Notes on the uncertainty principle

Phan Thanh Nam

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The notes review material on uncertainty principle discussed on Monday 9 May. Moreover, I have fixed two unclear points remained in the proof of Beurling Theorem (see the end of the notes)[1].

From the physical point of view, the uncertainty principle means that we can not determine both of localization and momentum of a particle at the same time. In mathematical language, it can be said that we can not localize both of a function f and its Fourier transform $\widehat{f}$ at the same time. Note that if a normalized function $f \in L^2$ is the wave function of a particle then $|f(x)|^2$ is the density of the particle in the configuration space and $\widehat{f}(p)$ is the density of the particlein the momentum space [2] .

There are many ways to demonstrate the uncertainty principle, and in our discussion we considered three theorems. For simplicity we shall restrict our attention in 1 dimension.

Theorem 1 (Heisenberg's uncertainty principle)**.** *If $f \in L^2(\mathbb{R})$, $||f||_{L^2} = 1$ then*

$$\left(\int\limits_{\mathbb{R}} x^2 |f(x)|^2 dx\right)\left(\int\limits_{\mathbb{R}} p^2 |\widehat{f}(p)|^2 dp\right) \geq \frac{\pi}{2}.$$

Proof. Use

$$\int\limits_{\mathbb{R}} p^2 |\widehat{f}(p)|^2 dp = \int\limits_{\mathbb{R}} |\widehat{f'}(p)|^2 dp = 2\pi \int\limits_{\mathbb{R}} |f'(x)|^2 dx,$$

the integral by part (by approximation we can assume f smooth enough)

$$\operatorname{Re} \int\limits_{\mathbb{R}} f'(x) x f(x) dx = -\frac{1}{2} \int\limits_{\mathbb{R}} |f(x)|^2 dx,$$

and the Hölder inequality. □

Theorem 2 (Hardy's uncertainty principle)**.** *Assume that $|f(x)| \leq Ce^{-\alpha x^2}$ and $|\widehat{f}(p)| \leq Ce^{-\beta p^2}$ for $\alpha, \beta > 0$.*

(i) If $\alpha\beta > 1/4$ then $f = 0$.

[1] If you have any further comment, please contact to me: ptnam@math.ku.dk.

[2] Maybe up to some constant dependent on the convention of Fourier transform. In this course we use $\widehat{f}(p) = \int\limits_{\mathbb{R}} f(x) e^{-ipx} dx$

(ii) If $\alpha\beta = 1/4$ then $f(x) = const.e^{-\alpha x^2}$.

Proof. The proof follows from Tao's blog [4], except for the last point which I will explain.

It suffices to show (ii). By scaling, we may assume $\alpha = \beta = 1/2$ (recall that the Fourier transform of $e^{-x^2/2}$ is it self multiplying with some constant).

Since f decays faster than any exponential type, its Fourier transform $\widehat{f}$ is an entire function. Using $|f(x)| \le Ce^{-x^2/2}$ we get

$$\left|\widehat{f}(z)\right| = \left|\int_{\mathbb{R}} f(x)e^{-izx}dx\right| \le C\int_{\mathbb{R}} e^{-x^2/2}e^{x.\operatorname{Im} z}dx = C'e^{|\operatorname{Im} z|^2}, \quad \forall z \in \mathbb{C}.$$

Thus the entire function $F(z) = e^{z^2/2}\widehat{f}(z)$ is bounded in both of the real axis and the imaginary axis. We want to show that $F(z)$ is bounded in the whole complex plane, and then by Liouville's Theorem we can conclude that $F(z)$ must be a constant, which ends the proof.

To show that f is bounded, we need to use the Phrafmén-Lindelöf Theorem. This version is taken from [2] (Theorem 1, page 37). In fact, this version is equivalent to the strip-version we learned in the course.

