{ H_k c^{x^2/2} \}_{k \in \mathbb{N}} = H^1(\mathbb{R})$$

$$\text{Span} \{ u_{\pm, k} \}_{k \in \mathbb{N}} \stackrel{?}{=} H^1(\mathbb{R})$$

$$u_{\pm, k} \in H_0^1(\mathbb{R}_k)$$