Theorem (Phrafmén-Lindelöf)**.** *Let D be an angle of opening π/λ, and let $f(z)$ be a function analytic in D satisfying*

(i) $|f(z)| \le M$ for all $z \in \partial D$, and

(ii) $|f(z)| \le Ce^{|z|^{\lambda-\varepsilon}}$ for some $\varepsilon > 0$.

Then $|f(z)| \le M$ for all $z \in D$.

If we apply the Phrafmén-Lindelöf Theorem directly to $F(z)$ with the angle of opening $\pi/2$, then we need an estimate $|F(z)| \le Ce^{|z|^{2-\varepsilon}}$. However, we cannot deduce such estimate from the bound $|F(z)| \le C'e^{|\operatorname{Im} z|^2}$.

To overcome that, we can do as in Tao's blog [4]. Let $\delta > 0$ small and consider the entire function

$$g(z) = e^{i\delta z^2}F(z) = e^{i\delta z^2 + z^2/2}\widehat{f}(z)$$

Of course $g(z)$ is still bounded in the real line. Moreover, using $|F(z)| \le C'e^{|\operatorname{Im} z|^2}$ we have

$$|g(z)| \le C'e^{-\operatorname{Re} z.\operatorname{Im} z + |\operatorname{Re} z|^2/2} \le C'$$

if $\arg z = \theta$ for some $\theta < \pi/2$ and near $\pi/2$ enough.

Now we can apply the Phrafmén-Lindelöf Theorem to $g(z)$ on the angle $0 < \arg z < \theta$. Now in the Phrafmén-Lindelöf Theorem $\lambda > 2$, while $|g(z)| \le Ce^{|z|^2}$ and everything is fine.

Last remark: In Tao's blog, he use the maximum modulus principle instead of the Phrafmén-Lindelöf Theorem, and hence he need to modify further the function g (the modification is the same spirit of the proof of the Phrafmén-Lindelöf Theorem). □

Theorem 3 (Beurling [3])**.** *If $f \in L^1(\mathbb{R})$ and*

$$\iint_{\mathbb{R}^2} |f(x)\widehat{f}(y)|e^{|xy|}dxdy < \infty$$

then $f = 0$.

Note that this result reproduces the first statement (i) in Hardy's uncertainty principle. This result was then generalized to contain the critical case as follows (the result holds in all dimension).

Theorem 4 (Bonami-Demange-Jaming [1])**.** *If* $f \in L^2(\mathbb{R})$ *and*

$$\iint_{\mathbb{R}^2} \frac{|f(x)\widehat{f}(y)|}{(1+|x|+|y|)^N} e^{|xy|} dxdy < \infty$$

then $f = const.P(x)e^{-ax^2}$ *for some polynomial* $P(x)$ *of degree* $< (N-d)/2$*. In particular, if* $N \le d$ *then* $f = 0$*.*

Return to Beurling's original version in Theorem [3]. In our discussion, we got though the main idea of the proof in Hörmander [3], but there were two points we did not justify, which I want to explain below.

1) I did not explain why f is well-determined from $\widehat{f}$ via the inverse Fourier transform.

Answer: It suffices to show that $\widehat{f} \in L^1(\mathbb{R})$. We can argue as follows. If $f \not\equiv 0$ then $\widehat{M}(y) = \int |f(x)|e^{|xy|}dx$ must grow at least exponentially. Since $\int |\widehat{f}(y)|\widehat{M}(y)dy < \infty$, we have that $\int_{|y|\ge y_0} |\widehat{f}(y)|dy < \infty$ for some y_0 big. Moreover, $\widehat{f}$ is uniformly bounded. Therefore, $\widehat{f} \in L^1(\mathbb{R})$.

2) Show that f is entire and $|f(z)| \le Ce^{c|\operatorname{Im} z|^2}$. In Hörmander [3], to get this bound he assumed that $M(x) \ge Ce^{cx^2/2}$. In the case that $M(x)$ does not grow as fast, he use the Phrafmén-Lindelöf Theorem to conclude that f is a bounded function, and hence $f = 0$ (since $f \in L^1$). However, I do not know which version of the Phrafmén-Lindelöf Theorem he mentioned here, and I found difficult to deduce his claim from the version we cited above.

Answer: A way to get out from this situation is to use an idea in [1]. This is to introduce a new function g by $\widehat{g}(y) = \widehat{f}(y)e^{-y^2/2}$ and then show that

$$\iint_{\mathbb{R}^2} |g(x)\widehat{g}(y)|e^{|xy|}dxdy < \infty$$

After showing that, we have $\widehat{g} \in L^1(\mathbb{R})$ and by the above argument (with f replaced by $\widehat{g}$) we have $g \in L^1(\mathbb{R})$. Thus by replacing f by g if necessary we may assume that $|\widehat{f}(y)| \le Ce^{-y^2/2}$. Hence f is an entire function and from $|\widehat{f}(y)| \le Ce^{-y^2/2}$ and the inverse Fourier transform we have also $|f(z)| \le Ce^{|\operatorname{Im} z|^2/2}$ (similarly to the first estimate in the Proof of Hardy's uncertainty principle). Then we may follow the rest in Hörmander [3].

Finally, it remains to show that if $\widehat{g}(y) = \widehat{f}(y)e^{-y^2/2}$ then $\iint_{\mathbb{R}^2} |g(x)\widehat{g}(y)|e^{|xy|}dxdy < \infty$. In fact, we have

$$\begin{aligned} g(x) &= (2\pi)^{-1} \int_{\mathbb{R}} \widehat{f}(u) e^{-u^2/2} e^{ixu} du \\ &= (2\pi)^{-1} \iint_{\mathbb{R}^2} f(t) e^{-itu} e^{-u^2/2} e^{ixu} dt du \\ &= C \int_{\mathbb{R}} f(t) e^{-(t-x)^2/2} dt. \end{aligned}$$

Here in the last identity we have taken the integral in u first, and use the Fourier transform of Gaussian. Thus by the assumption on f we have

$$\begin{aligned} \iint_{\mathbb{R}^2} |g(x)\widehat{g}(y)| e^{|xy|} dx dy &\leq C \iiint_{\mathbb{R}^3} |f(t)\widehat{f}(y)| e^{-(t-x)^2/2 - y^2/2 + |xy|} dt dx dy \\ &= C \iint_{\mathbb{R}^2} |f(t)\widehat{f}(y)| e^{|ty|} A(t,y) dt dy < \infty \end{aligned}$$

where

$$\begin{aligned} A(t,y) &= \int_{\mathbb{R}} e^{-(t-x)^2/2 - y^2/2 + |xy| - |ty|} dx \leq \int_{\mathbb{R}} e^{-(t-x)^2/2 - y^2/2 + |x-t|.|y|} dx \\ &= C \int_{\mathbb{R}} e^{-(|x|-|y|)^2/2} dx = C \left(\int_0^{\infty} e^{-(x-|y|)^2/2} dx + \int_{-\infty}^{0} e^{-(-x-|y|)^2/2} dx \right) \\ &\leq C \left(\int_{\mathbb{R}} e^{-(x-|y|)^2/2} dx + \int_{\mathbb{R}} e^{-(-x-|y|)^2/2} dx \right) = 2C \int_{\mathbb{R}} e^{-x^2/2} dx = C'. \end{aligned}$$

References

[1] A. Bonami, B. Demange, P. Jaming, Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms, Rev. Mat. Iberoamericana, 2003. Online available: http://arxiv.org/abs/math/0102111

[2] B.Y. Lewin, Lecture on entire functions, AMS, 1996. Online available: http://www.ams.org/publications/online-books/mmono150-index

[3] L. Hörmander, A uniqueness theorem of Beurling for Fourier transform pairs. Ark. Mat. 29 (1991), 237-240.

[4] T. Tao's blog http://terrytao.wordpress.com/2009/02/18/hardys-uncertainty-principle